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THE STROKE FUNCTION IN NATURAL DEDUCTION

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I will give three independent and consistent rules for the stroke function which yield the complete sentential calculus with suitable definitions. Two principal fragments, one of which yields the complete sentential calculus in implication and negation with suitable definitions, will also be discussed.

The conventions of derivation, which derive from G. GENTZEN, are due to F. B. FITCH[1]). Adequate formulations of the syntax are available in the literature and need not be repeated here. The reader should note that, as the system is axiomless and the rules are schemata, the usual rule of substitution is available. All rules are stated schematically.

I. The Rules of the System

Stroke Introduction ($|I$):

1	p	hypothesis
⋮	⋮	
2	$q \mid q$	assumption (i. e., it is assumed that this deduction can be completed).
3	$p \mid q$	1—2, $\mid I$

Stroke Elimination ($|E$):

1	p	hypothesis
2	$q \mid p$	hypothesis
3	$q \mid q$	1, 2, $\mid E$

Double Stroke Elimination ($||E$):

1	p	hypothesis
2	$(q \mid q) \mid p$	hypothesis
3	q	1, 2, $\mid\mid E$

[1]) For details confer F. B. FITCH, Symbolic Logic, New York 1952. Vertical lines to the right of other lines mark off subordinate proofs. Steps in a subordinate proof may not be repeated in a proof to which that subordinate proof is subordinate.

II. The Consistency of the System

Given the standard matrix for the PIERCE-SHEFFER stroke function conclusions reached by the elimination rules will have the value 1 when their hypotheses do. $(q|q)$ will have the value 1 when p does, but then q has the value 0, where p is premise and $(q|q)$ is conclusion of a subordinate proof immediately preceding a line reached by stroke introduction. But, if p and q cannot both have the value one, then $(p|q)$ must have the value one. Q. E. D.

The matrix:

	*1	0
*1	0	1
0	1	1

III. Independence

Stroke Introduction: If we interpret stroke as conjunction it can readily be seen that the elimination rules hold in any standard system but the introduction rule does not.

Double Stroke Elimination: It can be seen that only this rule will yield a conclusion of minimum length, where that conclusion is not included among the hypotheses.

Stroke Elimination: The matrix following, where * marks the designated truth values, can be seen to satisfy all rules except stroke elimination. (Where p has the value 2, and q has the value 1, $(q|q)$ has the value 3.)

The matrix:

	*1	*2	3
*1	3	2	1
*2	3	3	1
3	1	1	2

IV. The System is Complete

NICOD's Axiom and Rules may be derived within the system. But these are complete, hence, as the system is consistent, it is complete.

1) Substitution: This is available, as has been indicated above.

2) Detachment: This is C 3 among the schemata derived below.

3) Nicod's Axiom: This is C 5 among the schemata derived below.

V. Fragments

If we limit ourselves to the above three rules the following are especially interesting. In both cases $(\sim p)$ is introduced by definition as $(p|p)$ and $(p \supset q)$ is introduced as $((q|q)|p)$

1) Stroke Introduction and Stroke Elimination together will yield the standard introduction rules in implication and negation but only restricted forms of the elimination rules. These rules are derived schemata A 1—A 6. We have at least the full sentential calculus in implication and negation for negations. I. e., propositions of the form $(p|p)$.

2) Stroke Introduction and Double Stroke Elimination together will yield the full sentential calculus in implication and negation. These rules are derived schemata B 1—B 6.

In both cases we have sufficiently rich schemata to obtain the full sentential calculus by reintroducing "stroke" and the remaining "truth functions" with new definitions but our primitive stroke and our variables will need radical reinterpretation.

VI. Derived Schemata

A. The following proofs use Stroke Introduction and Stroke Elimination only.

A 1. Restricted Negation Elimination $(R \sim E)$

1	p	hypothesis
2	$p \mid p$	hypothesis
3	$\quad q$	hypothesis
4	$\quad p \mid p$	2, repetition
5	$q \mid p$	3—4, $\mid I$
6	$q \mid q$	1, 5, $\mid E$

A 2. Negation Introduction $(\sim I)$

1	$\quad p$	hypothesis
⋮	$\quad \vdots$	
2	$\quad q$	assumption
3	$\quad q \mid q$	assumption
4	$\quad p \mid p$	2, 3, $R \sim E$
5	$p \mid p$	1—4, $\mid I$

A 3. Double Negation Introduction ($\sim\sim I$).

1	p	hypothesis
2	$\quad p \mid p$	hypothesis
3	$\quad p$	1, repetition
4	$\quad (p \mid p) \mid (p \mid p)$	2, 3, $R \sim E$
5	$(p \mid p) \mid (p \mid p)$	2—4, $\mid I$

A 4. Restricted Double Negation Elimination ($R \sim\sim E$).

1	$((p \mid p) \mid (p \mid p)) \mid ((p \mid p) \mid (p \mid p))$	hypothesis
2	$\quad p$	hypothesis
3	$\quad (p \mid p) \mid (p \mid p)$	2, $\sim\sim I$
4	$\quad\quad p$	hypothesis
5	$\quad\quad ((p \mid p) \mid (p \mid p)) \mid ((p \mid p) \mid (p \mid p))$	1, repetition
6	$\quad p \mid ((p \mid p) \mid (p \mid p))$	4, 5, $\mid I$
7	$\quad p \mid p$	3, 6, $\mid E$
8	$p \mid p$	2—7, $\mid I$

A 5. Restricted Implication Elimination ($R \supset E$).

