Hamilton College
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Unit 5 Notes
Kant's Transcendental Idealism
(And a Bit of Newton and Leibniz)

§I. Approaching Kant's First Critique

Good studies in the history of philosophy must balance reading a wide range of texts with close analyses of them.
In this class, we have been mainly focusing on a few central topics, looking briefly at how various philosophers of the modern era think about them.
We have analyzed the most important passages and examined the central arguments in class.
Starting with the key claims, the philosophers' main conclusions, we have been working outwards to evaluate the arguments in as much detail as we can get to in the short time we have.
If you have done all of the readings, you have done a lot of reading.

Our approach contrasts with what one might call a deductive or synthetic approach.
On the synthetic approach, we would start at the beginning of a work, and trace the argument carefully through a text.
We took that approach to Descartes's *Meditations*, but abandoned it with the difficult arguments in Spinoza's *Ethics* and the dense repetitiveness of Locke's *Essay*.
The center-out approach is nearly unavoidable given our syllabus.
We return briefly to the synthetic approach with our study of Kant's work.

Kant's master work in metaphysics and epistemology is called the *Critique of Pure Reason*.
A critique is not merely a criticism, though people often misuse the term.
A critique is an extended review or commentary.
See the usage note in the American Heritage Dictionary which calls many of the common uses of 'critique' pretentious jargon.

Kant wrote three Critiques late in his life.
The First Critique is devoted to the questions, "Is metaphysics possible?" and, "If so, how?"
We can see Kant's First Critique as attempting to define the limits of human knowledge.
The rationalists over-reached, claiming knowledge where none could really be had.
But the empiricists fell short, ending as skeptics or idealists.
Kant's work attempts to bridge the two approaches.
The Second Critique (*Critique of Practical Reason*) concerns moral philosophy.
The Third Critique (*Critique of Judgment*) concerns aesthetics.

The First Critique may be seen as the dying gasp of the representationalist theory of ideas that characterizes most work of the modern period.
It certainly marks the end of the modern era.
Western philosophy for about a century after Kant mainly focused on the consequences of his so-called transcendental idealism.
After that, philosophy sort of branches into two schools.
To oversimplify, the first follows Nietzsche and Kierkegaard into twentieth-century continental philosophy and literary theory.
The second follows Mill and Frege into the linguistic revolution and twentieth-century analytic philosophy.

Kant wrote the first edition of the First Critique, now called the A version, in 1781.

He published a second edition, now called the B version, in 1787.
Most people now read the two editions together.
Some of the B version extends and clarifies Kant's original arguments.
We will not spend time on the distinction between the two versions.
Both are presented in the Ariew and Watkins, and you can see the marginal page numbers for each.

Proper study of the First Critique really should take a full year of hard work.
So, as with much of the work we study in this course, we can barely get to the most central arguments.
We will start at the beginning of the *Critique*, working forward just until we get a sense of his project, and then skip a head to a few consequences for the topics we have been studying this term.
We will focus on Kant's claim about synthetic *a priori* judgments and how Kant's novel transcendental approach to philosophy allows him a new perspective on some of the core themes of this course, including questions about the self, free will, mathematics, causation, minds and bodies, and the existence of God.

§II. Metaphysics and the Synthetic *A Priori*

II.1 Reason

We'll start by thinking about the concept of reason as Kant invokes it in the title of the First Critique.
Every philosopher we have read accepts that we have some kind of ability to reason.
Philosophers disagree about the matter for reason and the limits and extent of what we can learn from sense experience.
The rationalists believe that the content of our judgments is provided by both innate ideas and sense experiences.
Rationalists disagree amongst themselves about the role and veridicality of sense experience, but not about whether we are presented with sense experience.
The empiricists believe that the contents of our minds are provided primarily by sense experiences and secondarily by reflection on those experiences.
They sometimes even seem to try to reduce reasoning to psychological associations among images.
Kant rejects rationalism for being dogmatic beyond our true abilities.
He rejects empiricism for its skeptical conclusions.
Kant seeks a new parth which combines insights of both camps.

One might think of pure reason as logic.
If we take logic, as Kant does, to be the rules of reasoning and thought, as a psychological discipline, then Kant's project is a logical project.
He looks in part at how reason can determine, or structure, an object.
He also examines how reason can make objects actual through the application of pure thought.
Kant thus claims that some cognition is pure, consisting of reason acting on itself.
That's not the rationalists' claim that we are given ideas carrying significant particular content innately.
Kant's view is that we must get clear about the role and extent of our reasoning.
The rules which govern reason are logical.
Logic is thus psychological.

II.2. Kant's Copernican Revolution

Kant compares his First Critique to Copernicus's revolution, the shift from a geocentric model of the universe to a heliocentric one.
Aristoteleans believed that the sun, stars, and other celestial bodies circled the earth.

Astronomical discoveries complicated the descriptions of the cycles of those bodies.
Copernicus and others found that astronomical mathematics became tractable if we posit a moving Earth.

> Having found it difficult to make progress there when he assumed that the entire host of stars revolved around the spectator, he tried to find out whether he might not be more successful if he had the spectator revolve and the stars remain at rest (Kant, *Critique of Pure Reason*, Bxvi, AW 720a).

Kant argues that the empiricists found it impossible to justify knowledge of a material world because they assumed that our cognition has to conform to objects.
They started with an assumption of a structured world independent of us and tried to account for knowledge of that world.
Locke counseled humility.
Berkeley denied the existence of a material world.
Hume ended up a skeptic.
They could not find a way to explain knowledge of a transcendent world.
Kant claims that the problem with all of these approaches is that they attempt to mold our understanding and experience to the external world.
Instead, he argues, objects must conform to our cognition.
Then we can explain how we have *a priori* knowledge of those objects.

> Locke left the door wide open for *fanaticism*; for once reason has gained possession of such rights, it can no longer be kept within limits by indefinite exhortations to moderations. Hume, believing that he had uncovered so universal a delusion— regarded as reason—of our cognitive faculty, surrendered entirely to *skepticism*. We are now about to try to find out whether we cannot provide for human reason safe passage between these two cliffs, assign to it determinate bounds, and yet keep open for it the entire realm of its appropriate activity (Kant, *Critique of Pure Reason* B128, AW 745b)

One way in which objects conform to our cognition is in imagination, when we fantasize.
If all of the world were merely one person's fancy, say, then the objects of that world would necessarily conform to that person's cognition.
Such a view of the world would be an unacceptable, subjective idealism.

In contrast, Kant defends a transcendental idealism.
In Kant's idealism, the world conforms to our cognition because we can only cognize in certain ways.
The world of things-in-themselves, or what Kant calls the noumenal world, remains, as it did for Hume, inaccessible, completely out of range of our cognition.
The noumenal world is beyond the limits of possible experience.
But any possible experience has to conform to our cognitive capacities.
We do not need experiences of objects as they are in themselves to know about the world.
Such a possibility is not even sensible.
The world that we can know is, by definition, not inaccessible.
The phenomenal world, the world of possible experience, is necessarily structured according to our capacities.
A proper understanding of that world must include a full examination of those structuring capacities.

