Geometry and Philosophy

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Plato’s Academy

ΑΓΕΩΜΕΤΡΗΤΟΣ
ΜΗΔΕΙΣ ΕΙΣΙΤΩ

Let no one ignorant of geometry enter here

The Big Question

How do we know about mathematical objects, if we do not sense them?

- One type of account ascribes a special, non-sensory, ability to know.
- Guiding principle: Avoid the crazy view!

The Divided Line

Plato's Analogy of the Divided Line

	Objects	Mental States
Intelligible World	The Good Forms	Intelligence (*noēsis*) or Knowledge (*epistēmē*)
	Mathematical Objects	Thinking (*dianoia*)
World of Appearances	Visible Things	Belief (*pistis*)
	Images	Imagining (*eikasia*)

Euclid
The Primacy of Geometry

- We have to go beyond sense experience.
- If mathematics is essentially geometry, then we do not have to go too far beyond.
- Books VII-IX of the *Elements* concern number theory.
- The proofs are derived from geometric relations.
- Even the basic properties of numbers are considered geometrically (e.g. square numbers).

Euclid's *Elements*
Book IX
Proposition 7

If a composite number multiplied by any number makes some number, then the product is solid.

Let the composite number A multiplied by any number B make C.
I say that C is solid.
[A solid number has at least three distinct factors.]

Since A is composite, it is measured by some number D.
[We now use 'divided by' for 'measured']
Let there be as many units in E as times that D measures A.
Since D measures A according to the units in E, therefore E multiplied by D makes A.
And, since A multiplied by B makes C, and A is the product of D and E, therefore the product of D and E multiplied by B makes C.
Therefore C is solid, and D, E, and B are its sides.
Therefore, if a composite number multiplied by any number makes some number, then the product is solid.

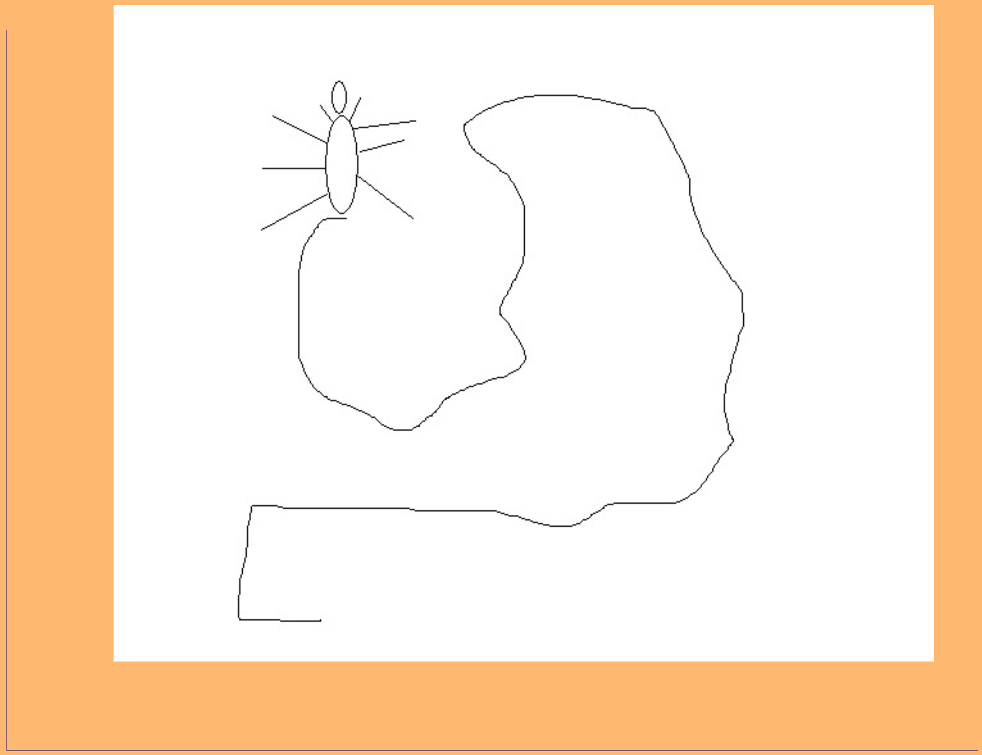
http://aleph0.clarku.edu/~djoyce/java/elements/bookIX/propIX7.html

Analytically

If ab=c, where a is composite, then c has three factors.

- Proof:
  - Since a is composite, a=de, for some d,e,<a.
  - So, c=deb.
  - So, c has three factors.