1	p	hypothesis
2	$((q \mid q) \mid (q \mid q)) \mid p$	hypothesis
3	$((q \mid q) \mid (q \mid q)) \mid ((q \mid q) \mid (q \mid q))$	1, 2, $\mid E$
4	$q \mid q$	3, $R \sim\sim E$

A 6. Implication Introduction ($\supset I$).

1	$\quad q \mid q$	hypothesis
2	$\quad\quad p$	hypothesis
⋮	$\quad\quad \vdots$	⋮
3	$\quad\quad q$	assumption
4	$\quad\quad q \mid q$	1, repretition
5	$\quad p \mid p$	2—4, $\sim I$
6	$(q \mid q) \mid p$	1—5, $\mid I$

B. The following proofs use Stroke Introduction and Double Stroke Elimination only.

B 1. Negation Elimination ($\sim E$).

1	p	hypothesis
2	$p \mid p$	hypothesis
3	$\quad q \mid q$	hypothesis
4	$\quad p \mid p$	2, repetition
5	$(q \mid q) \mid p$	3–4, $\mid I$
6	q	1, 5, $\mid\mid E$

B 2. Negation Introduction ($\sim I$).

B 3. Double Negation Introduction ($\sim\sim I$).

These are proofs identical to proofs of A 2 and A 3 above except that Negation Elimination replaces Restricted Negation Elimination in step 4 of both. B 6 below is simply identical to A 6.

B 4. Double Negation Elimination ($\sim\sim E$).

1	$(p \mid p) \mid (p \mid p)$	hypothesis
2	$\quad (p \mid p)$	hypothesis
3	$\quad ((p \mid p) \mid (p \mid p)) \mid ((p \mid p) \mid (p \mid p))$	2, $\sim\sim I$
4	$(p \mid p) \mid ((p \mid p) \mid (p \mid p))$	2, 3, $\mid I$
5	p	1, 4, $\mid\mid E$

B 5. Implication Elimination ($\supset E$).

1	p	hypothesis
2	$(q \mid q) \mid p$	hypothesis
3	q	1, 2, $\mid\mid E$

B 6. Implication Introduction ($\supset I$). See above.

C. These proofs use Stroke Introduction, both elimination rules, and schemata derived above.

C 1. Commutivity for Stroke (|Comm).

1	$p \mid q$	hypothesis
2	q	hypothesis
3	$p \mid q$	1, repetition
4	$p \mid p$	2, 3, $\mid E$
5	$q \mid p$	2–4, $\mid I$

C 2. Tautology (Taut).

1	p	hypothesis
2	$(p \mid p) \mid (p \mid p)$	1, $\sim\sim I$
3	$p \mid (p \mid p)$	1–2, $\mid I$

C 3. Rule of Detachment (Nicod).

1	p	hypothesis
2	$p \mid (q \mid r)$	hypothesis
3	$r \mid r$	hypothesis
4	q	hypothesis
5	$r \mid r$	3, repetition
6	$q \mid r$	4–5, $\mid I$
7	$(q \mid r) \mid p$	2, repetition, |Comm
8	$(q \mid r) \mid (q \mid r)$	1, 7, $\mid E$
9	$(r \mid r) \mid (r \mid r)$	3–8, $\sim I$
10	r	9, $\sim\sim E$

C 4. Second Rule of Detachment (Docin).

1	p	hypothesis
2	$p \mid (q \mid r)$	hypothesis
3	$q \mid q$	hypothesis
4	r	hypothesis
5	$q \mid q$	3, repetition
6	$r \mid q$	4–5, $\mid I$
7	$q \mid r$	6, |Comm
8	$(q \mid r) \mid p$	2, |Comm, repetition
9	$(q \mid r) \mid (q \mid r)$	1, 8, $\mid E$
10	$(q \mid q) \mid (q \mid q)$	3–9, $\sim I$
11	q	10, $\sim\sim E$

C 5. Nicod's Axiom.

1	$(t \mid (t \mid t)) \mid ((s \mid q) \mid ((p \mid s) \mid (p \mid s)))$	hypothesis
2	$p \mid (q \mid r)$	hypothesis
3	$t \mid (t \mid t)$	Taut
4	$s \mid q$	3, 1, repetition, Docin
5	p	hypothesis
6	q	5, 2, repetition, Docin
7	$s \mid q$	4, repetition
8	$s \mid s$	6, 7, $\mid E$
9	$p \mid s$	5–8, $\mid I$
10	$(p \mid s) \mid (p \mid s)$	3, 1, repetition, Nicod
11	$(p \mid (q \mid r)) \mid (p \mid (q \mid r))$	2–10, $\sim I$
12	$((t \mid (t \mid t)) \mid ((s \mid q) \mid ((p \mid s) \mid (p \mid s)))) \mid (p \mid (q \mid r))$	1–11, $\mid I$
13	$(p \mid (q \mid r)) \mid ((t \mid (t \mid t) \mid ((s \mid q) \mid ((p \mid s) \mid (p \mid s))))$	12, $\mid$ Comm

(Eingegangen am 19. April 1961)