Kant believes that our cognitive capacities come under two general headings: intuition (or sensibility) and understanding.
Intuition is our mental faculty for having something presented to us.
Understanding, which is structured according to certain basic concepts, is our mental faculty for determining or thinking about objects.
All objects must be presented in intuition and determined by concepts in the understanding in order for us to

think about them.
Thus, all of experience necessarily conforms to the two aspects of our cognition.
Logic, as the laws of thought, will help us understand our faculty of cognizing and will thus help us understand the phenomenal world.

The distinction between the realm of objects of possible experience and that of transcendent objects helps Kant deny the legitimacy of much of the work of the continental rationalists.
For example, God is, according to Kant, outside the range of possible experience, and thus can not be an object of knowledge.

> In order to reach God, freedom, and immortality, speculative reason must use principles that in fact extend merely to objects of possible experience; and when these principles are nonetheless applied to something that cannot be an object of experience, they actually do always transform it into an appearance, and thus they declare *all practical extension* of reason to be impossible. I therefore had to deny *knowledge* in order to make room for *faith* (Kant, *Critique of Pure Reason* Bxxx, AW724a–b).

Similarly, *a priori* knowledge of a mind-independent noumenal world is impossible.
But *a priori* knowledge of our world of possible experience is possible if we pay attention to the conditions of that experience.
By reasoning about the underlying framework of our experiences, we can unmask the conditions for those experiences, the metaphysical structure of our knowledge.

II.3. The Analytic and the Synthetic

Kant's fundamental claim is that metaphysics is possible, and that it consists of synthetic *a priori* judgments.
To understand this claim, we have to contrast two distinctions, between analytic and synthetic claims and between *a priori* and empirical (or *a posteriori*) claims.

For Kant, analyticity and syntheticity are characterizations of judgments, which are mental acts.[1]
A judgment is analytic if the concept of the predicate is contained in the concept of the subject.
A judgment is synthetic if the concept of the predicate goes beyond the concept of the subject.
'Bachelors are unmarried' is analytic because the concept of a bachelor contains the concept of being unmarried.
'Bachelors are unhappy' is synthetic because the concept of a bachelor does not contain the concept of being unhappy.
It may be the case that all bachelors are unhappy, but that depends on the way the world is and not on the

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[1] The analytic/synthetic distinction is today generally taken to be a linguistic distinction between kinds of propositions (or statements). A proposition is, roughly, the meaning of a sentence. Sentences are expressions of propositions in particular languages. Whether we take analyticity and syntheticity to be properties of judgments (as Kant does) or propositions (as most contemporary philosophers do), they almost always are taken to be dependent on concepts, which, who knows what those are? This topic comes up a lot in *The Language Revolution*. Keep reading these notes for more on concepts.

Also, judgments, for Kant, following Aristotle, are all of subject-predicate form. Today, it's widely accepted that some statements are best understood differently. For example, while 'I give a rose to Emily' may be understood as ascibing a predicate ('gives a rose to Emily') to a subject (me), it may be better understood as a three-place relation among giver, gift, and recipient (me, the rose, and Emily). We discuss the difference in *Symbolic Logic*.

way that language or concepts are.
Other analytic judgments include, 'If you're running then you're moving' and 'all neurologists are doctors'.

I won't much pursue the questions of how concepts contain other concepts and what the relation of containment is, but a few words are worth our time.
First, concepts may be taken either as mental objects (thoughts) or as abstract objects.
If we take concepts to be thoughts, then different people can not share concepts.
My thoughts are not your thoughts and your thoughts are not mine.
Still, it seems that we can think about the same thing.
It is thus preferable to take concepts as abstract objects and to take our private thoughts to be about public (if abstract) concepts.
When I think of a concept, like the concept of a bachelor, I perform a mental act which we can call grasping the concept.
These concepts are structured so they can contain, or not contain, other concepts.

Second, there are at least two different notions of conceptual containment that philosophers have used.
Kant uses what Frege (in the late nineteenth century) called beams-in-the-house analyticity.
When we look at a house, if we want to see if it contains a certain structure, we can merely peel back the walls, and literally see the beams.
In contrast, Frege defends a plant-in-the-seeds analyticity.
According to Frege, a statement is analytic as long as it follows from basic axioms according to analyticity-preserving rules of inference.
For Frege, the claim that seven plus five is twelve is analytic even if we can't literally see the twelve in the sum of seven and five or the seven and the five in the twelve.

One of the advantages of Frege's views over Kant's is that he can handle statements that are not in subject-predicate form, like AW and SC.

AW	Astrid walks with those with whom she strolls.
SC	If it is snowing, then it is cold.

Sentences like AW and SC seem analytic, true in virtue of the conceptual containments of their parts.
Yet, they are not of simple subject-predicate form.
The concept of 'walking with those with whom one strolls' is not contained in the concept 'Astrid'.
Any parsing of SC that puts it into subject-predicate form seems forced.
Again, I won't pursue this worry about Kant's account of analyticity.

II.4.. Linguistics, Epistemology, and Metaphysics

Analyticity and syntheticity concern relations among concepts, whatever we take them to be.
The linguistic or conceptual (or even psychological) distinction between analytic and synthetic judgments is independent of the epistemological distinction between *a priori* justifications and empirical (or *a posteriori*; these are synonymous terms, as I am using them) ones.
A statement is justified empirically if we appeal in our account of how we know it to particular sense experiences.
Our belief that snow is white is empirical since we have to see snow to justify knowledge of its whiteness.
In contrast, our belief that 3+2=5 may be justified *a priori*, independent of sense experience.
We need to see snow in order to know that snow is white.
We need experiences with no particular objects in order to know that 2+3=5.

Further, no empirical experiences undermine *a priori* claims.
When we add 2 cups of water to 3 cups of salt, and fail to come up with 5 cups of anything, we don't abandon the claim that 2 + 3 = 5.
Similarly, two chickens added to three foxes doesn't produce five animals; it just yields three fat foxes and a pile of feathers.
Arithmetic claims remain true independent of their failure to apply in particular cases.

So the analytic/synthetic distinction is linguistic/conceptual and the *a priori*/empirical distinction is epistemological.
A third distinction, between necessary and contingent claims, is metaphysical.
Some claims hold necessarily, like mathematical claims.
Other claims are merely contingent, like the claim that snow is white.
Many philosophers typically and traditionally considered claims to be necessary only if they are believed *a priori*.
Discussing the apriority of physical laws, Kant makes that claim explicitly.

> [Such] propositions are clearly not only necessary, and **hence** of *a priori* origin, but also synthetic (Kant, *Critique of Pure Reason* B18, AW 726b–727a, emphasis added).

As Hume argued, one can not arrive at a necessary truth from contingent experiences.
Further, one might think that all *a priori* claims must be analytic, since one reasons to the truth of an analytic claim without appeal to experience.
Similarly, one might align contingency with empirical justification and syntheticity.
A claim is contingent when it is justified by appeal to sense experience and it brings together concepts that are not necessarily related.

