Some Entries for *Key Terms in Logic*
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Drafts

Model Theory (491):

All communication involves the production and interpretation of statements. Semantics studies the interpretations of statements. The study of the statements of a formal system is called metatheory. Metatheory may be divided into proof theory and model theory. Proof theory studies the rules guiding inferences within the system. Model theory is a mathematical approach to semantics, in particular to the assignment of truth values to the statements of a theory.

A formal system consists of a language (vocabulary and formation rules) as well as axioms and rules for generating theorems. Given a formal system, the first step in model theory is to specify an interpretation of each particle of the system. Then, we provide rules governing the assignments of truth values to complex expressions on the basis of assignments of truth values to their component parts. A model is an interpretation of a system on which its theorems are true.

The semantics for the propositional calculus are easily given without model theory. Truth tables suffice to interpret the connectives, and propositional variables can be replaced by propositions or sentences.

The semantics for predicate logic normally proceeds using set theory. We specify a domain of interpretation for the variables of the system. For example, it is natural to use the domain of natural numbers to model the Peano Axioms, and to use sets to model the axioms of set theory. Models of physical theories naturally take the physical world as their domains. Non-standard models, using unintended domains of quantification, are available.

The next step in constructing a model is to assign elements of the domain to particles of the system. We assign particular objects to the constants. Predicates are normally interpreted as sets of objects in the domain, and n-place relations are taken as ordered n-tuples within the domain. An existentially quantified sentence is true in a model if there is an object in the domain of interpretation with the properties mentioned in the sentence. A universally quantified expression is true if the properties mentioned hold of every object in the domain.

For modal logics, Kripke models provide possible-worlds semantics. In a Kripke model, we start with a set of ordinary models, one for each possible world, and an accessibility relation among them. A statement is taken to be possible if there is an accessible possible world in which the statement is true. A statement is taken to be necessary if it is true in all possible worlds.

Model theory, developed in large part by Alfred Tarski and Abraham Robinson in the mid-twentieth century, has become a standard tool for studying set theory and algebraic structures. Major results of model theory include Paul Cohen's proof that the continuum hypothesis is independent of the axioms of Zermelo-Fraenkel (ZF) set theory including the axiom of choice, and that the axiom of choice itself is independent of the other axioms of ZF. Model theory is responsible for the so-called Skolem paradox, one of its earliest results.

See also: completeness, metalanguage, soundness, Hilbert

Semantic Tree (284):

In propositional logic, we can determine whether a set of formulas is consistent by examining the truth tables for the set. Alternatively, we can construct a semantic tree. Semantic trees may be used to test an argument for validity, since an argument is invalid if, and only if, the negation of its conclusion is consistent with the truth of its premises. Semantic trees are less cumbersome than truth tables, providing an easy method for testing a large set of formulas. They require less creative construction than natural deductions. There are decision procedures for semantic trees for propositional logic, which means that the procedure will always terminate in a solution.

To construct a semantic tree, we replace compound formulas with simpler sub-formulas with the same truth conditions. For example, we can replace '$\neg(A \vee B)$' with '$\neg A$' and '$\neg B$', since the longer formula is true if, and only if, the shorter ones are true. Some replacement rules branch, giving the construction the appearance of a tree. For example, any branch on which '$A \rightarrow \neg B$' appears divides into two branches, one which contains '$\neg A$' and the other which contains 'B'. The tree is completely constructed when all compound formulas have been replaced by either simple formulas or negations of simple formulas. If all branches contain a contradiction (a simple propositional formula and its negation) then the original set of formulas is consistent.

Semantic trees are useful in predicate logic, as well. For sets of formulas with only monadic predicates, the rules determine a decision procedure. There is no decision procedure for some sets of formulas with relational predicates. Semantic trees are useful in modal logic.

Semantic trees are also called truth trees, semantic tableau, or semantic tableaux.

See also: deduction, proof

Syntax (50):

Providing the syntax of a formal theory is the first step in its construction. The syntax consists of the language, or alphabet, in which the theory is written, along with rules for constructing well-formed formulas. Syntax is opposed to semantics, which governs the assignment of truth values to those formulas.

Variable (48):

Variables in formal systems function similarly to pronouns in natural language; their reference varies. Variables stand for other logical particles. Sentences or propositions may substitute for propositional variables. Objects or names of objects substitute for variables in first-order predicate logic. Higher-order logics also contain variables in predicate positions.

Church (391):

In 1936, Alonzo Church, Emil Post, and Alan Turing each proposed independent explications of the informal notion of an effectively computable function, or algorithm. The three formal notions were later shown to select the same class of mathematical functions. Further equivalent formulations have been produced by Gödel and others. The resulting thesis, that the computable functions are the recursive functions, has become known as Church's Thesis, or the Church-Turing thesis. (The notion of a recursive function traces to Gödel; Church considered a related class of functions called λ-definable.)