Descartes and the Spider



Descartes and Analytical Geometry

- *Geometry*, 1637
- Avoid appeals to geometric intuition, and diagrams!
- Analysis allows us to present results in their most general form.
- Rule 16: Representing everything algebraically, abstracting from specific numerical magnitudes as well as from geometrical figures, allows us to appreciate just what is essential.
- Algebraic techniques lead to the Calculus of Newton and Leibniz, 1665-1687.
- But, we're back to the crazy view.

Newton and Space

- The Science of the 17th Century relied on the calculus and its analytic methods.
- The background geometry of Newtonian space is Euclidean.
- Newton actually thought of the calculus as essentially geometrical.
- We seem to be stuck between Descartes's crazy view that we have a special mathematical capacity distinct from sensation, and Newton's reactionary view that all of mathematics is essentially geometry.

Kant, late 18th Century

- Euclidean space and our concepts of numbers are built into the way we see the world.
- We can hold the crazy view, because our ability to know about mathematics is built into us.
- The development of non-Euclidean geometries was unfortunate for the Kantian view.
- The Kantian must hold that non-Euclidean geometries are merely empty formalisms, and that Euclidean geometry is primary.

The Special Theory of Relativity

- Oops!
- Einstein's special theory of relativity includes curvatures in space-time.
- Consider the triangle formed by light rays emitted from three stars.
- The gravitational pull of bodies among these triangles will warp the lines.
- The sum of the angles of such interstellar triangles will be less than pi.

Hilbert, and the Independence of Geometry from Analysis

- Around the same time that Einstein was finding a physical instantiation for hyperbolic geometry, David Hilbert was working to separate geometry from arithmetic.
- Euclid had derived arithmetic from geometry.
- Descartes had derived geometry from arithmetic.
- Hilbert's axioms for geometry (1899) seek an independent formulation of geometry.

Hilbert’s Motivations

From a letter to Gottlob Frege, ~1903

I wanted to make it possible to understand those geometrical propositions that I regard as the most important results of geometric inquiries: that the parallel axiom is not a consequence of the other axioms, and similarly Archimedes’ axiom, etc. I wanted to answer the question whether it is possible to prove the proposition that in two identical rectangles with an identical base line the sides must also be identical, or whether as in Euclid this proposition is a new postulate. I wanted to make it possible to understand and answer such questions as why the sum of the angles in a triangle is equal to two right angles and how this fact is connected with the parallel axiom...” (Frege 1980: 38-9)

The Status of the Big Question

- We can keep mathematical knowledge and avoid the crazy view only if mathematics is essential (indispensable) to our best scientific theories.
- Willard van Orman Quine, June 25, 1908 – December 25, 2000
- Ockham’s razor
- If we can re-write physical theories without mathematical commitments, we can eliminate mathematics.
- (Perhaps we can save a bit on the basis of logical entailments.)

Hartry Field, Science without Numbers

- Step 1. Recast Newtonian Gravitational Theory replacing quantification over sets with quatification over regions of space-time.
  - Field's model is Hilbert's axiomatization of geometry.
  - Hilbert showed geometry was independent of analysis.
  - Field wants to show that physics is independent of analysis, too.
  - Field extends Hilbert's representation theorems from two to four dimensions, and adds predicates for dynamic notions of mass density and gravitational potential.
  - Representation theorems map space-time points onto real numbers, showing how to translate statements about the real numbers into statements about space-time geometry.
- Step 2. Argue that geometry is really an empirical theory of space-time.
- Step 3. Extend Step 1 to relativity (curved space-time), QM (probabilistic inferences), etc.

The recasting of NGT

Continuity

- In standard physics, the continuity of the reals is assumed via the Dedekind axiom of continuity.
  - A function f is continuous at a point c if for every neighborhood V of f(c) there exists a neighborhood U of c such that f(x) ε V whenever x ε U.

- For Field's project, continuity must be defined without real numbers. His continuity axiom second-order quantifies over collections of space-time points.
  - A region R is scalar-basic iff there are distinct points x and y such that either
    - a) R contains precisely those points z such that z Scal-Bet xy and not ($z \approx_{scal} x$) and not ($z \approx_{scal} y$); or
    - b) R contains precisely those points z such that y Scal-Bet xz and not ($z \approx_{scal} y$).
  - A function is continuous at c if, for any scalar-basic region that contains c, there is a spatio-temporally basic subregion that contains c.

My Opinion

- The crazy view is right: we have special mathematical insight.
- Field is right that we can re-write physics without sets.
- But, those re-formulations are unimportant, because we have independent mathematical knowledge.

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