In particular, Hume makes these two claims.
Relations of ideas are necessary, justified *a priori*, and analytic.
Matters of fact are contingent, justified empirically (by tracing ideas back to initial impressions) and synthetic.
Puttiing aside the necessary/contingent distinction, since Hume and Kant agree on it, we can depict Hume's claim in the following chart.
The upper-right and lower-left cells are empty.

Hume's Rubric	*A priori*	Empirical
Analytic	Relations of Ideas	
Synthetic		Matters of Fact

Kant's big claim, essential to his assertion that metaphysics is possible, is that the lower-left cell is non-empty.

Kant's Rubric	*A priori*	Empirical
Analytic	Logic / Beams in the house	
Synthetic	Most Mathematics, Metaphysics, and Some Physics	Empirical Judgments

Kant agrees with Hume that matters of fact are all synthetic.

> Experiential judgments, as such, are one and all synthetic (Kant, *Critique of Pure Reason*, A7/B11, AW 725a).

Thus the upper-right cell remains empty.
But Kant disagrees with Hume that the converse holds.
There are synthetic claims that are not experiential, or empirical.

Kant's argument that metaphysics is possible thus relies on his claim that it consists of synthetic *a priori* judgments.
Kant makes that claim by providing examples of claims which he says that we can see are synthetic *a priori*.

II.5. The Synthetic *A Priori*

Kant's least contentious examples of synthetic *a priori* claims are mathematical.
In particular, he claims that '7 + 5 = 12' is not analytic, contrary to Leibniz and Hume before him, and contrary to Frege afterward.

> Mathematical propositions, properly so called, are always *a priori* judgments rather than empirical ones; for they carry with them necessity, which we could never glean from experience...It is true that one might at first think that the proposition 7 + 5 = 12 is a merely analytic one that follows, by the principle of contradiction, from the concept of a sum of 7 and 5. Yet if we look more closely, we find that the concept of the sum of 7 and 5 contains nothing more than the union of the two numbers into one; but in [thinking] that union we are not thinking in any way at all what that single number is that unites the two. In thinking merely that union of 7 and 5, I have by no means already thought the concept of 12; and no matter how long I dissect my concept of such a possible sum, still I shall never find in it that 12. We must go beyond these concepts and avail ourselves of the intuition corresponding to one of the two... (Kant, *Critique of Pure Reason* B14-5, AW 726a).

Extending the claim that there are synthetic *a priori* judgments to metaphysics, Kant claims that 'every effect has a cause' is also synthetic *a priori*.
The universality of the statement entails that it is not an empirical judgment.
But, Kant claims that it is not an analytic judgment.

> In the concept of something that happens I do indeed think an existence preceded by a time, etc., and from this one can obtain analytic judgments. But the concept of a cause lies quite outside that earlier concept and indicates something different from what happens... (Kant, *Critique of Pure Reason* A9/B13, AW 725b).

In addition to mathematics and metaphysics, Kant claims that physics also proceeds according to synthetic *a priori* principles.
The claim that some scientific propositions are synthetic *a priori* shows that Kant's conception of physics is closer to that of Galileo and Descartes than it is to that of contemporary physicists.
The science of the scientific revolution was more speculative, whereas much of contemporary science is more experimental.
While some contemporary physics is highly speculative, it is generally held that a mark of a good theory is whether it is testable, or refutable, or otherwise confirmed or contravened by experimental results.
String theory, which is a purported unifying theory for physics, has been controversial because its proponents

have not been able to formulate tests for it.
Kant agrees that some portions of physics must be empirically testable.
But he also believes that certain physical principles are synthetic *a priori*.

> *Natural science contains synthetic* a priori *judgments as principles*. Let me cite as examples just a few propositions: e.g., the propositions that in all changes in the corporeal world the quantity of matter remains unchanged; or the proposition that in all communication of motion, action and reaction must always be equal to each other (Kant, *Critique of Pure Reason* B17-18, AW 726b).

Kant's latter example is Newton's third law of motion.
His claim is that such laws hold necessarily and so can not be learned from experience.

> For experience would provide neither strict universality nor apodeictic certainty... (Kant, *Critique of Pure Reason* A31/B47, AW 733b).

Berkeley and Hume agreed that universal physical laws could not be learned from experience.
From that claim and the empiricist's belief that all knowledge comes from experience, Berkeley concluded idealism and Hume concluded skepticism.
Kant, working in the other direction, starts by accepting that there are mathematical, metaphysical, and even physical laws that hold necessarily, that are known *a priori*.
Working backwards, or transcendentally, he argues that our cognitive abilities must be such that they allow us to know those principles *a priori*.
Kant does not argue that innate ideas are built into our minds as Descartes and Leibniz alleged.
Instead, he argues that there are certain cognitive structures that impose an order to our possible experience.
The mind has templates for judgments which are imposed and can be known *a priori*.
But against those who defend innate ideas, it does not contain judgments themselves.
If we look at our cognitive structures, turning our reason on itself, we can find the necessary structure of our reasoning and grounds for synthetic *a priori* claims.
That process, which Kant calls transcendental reasoning, is the essence of Kant's Copernican revolution.
Kant's transcendental arguments lead to a description of our subjective conceptual framework which nevertheless holds necessarily for all possible experience.

So to make room for metaphysics, Kant argues that, like much of mathematics and physics, it consists of synthetic *a priori* judgments.
Since these judgments are synthetic, they do not follow simply from conceptual analysis.
Since these judgments are *a priori*, they can not be learned from experience.
Kant works backwards from experience to the conditions that must obtain in order for us to have such knowledge.
Such conditions will be the necessary structures of our logic or reasoning.

We will look at the first two parts of the First Critique: the transcendental aesthetic and the transcendental analytic.
These two parts correspond to two distinct functions of our psychology.
In the transcendental aesthetic, Kant discusses how objects and the world are given to us.
In the transcendental analytic, Kant discusses how our minds determine and understand what is given.
We are presented with a world having certain properties.
Kant calls this aspect of human cognition our sensibility.
Then, we cognize that world according to certain concepts.
Kant calls this aspect of human cognition the understanding.
By examining the properties that form the foundations of all our experiences, we will find the necessary

properties of our experience.
By examining the concepts that determine all our understanding, we will find the necessary properties of our thought.

§III. The Transcendental Aesthetic

III.1. Intuition

Let's start our look at the transcendental aesthetic with a few definitions.
Intuition is our capacity for receiving or barely representing experience.

> The effect of an object on our capacity for representation, insofar as we are affected by the object, is *sensation*. Intuition that refers to the object through sensation is called *empirical* intuition. The undetermined object of an empirical intuition is called *appearance* (Kant, *Critique of Pure Reason* A19-20/B34, AW 729b).