Church's Thesis is important because we want to know whether some problems have algorithmic solutions. For example, Church initially formulated the thesis in an attempt to answer the question of whether first-order logic was decidable. A theory is decidable if there is a procedure for determining whether any given formulais a theorem. Since recursion is formally definable, Church's Thesis provides a method for determining whether a particular problem has an effective solution. It provides a formal characterization of an intuitive concept.

Church's Thesis is also important because of its relation to Turing's formulation. Turing selected the recursive functions by considering the abilities of logical computing machines, or Turing Machines. Some writers who have compared Turing machines to human minds have used Church's Thesis with excessive enthusiasm, making broader claims about its implications than are supported by the thesis properly construed. In contrast, Church's Thesis is entirely silent about the nature and limitations of both the human mind and computing machines.

Church's Thesis is widely accepted. It appears that every recursive function is effectively computable, and it also appears that every effectively computable function is recursive. Still, there is some debate over whether Church's Thesis is provable. This debate has focused on whether any identification of an informal concept with a formal notion can be proven. Some philosophers consider Church's Thesis to be a working hypothesis. Others take it to be merely another mathematical refinement of a commonsense notion like set, function, limit, or logical consequence.

Church's Thesis is independent of the purely technical result called Church's Theorem. Church's Theorem shows that first-order logic is recursively undecidable, as long as it contains non-monadic predicates. (A monadic predicate, like 'is blue' takes only one variable, in contrast to relational predicates, like 'is bigger than' or 'is between'.) Appending Church's Thesis to Church's Theorem, as Church did, we can show that there is no effective decision procedure for first-order logic, no sure method for deciding whether a formula of first-order logic is valid. Though, there is a decision procedure for monadic predicate logic.

See also: computability, decidability

Hilbert (496):

David Hilbert, 1862-1943, should be viewed as the progenitor of metatheory in logic and mathematics. He was among the most prominent mathematicians of his time. His achievements in mathematics include work on geometry, number theory, algebra and analysis; he also contributed to the theory of relativity. Hilbert shaped the direction of mathematical thought in the twentieth century, most famously by framing the Paris Problems: twenty-three open questions presented at the 1900 International Congress of Mathematicians.

In philosophy of mathematics, Hilbert is variously characterized as a formalist and as a finitist. While Hilbert's views contain elements of both formalism and finitism, neither of these terms effectively captures the subtlety of his thought. Some formalists hold that mathematical theories are best understood as uninterpreted systems. Some finitists reject all infinitary results. In contrast, Hilbert believed both that some mathematical statements were true of real objects and that transfinite mathematics was legitimate.

In the early twentieth century, philosophers of mathematics struggled to understand the ramifications of various oddities of set theory, including Cantor's paradox (arising from consideration of the set of all sets), the Burali-Forti paradox (arising from consideration of the well-ordering of the ordinals), and Russell's paradox (arising from the assumption that every property determines a set). Intuitionists, e.g. Brouwer, concluded that the infinitary mathematics which leads to these paradoxes was illegitimate. Hilbert, in contrast, wished to establish finitistic foundations for infinitary mathematics.

Hilbert distinguished between real and ideal mathematical formulas. Real formulas are generally finitistic, and may be directly verified. Mathematical theories which included ideal elements were instead to be tested for their consistency. Unlike logicists like Frege, for whom the consistency of mathematics follows from the presumed truth of its axioms, Hilbert took the consistency of a set of axioms as sufficient evidence for mathematical legitimacy. Further, Hilbert took ideal formulas to be meaningless. Hilbert's emphasis on consistency and his claims about the meanings of ideal formulas have led people to consider him a formalist.

In addition to consistency, if one can show completeness, that every valid theorem is provable, then one could hope for a solution to all open mathematical problems. Hilbert tried to establish that mathematical theories were both consistent and complete by studying mathematical systems themselves. He thus founded the metamathematics and metalogic that characterize much of contemporary logical research.

Many logical theories are provably consistent and complete. In contrast, Gödel's incompleteness theorems struck a decisive blow against Hilbert's pursuits these results for mathematics. Gödel's first theorem showed that, for even quite weak mathematical theories, a consistent theory could not be complete. Gödel's second theorem proved that the consistency of a theory could never be proven within the theory itself. We can only prove that mathematical theories are consistent relative to other theories.

Hilbert's views survive in Hartry Field's fictionalism, which emphasizes the consistency of mathematical theories; in Mark Balaguer's plenitudinous platonism, which asserts that every consistent mathematical theory truly describes a mathematical realm; and in defenses of limited versions of Hilbert's Programme.

See also: intuitionism, logicism

Leopold Löwenheim (49):

Leopold Löwenheim, 1878-1957, is known for his work on relations and for an early negative result in logic, which has come to be known as the Löwenheim-Skolem theorem. The theorem says that if a first-order theory requires models of infinite size, models can be constructed of any infinite size.

See also: model

Thoraf Skolem (50):

Thoraf Skolem, 1887-1963, simplified Löwenheim's theorem which states that if a first-order theory requires models of infinite size, models can be constructed of any infinite size. 'Skolem's paradox' refers to the observation that first-order theories which yield theorems asserting the existence of uncountably many (or more) objects have denumerable models.

See also: model