Not all intuitions are empirical, though.
Some intuitions are pure, about the form of intuition itself.
In empirical intuitions we can divide the matter from the form.
The matter is what corresponds to sensation in the strictest sense possible.
If I am holding a pen and looking at it, I am given some appearance in intuition.
Additionally, this appearance has certain abstract properties, a form.
The particulars of the form of this appearance are unique to my experience of the pen.
But the general properties of the form of appearances are properties of all experiences, of pens and papers and trees and buildings.
All such experiences take place in space and in time, which is part of what connects and relates them.
My experiences of both the pen and the paper are necessarily given in both space and time, since all experiences are so given.
That's in part what allows us to put the pen to the paper.

Intuitions which contain no empirical matter are pure.

> If from the representation of a body I separate what the understanding thinks in it, such as substance, force, divisibility, etc., and if I similarly separate from it what belongs to sensation in it, such as impenetrability, hardness, color, etc., I am still left with something from this empirical intuition, namely, extension and shape. These belong to pure intuition, which, even if there is no actual object of the senses or of sensation, has its place in the mind *a priori*, as a mere form of sensibility (Kant, *Critique of Pure Reason* A20-1/B15, AW 730a).

Note Kant's method here.
We arrive at our consideration of pure forms of intuition by a method something like abstraction,
But Kant does not claim that our knowledge of space and time are derived by abstraction.
We are discovering that knowledge of space and time is presupposed in any empirical intuition.
Kant is not, like Locke, pretending that we have a capacity to form abstract ideas.
He is also not claiming that we arrive at the concept of space (or time) by induction over our experiences.
He is looking for conditions which necessarily underlie all of our experiences.
It is a transcendental argument.

III.2. The Intuition Installment of the Copernican Revolution

Kant claims that space and time are underlying forms of all intuitions.
All objects outside of us are represented as extended in space, the form of outer sense.
We represent objects according to our inner sense as in time.
Both space and time are presupposed in all experiences.

> The representation of space must already be presupposed in order for certain sensations to be referred to something outside me (i.e. referred to something in a location of space other than the location in which I am)... We can never have a representation of there being no space, even though we are quite able to think of there being no objects encountered in it. Hence space must be regarded as the condition for the possibility of appearances... (Kant, *Critique of Pure Reason* A23-4/B38-9, AW 730b-731a).

Similarly, time must be presupposed for all experiences.

> Simultaneity or succession would not even enter our perception if the representation of time did not underlie them *a priori* (Kant, *Critique of Pure Reason* A30/B46, AW 733a).

Kant's argument for the presupposition of space and time echoes Plato's argument for the doctrine of recollection, or *anamnesis*.
In *Phaedo* 74 et seq., Plato argues that our knowledge of equality can not come from looking at equal things.
All things are unequal in some way.
Even if we were to find some perfectly equal things, like atoms, our concept of equality could not come from our experiences with them.
We must presuppose an idea of the equal in our claims that two objects are equal and can not learn that concept from unequal objects.
Descartes or Spinoza might say that such knowledge is innate.
Similarly, Kant argues that our experiences with objects presuppose that they are given in space and time.
We do not derive knowledge of space and time from experience.
We bring knowledge of space and time to experience.

The argument for space and time being *a priori* forms of intuition is thus Kant's Copernican revolution applied to intuition.
The idea of a possible experience occurring outside of space or time is, he says, nonsense.
And we cannot learn of it from experience either, as Berkeley and Hume show.
But instead of despairing of learning of space and time from experiences, Kant inverts his account to make space and time subjective forms of intuition, presuppositions of experience.
Space and time are ways in which we structure the world of things in themselves.
They are not ways in which the world exists in itself.
They are properties of appearances which are the objects of our empirical intuition.
Our knowledge of the world and its laws concerns these appearances, not the things in themselves.

III.3. Transcendental Idealism and Empirical Realism

We might see taking space and time to be forms of intuition as extending Hume's claims about causation.
Hume reinterpreted 'cause' as referring to a mental phenomenon.
Kant, a bit more generally, takes space and time to be forms of our intuition, rather than things in themselves.
Consequently, Kant is able to take objects in space and time to be empirically real.

> Our exposition teaches that space is *real* (i.e. objectively valid) in regard to everything that we can encounter externally as object, but teaches at the same time that space is *ideal* in regard to things when reason considers them in themselves, i.e., without taking into account the character of our sensibility. Hence we assert that space is *empirically real* (as regards all possible outer experience), despite asserting that space is *transcendentally ideal*, i.e., that it is nothing as soon as we omit [that space is] the condition of the possibility of all experience and suppose space to be something underlying things in themselves (Kant, *Critique of Pure Reason* A28/B44, AW 732b).

The twin doctrines of empirical realism and transcendental idealism are at the center of Kant's work.
We can say nothing of the noumenal world of things in themselves, not even that they are in space and time.
This is not a limitation on our speech or cognition.
Space and time are, for Kant, literally not properties of things in themselves.
They are properties of our representations of the world.
Berkeley's empirical idealism made the mistake of denying an outer, material world on the basis of the transcendence of the noumenal world.
The rationalists, as transcendental realists, erred by asserting knowledge of things in themselves.
Kant claims that we can have significant knowledge of an external world (of appearances) without claiming any knowledge of the noumenal world.
The outer world is the world of representations, not a world as it is in itself.

III.4. Empirical Realism and Mathematics

Kant's transcendental exposition of space and time explains how we can have certainty of both geometry and pure mechanics.
Geometry is the study of the form of outer sense, of pure *a priori* intuitions of space.
Pure mechanics is the study of the form of inner sense, time.

> Only in time can both of two contradictorily opposed determinations be met with in one thing: namely, *successively*. Hence our concept of time explains the possibility of all that synthetic *a priori* cognition which is set forth by the—quite fertile—general theory of motion (Kant, *Critique of Pure Reason* A32/B48-9, AW 734a).

Arithmetic, too, depends essentially on construing addition as successions in time.
But constructing numbers in intuition requires the synthetic unity of apperception behind the categories of the understanding.
That sentence may not make any sense at this point.
To explain it, we will have to spend a little time on our other cognitive faculty, the understanding, which Kant discusses in the transcendental analytic, and which we will discuss next.

III.5. From Intuition to Understanding

We saw that Kant separates two faculties of cognition: sensibility (the faculty of intuition) and understanding.
There are two pure forms of intuition, space and time, which are neither things in themselves nor properties of things in themselves.
Instead, they are formal presuppositions we impose on all our possible experience.

The faculty of intuition gives us appearances.
But appearances are just the raw data, the content of experience.

Our intuitions are passive.
The raw data of intuition is processed in the understanding by the imposition of concepts.

> All our intuitions, as sensible, rest on our being affected; concepts, on the other hand, rest on functions. By *function* I mean the unity of the act of arranging various representations under one common representation (Kant, *Critique of Pure Reason* A68/B93, AW 738b).

This act of arranging what is given in intuition is what Kant calls synthesis of the manifold.
We are given separate, individual, atomic bits of perception.
Recall the particularism we saw in the work of Locke (e.g. the Molyneux problem) and Berkeley (e.g. the apple of *Principles* §1).
Kant agrees.
But we do not perceive all of the little bits separately.
We unite, or synthesize, the separate bits of perceptions into a single experience.
This synthesis is cognized by the structured application of concepts in the understanding.
If the synthesis is empirical, then we have an ordinary empirical cognition like the judgment that it is raining.
If the synthesis is pure, then we can arrive at pure concepts of the understanding, which are nevertheless the conditions of possible experience.
Intuition and understanding thus work together to produce experience.

> Thoughts without content are empty; intuitions without concepts are blind (A51/B76, AW 737b).

The transcendental aesthetic, we have seen, consists of Kant's explications of the pure intuitions of space and time.
The transcendental analytic is the much-longer explication of the categories of the understanding, how we impose our conceptual apparatus on what is given in intuition.

What is given in intuition is not immediately structured by the understanding.
We are given appearances without any conceptual structure.
We are just given appearances in space and time.

> Appearances might possibly be of such a character that the understanding would not find them to conform at all to the conditions of its unity. Everything might then be so confused that, e.g., the sequence of appearances would offer us nothing providing us with a rule of synthesis and thus corresponding to the concept of cause and effect, so that this concept would then be quite empty, null, and without signification. But appearances would nonetheless offer objects to our intuition; for intuition in no way requires the functions of thought (Kant, *Critique of Pure Reason* A90-1/B 123, AW 744a).

In order to think about appearances, we have to cognize them.
We cognize using whatever conceptual apparatus we have.
That conceptual apparatus is subjective in that it belongs to us individually.
But it is also objective, though not noumenal, because the world of objects is precisely the world of appearances, what is given in intuition.

§IV. The Transcendental Analytic
IV.1. Aristotle and the Categories

The Transcendental Analytic contains Kant's transcendental derivation of the concepts we impose on appearances given in intuition.
The transcendental method, as we have seen, starts with our cognitions and works backwards towards the conditions that must exist in order for us to have those cognitions.
Kant presents what he takes to be a complete table of concepts, dividing them into four classes.
In presenting the table of categories, he recalls Aristotle's work on the categories.

Aristotle said that being is said in many ways.
He delimited ten categories of being, ten ways to be.

A1 substance (e.g. man, horse)
A2 quantity (e.g. four-foot)
A3 quality (e.g. white, grammatical)
A4 relation (e.g. double, larger)
A5 where (e.g. in the market)
A6 when (e.g. yesterday)
A7 being-in-a-position (e.g. is-standing)
A8 having in addition (e.g. has-hat-on)
A9 doing (e.g. cutting)
A10 being affected (e.g. suffering, passion)

For Aristotle, all language, indeed all thought, belongs to one of these categories.
When we say, or think, something, we combine instances from two or more of the categories.
If Aristotle's list were complete, we could adopt it as a fundamental theory about our thought.
If, further, this list were not merely accidentally complete, but necessarily complete, we might see it as indicating *a priori* conditions of human cognition.
Kant does not adopt Aristotle's categories uncritically, in large part because he wants to make sure that the list is complete.
He adjusts and organizes Kant's list.
Moreover, he argues that his list is not derived from observation.
It is not inductive, arbitrary, or haphazard.
It is a transcendental deduction of preconditions gathered by examining all possible forms of judgment and the ways in which any judgment must presuppose such conditions.
The categories are logical laws of thought.

> [The categories] are concepts of an object in general whereby the object's intuition is regarded as *determined* in terms of one of the *logical functions* in judging (Kant, *Critique of Pure Reason* B128, AW 745b).

Kant's logic is thus a psychological program.

IV.2. Kant's Categories and Their Origins

Kant provides four conditions for the transcendental analytic.

> (1) The concepts must be pure rather than empirical.
> (2) They must belong not to intuition and sensibility, but to thought and the understanding.
> (3) They must be elementary concepts, and must be distinguished carefully from concepts that are either derivative or composed of such elementary concepts.
> (4) Our table of these concepts must be complete, and the concepts must occupy fully the whole realm of the pure understanding (Kant, *Critique of Pure Reason* A64/B89, AW 737b).

On this basis, Kant develops twelve categories in four classes:

Quantity	Relation
Unity	Inherence and Subsistence (substance)
Plurality	Causality
Totality	Community (Interaction)
Quality	Modality
Reality	Possibility and Impossibility
Negation	Existence and Non-Existence
Limitation	Necessity and Contingency

In developing these categories transcendentally, rather than empirically, in what is called the metaphysical deduction of the categories, Kant distinguishes his work not just from Aristotle.
Hobbes, Locke, and Hume produced rudimentary results about the structure of our psychology empirically.
Hobbes discusses the train of our thoughts, dividing them into regulated and unguided mental discourse in Chapter 3 of *Leviathan*.
Locke discusses ideas of reflection and isolates our ability to abstract as an important psychological capacity for philosophical purposes.
For Hume, connections among ideas are either resemblance, contiguity, or cause and effect relations.
None of these earlier philosophers insist that their categories are comprehensive.
All of them proceed empirically, looking at our psychological processes and generalizing.

Kant insists that such empirical deductions could never yield the necessity that underlies synthetic *a priori* reasoning.

> Experience contains two quite heterogeneous elements: namely, a *matter* for cognition, taken from the senses; and a certain *form* for ordering this matter, taken from the inner source of pure intuition and thought. It is on the occasion of the impressions of the senses that pure intuition and thought are first brought into operation and produce concepts. Such exploration of our cognitive faculty's first endeavors to ascend from singular perceptions to universal concepts is doubtless highly beneficial, and we are indebted to the illustrious *Locke* for first opening up the path to it. Yet such exploration can never yield a *deduction* of the pure *a priori* concepts, which does not lie on that path at all. For in view of these concepts' later use, which is to be wholly independent of experience, they must be able to display a birth certificate quite different from that of descent from experiences (Kant, *Critique of Pure Reason* A86–7/B118-9, AW 742b–743a).

Consider causality.
There is a difference between an instance of causal connection, say a massive object falling to the surface of

the Earth, and accidental conjunction, like my checking my mail and then having lunch at the diner.
The causal relation has an element that necessitates the effect; the massive object must fall.
The accidental relation has no such aspect; I could check my mail without going to the diner and I could have lunch without checking my mail.
If the world were Humean (i.e. a world of conjunction rather than connection), then all relations among events would be like that between the mail and diner.
But the world is full of causal connections.

> This concept [causation] definitely requires that something, A, be of such a kind that something else, B, follows from it *necessarily* and according to an *absolutely universal rule*. Although appearances do provide us with cases from which we can obtain a rule whereby something usually happens, they can never provide us with a rule whereby the result is *necessary* (Kant, *Critique of Pure Reason* A91/B124, AW 744a).

Kant may be a little too generous with Hume's account of causation here.
If we had no *a priori* knowledge of causes, I'm skeptical that we could even infer rules about things usually happening.
The entire world could seem loose, unconnected, haphazard.

Locke thought that he could abstract the requisite concepts from experience.
Hume and Berkeley showed that such abstraction was not justified on the basis of experience.
Hume agrees with Kant that an empirical deduction of our psychological capacities could never yield the necessity that we need for metaphysics and science.
Thus Hume yields to skepticism.
Kant's Copernican revolution consists of the rejection of skepticism, the embrace of synthetic *a priori* knowledge, and, consequently, the transcendental deduction of the categories.

IV.3. The Transcendental Deduction: An Overview

We have seen that intuition presents us with bare appearances, the manifold of representation.
Without conceptual structures, the manifold is unintelligible.
We would see the world as a baby might: chaotic, arbitrary, and magical.
These bare appearances have to be structured in order to be thought.
In order to think about what is given, we impose concepts on the manifold which give it structure and thus intelligibility.

So far, in this section, we have been setting up the deduction of the pure concepts of the understanding, describing the role of the concepts, without discussing Kant's argument for their subjective and objective validity.
Now, we have reached the deduction.
Kant rewrote the deduction significantly in the B version of the *Critique*.
Ariew and Watkins present only the B version, which we will follow.
We can describe the goal of the deduction as to show that the categories necessarily apply to the manifold given in intuition.
The deduction is Kant's attempt to show how the sensible and intellectual functions of our cognitive capacities align.

The deduction is divided into two stages.
In the first stage, §15-§21, Kant argues that the categories apply to any being with sensible intuition.

In the second stage, §24-§26, Kant argues that they apply to any being with human sensible intuition, i.e. with our sensory apparatus.

Kant presents a summary of the first stage of the deduction in §20, AW 749b–750a.
James van Cleve, who compares the deduction to a tropical jungle, clears away the growth and reduces it to the following argument:

> 1. *The Unity Premise*: All representations of which I am conscious have the unity of apperception.
> 2. *The Synthesis Premise*: Representations can have such unity only if they have been synthesized.
> 3. *The Category Premise*: Synthesis requires the application of Kant's categories.
> *Conclusion*: The categories apply to all representations of which I am conscious.

I'll roughly follow van Cleve, though not in detail.

IV.4. The Synthetic Unity of Apperception

Kant begins the first stage by recognizing that the application of categories which results in a thought presupposes more structure and organization than is given in bare, thin, messy intuition.
Kant calls the imposition of concepts on the manifold of representation by the understanding combination.
Raw appearances come to us as an unordered, unstructured, mess.
The imposition of concepts on that manifold turn that mess into an orderly thought.
But we can apply the categories on a representation only if it is already synthesized and orderly.
Otherwise, it would be too disorganized for a single thought.
A representation must be synthesized (or combined) in order even to be a thought.
We can see that synthesis is required before thought, transcendentally.

> Combination is representation of the *synthetic* unity of the manifold. Hence this unity cannot arise from the combination; rather by being added to the representation of the manifold, it makes possible the concept of combination in the first place... Hence a category already presupposes combination (Kant, *Critique of Pure Reason* B131, AW 746b).

Concomitant with synthesis comes a synthesizing agent.
As Descartes noticed, a thought requires a thinker.
Kant adopts the Cartesian insight: a thought has a cognizer, something to perform the combination, as an implicit component.
The implicit thinking is what Kant calls apperception.
Apperception has to unify the messy manifold into an orderly cognition.
As it does so, it presupposes a particular thinker or apperceiver.

> For the manifold representations given in a certain intuition would not one and all be *my* representations, if they did not one and all belong to one self-consciousness (Kant, *Critique of Pure Reason* B132, AW 746b).

We proceed from a diverse manifold given in intuition to a single thought of a single, conscious person.
When we do so, we combine (either by synthesis or otherwise) the manifold.
This combination is an active function of our cognition in contrast to the passivity of intuition.
We act on the manifold in intuition, unifying it and subjecting it to the conditions of the synthetic unity of apperception.

> The understanding is nothing more than the faculty of combining *a priori* and of bringing the manifold of a given intuition under the unity of apperception—the principle of this unity being the supreme principle in all of human cognition (Kant, *Critique of Pure Reason*, B135, AW 747a-b).

Since our synthesizing action is subjective, the application of the conditions of unity are also subjective.
When we determine an intuition, we make it ours.
An empirical unity is subjective; everyone's individual experiences are independent.

> The empirical unity of apperception...is only derived from the original unity under given conditions *in concreto*, has only subjective validity. One person will link the representation of a certain word with one thing, another with some other thing; and the unity of consciousness in what is empirical is not, as regards what is given, necessary and universally valid (Kant, *Critique of Pure Reason* B140, AW 749a).

But the unity is also objective, since it determines objects for us.
Kant contrasts 'if I support this body, then I feel a pressure of heaviness' with 'this body is heavy'.
Or consider, "These lines look like they differ in length, but they are actually the same length."
We are able to make the latter claims about the world, and to distinguish them from subjective claims about our particular experiences of the world.
The relation among appearances is not merely arbitrary or accidental.
Even Hume marveled at the regularity in nature.
But, unless the subjective unity of apperception were also objective, we could only make the former, subjective claims.

We know of causal relations, Kant says.
Thus, we must be able to make objective claims about the world, not merely subjective claims.
Moreover, every act of cognition presupposes the synthetic unity of apperception as an *a priori* condition of judgment.
It is only by combining representations objectively that relations can hold *a priori* or necessarily.

> Bodies are heavy. By this I do not mean that these representations belong *necessarily to one another* in the empirical intuition. Rather, I mean that they belong to one another *by virtue of the necessary unity* of apperception in the synthesis of intuitions; i.e., they belong to one another according to principles of the objective determination of all representations insofar as these representations can become cognition—all of these principles being derived from the principle of the transcendental unity of apperception (Kant, *Critique of Pure Reason*, B142 AW 749b)

Intuitions become objects for an individual through the synthesis of the manifold.
But they are still objects.
We can distinguish between fantasies and appearances, between merely empirical judgments and objective *a priori* ones.

IV.5. The Self and the World: Applying the Categories

So all of our cognitions have two aspects: the matter given in intuition and the structure imposed by the understanding on what is combined in apperception.
The matter may be pure and *a priori*, as when we reflect on the structure of intuition itself.
It may be empirical, as when we have an ordinary sense experience.
The imposition of concepts by the understanding presupposes a self which unites the raw matter and, by doing so, makes it objective.

The process of turning pure intuition into conceptual content involves the application of the categories.
The imposition of categories is, roughly, what turns us from passive receivers of the passing show into thinkers.
Any creature that thinks about the world will have to apply the categories in order to have experiences.

Kant does not argue that any being must apply the categories.
They are only useful for those of us who have intuition which is separate from thought.
The categories are used to unify, through synthesis, the manifold given in intuition.
They apply to creatures whose relation with the world essentially involves representation.
An infinite mind might, in contrast, work not by representation but by direct awareness.
That mind would have no use for the categories.

> For if I were to think of an understanding that itself intuited (as, e.g., a divine understanding that did not represent given objects but through whose representation the objects would at the same time be given or produced), then in regard to such cognition the categories would have no signification whatever. The categories are only rules for an understanding whose entire faculty consists in thought, i.e. in the act of bringing to the unity of apperception the synthesis of the manifold that has been given to it from elsewhere in intuition (Kant, *Critique of Pure Reason* B145, AW 750a-b).

Moreover, we can not explain why we are constructed as we are, with these two aspects of cognition or with these particular categories of understanding or forms of intuition.
Such questions are unanswerable and any attempt to provide answers extends reason beyond its bounds.

> But why our understanding has this peculiarity, that it brings about unity of apperception *a priori* only by means of the categories, and only by just this kind and number of them—for this no further reason can be given, just as no reason can be given as to why we have just these and no other functions in judging, or why time and space are the only forms of our possible intuition (Kant, *Critique of Pure Reason*, B145–6 AW 750b).

All we can do is describe our experiences and their *a priori* preconditions.
Such descriptions will have limits.
They will only describe our experiences and our possible experiences.

Since the categories only apply to those with some sort of intuition, any pure concepts will apply only to objects of possible experience.
Mathematical propositions, for example, are not claims about a transcendent world.
They hold only for objects of possible experience.

> The pure concepts of the understanding, even when they are (as in mathematics) applied to *a priori* intuitions, provide cognition only insofar as these intuitions...can be applied to empirical intuitions... Consequently the categories cannot be used for cognizing things except insofar as these things are taken as objects of possible experience (Kant, *Critique of Pure Reason* B147-8, AW 751a).

Even my own existence is known only through the categories and so only as an appearance, not as it is in itself (or noumenally).

> Although my own existence is not appearance (still less mere illusion), determination of my existence can occur only in conformity with the form of inner sense and according to the particular way in which the manifold that I combine is given in inner intuition (Kant, *Critique of Pure Reason* B157-8, AW 752b).

These are just facts about our cognition, ones we can discover by transcendental analysis (or deduction) and ones which must apply to any cognizer with a separation between intuition and understanding.

IV.6. Making Nature Possible

In the second stage of the transcendental deduction, mainly §26, Kant shows that the categories apply not merely to any cognizer with intuitive and conceptual functions, but specifically to human sensibility.
It might be possible, one supposes, to imagine creatures who also have both intuitions and conceptual cognition, but with different forms of intuition.
We experience the world in space and time.
All of our cognition presupposes these forms.
Anything not in space and time is not an object of our possible experience.
But our concepts might be broader.
They might apply to another kind of intuition.
They are mere forms of thought, after all.

Still, such applications of concepts would not yield a thought for us.
We could not cognize such an object, given its alien representative character.

So beyond merely showing, as Kant does in the first stage of the deduction, that we have some conceptual categories which we impose, *a priori*, on some intuitions, Kant wants to show how these particular categories apply to our particular intuition to create a world which is both objective and knowable.

> We must now explain how it is possible, through *categories*, to cognize *a priori* whatever objects *our senses may encounter*—to so cognize them as regards not the form of their intuition, but the laws of their combination—and hence, as it were, to prescribe laws to nature, and even to make nature possible (Kant, *Critique of Pure Reason* B159–60, AW 753a).

Notice the strength of Kant's claim: we make nature possible.
We do not make the noumenal world possible.
But nature is not a property or aspect of the noumenal world.
It is a result of our structuring the raw data of experience that we are given in intuition.

We intuit the world in space and time.
Space and time do double duty.
Besides being forms of intuition, ways in which we are presented with the raw data of experience, they are themselves available for intuition.

> Space and time are represented *a priori* not merely as *forms* of sensible intuition, but as themselves *intuitions* (containing a manifold), and hence are represented with the determination of the *unity* of this manifold in them... (Kant, *Critique of Pure Reason* B160-1, AW 753b).

Since space and time are pure forms of intuition, they are presupposed in all experience.
Since any experience is already structured, or determined, space and time, as we experience them, are deeply embedded in those experiences.
Since any experience also presupposes the application of the categories, space and time themselves must be subject to the categories.
(This is pretty trippy, right?)

> When I turn the empirical intuition of a house into a perception by apprehending the intuition's manifold, then in this apprehension I presuppose the *necessary unity* of space and of outer sensible intuition as such; and I draw, as it were, the house's shape in conformity with this synthetic unity of the manifold in space. But this same unity, if I abstract from the form of space, resides in the understanding, and is the category of the synthesis of the homogeneous in an intuition as such, i.e. the category of *magnitude*. Hence the synthesis of apprehension, i.e. perception, must conform throughout to that category (Kant, *Critique of Pure Reason* B 162, AW 754a).

Kant presents a parallel example, apprehending the freezing of water, to illustrate the applicability of the categories to representations given in inner sense (time).
When I perceive the water changing states, I presuppose time, in order that I can represent change.

> This synthetic unity, as an *a priori* condition under which I combine the manifold of an *intuition as such*, is—if I abstract from the constant form of *my* inner intuition, i.e., from time—the category of cause; through this category, when I apply it to my sensibility, *everything that happens is, in terms of its relation, determined* by me *in time as such*. Therefore apprehension in such an event, and hence the event itself, is subject—as regards possible perception—to the concept of the *relation of effects and causes*; and thus it is in all other cases (Kant, *Critique of Pure Reason* B163, AW 754a).

In other words, not only do the categories apply to any intellect which receives appearances in intuition.
They apply specifically to our intuition which is sensible in the forms of outer sense (space) and inner sense (time).
Abstracting space and time, we find that the categories were presupposed.
We do not, via abstraction, create the categories.
We discover them already imposed on our experiences.

> The *possibility of experience* is what provides all our *a priori* cognition with objective reality. Now experience rests on the synthetic unity of appearances, i.e., on a synthesis of appearances in general performed according to concepts of an object. Without such synthesis, experience would not even be cognition, but would be a rhapsody of perceptions (Kant, *Critique of Pure Reason* A156/B195, AW 761a).

I mentioned earlier that one way of putting the goal of the transcendental deduction was to show how the sensible and intellectual functions of our cognitive capacities align.
Appearances conform *a priori* both to the forms of sensible intuition and to the categories of the understanding which combine the manifold.
Kant's idealism may, at this point, seem prominent.

> Just as appearances exist not in themselves but only relatively to the subject in whom the appearances inhere insofar as the subject has senses, so the laws exist not in the appearances but only relatively to that same being insofar as that being has understanding (Kant, *Critique of Pure Reason* B164, AW 754b).

The forms of intuition meet up with the categories of the understanding in large part because they are both *a priori* impositions of the subject.
We don't know about the conditions in the noumenal world.
There may be some lawlike connections.

> Things in themselves would have their law-governedness necessarily, even apart from an understanding that cognizes them (Kant, *Critique of Pure Reason* B164, AW 754b).

But our representations of laws hold for our structured cognition.
For us, experiences (i.e. appearances of objects in nature) must have certain abstract features.

> What connects the manifold of sensible intuition is imagination, and imagination depends on the understanding as regards the unity of its intellectual synthesis, and on sensibility as regards the manifoldness of apprehension (Kant, *Critique of Pure Reason* B164, AW 754b).

The laws of cognition are very general.
Kant's claim is not the overly dogmatic and implausible claim, held by Descartes, that the laws of nature are innate.
Instead, Kant argues that some laws of nature are synthetic *a priori*, arising from the general conditions for experience.

> Nature (regarded merely as nature in general) depends...on the categories as the original basis of its necessary law-governedness. But even the pure faculty of the understanding does not suffice for prescribing *a priori* to appearances, through mere categories, more laws than those underlying a *nature in general* considered as the law-governedness of appearances in space and time. Particular laws, because they concern appearances that are determined empirically, are *not completely derivable* from those laws... (Kant, *Critique of Pure Reason* B165, AW 754b-755a).

Only the most general laws of nature, those which arise from structuring our experience, can be known *a priori*.
Particular aspects of the laws, the gravitational constant, say, or the ideal gas law constant, would be empirical.

IV.7. After the Deduction

Kant has argued that the categories make experience possible for us.
Our experience is not whimsical or rhapsodic or fantastic.
It is ordered and structured and lawlike.
Such experience presupposes certain cognitive faculties as conditions, both intuitions and conceptual structure along with a unifying self which we can know, like everything else, only as an object of possible experience and not as it is in itself.

After the Deduction in the *Critique*, Kant explains, or transcendentally deduces, all of the particular categories.
We lack time for studying that heady conceptual psychology here.
But there's no reason that you couldn't dive into it on your own.

Then, Kant shows in greater detail how his transcendental idealism applies to a variety of traditional philosophical problems and paradoxes, including the question of the existence of an external world, whether space and time are absolute or relational, and whether we have free will.
In some cases, Kant sides with the rationalists, claiming that we have knowledge.
For example, Kant argues for the certainty of mathematics and knowledge of an external world.
In other cases, Kant finds the rationalists' claims overly dogmatic, exceeding the limits of pure reason.
For example, Kant argues that God is unknowable, nto an object of possible experience.
Again, Kant is clearing room for faith in God, in contrast to arguments for belief in God.

We are going to look at two of Kant's arguments from later in the *Critique*.

1. Two antinomies: on space and time and on freedom
2. Kant's criticisms of the ontological argument for the existence of God

§V. Space and Time
V.1. Antinomies

Kant presents four antinomies, or paradoxes, to supplant his claim that reason has limits, though Ariew and Watkins only present the first three, omitting the fourth on the existence of God.
While Kant believes that some proper metaphysics can be established using synthetic *a priori* reasoning, other topics (e.g. God, free will) are beyond our ken.
Our reason, wanting answers to such questions, speculates.
The problem with such speculation is that we can argue on either side of the debate.

For example, we can establish that the universe is infinite.
We can also establish that it is finite.
Since contradictions (or antinomies) can not hold, Kant sees the existence of such contrary proofs as demonstrating that reason has exceeded its limits.
Again, Kant has the revolutionary Hume as an influence: we can commit such arguments to the flames.

Kants claims in the first antinomy embody the views of both Newton, who defended an absolutist theory of space and time, and Leibniz, who defended a relationalist theory of space and time.
We will contextualize Kant's work by starting with Newton and Leibniz.

V.2. Absolute and Relational Notions of Space and Time

Theories of space and time have their roots in our observations about change.
Most or all change appears due to some sort of motion, of the change of place of some objects over time.
Motion is ordinarily measured relative to some external object.
When I am traveling on the highway, I am moving, with respect to the world outside the car, and sitting with respect to the car itself.
We use terms like 'up' and 'down', relative to the Earth.
But, even the Earth itself is moving, spinning on its axis.
The axis of the Earth is shifting as well, in the annual revolution of our planet around the sun.
The solar system is moving relative to our Milky Way Galaxy, and the Milky Way is moving within our local system of galaxies.
And so on, one supposes.
I am driving 50 mph west, while the Earth is spinning at 650 miles per hour East, and the whole system is flying through space in its revolution around the sun at around 66,000 miles per hour.
Further, our solar system is moving within our galaxy, which is moving in relation to other galaxies.
Is there some fixed point, some privileged reference frame, to which all motion can be measured?
For most practical purposes, we can pick a frame of reference outside of our solar system, measuring motion with respect to distant stars.
But, is there an absolute sense in which we can be said to be moving or not?
If so, can we measure this motion relative to some special body or substance, like absolute space?
Are we located in some special, privileged place?
Or are there only physical objects and spatial locations?
Is there space, in addition to places?
Where are we? Really, where? More on relational space

Newton and Leibniz clashed over whether space and time had absolute reality, or whether they were merely relational concepts.
Newton's view is that space is something distinct from the bodies that occupy it, and that time is something

that passes uniformly without regard to events in the world.
Space is an empty container, and time marches inexorably forward.
Though we measure space and time using bodies and events, these are only indicative of relative motions.

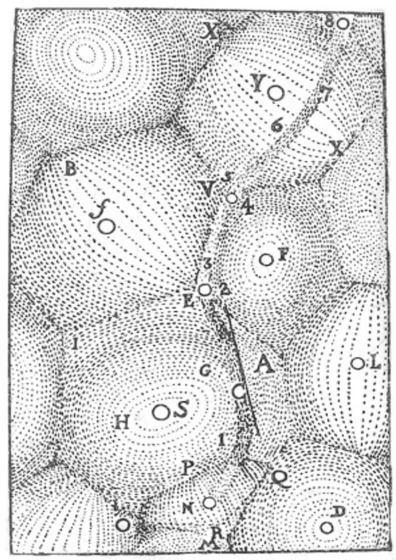
In contrast, Leibniz's relationalist view is that space and time are idealizations, abstractions from the realities of the material world.

> I hold space to be something merely relative, as time is...an order of coexistences, as time is an order of successions (Leibniz, Third Letter §4, AW 297b).

The differences between Newton and Leibniz over the nature of space and time are tied to their different conceptions of motion, and acceleration.
If motion is change of place over time, then to define motion, we have to know if we are appealing to absolute motion, the change in the place in absolute space of an object, or relative motion, the mere rearrangement of bodies.

Newton and Leibniz were influenced by two distinct views.
On the one hand, Descartes's physics denied the possibility of a void, or vacuum.
This view was inherited from the Aristoteleans who believed that a void is nothing, and what is nothing does not exist.
Descartes incorporated the opposition to a vacuum into the new science by taking the world to be a plenum, in which space is not distinct from the bodies which fill it.

> All places are full of bodies... Each body can move only in complete circle of matter, or ring of bodies which all move together at the same time: a body entering a given place expels another, and the expelled body moves on and expels another, and so on, until the body at the end of the sequence enters the place left by the first body... (Descartes, *Principles of* P *hilosophy*, II.33).



(Right: Descartes's depiction of the plenum, *Principles of Philosophy*, II.553)

Despite his many differences with Descartes, both in physics and metaphysics, Leibniz adopts Descartes's views on the completeness of the material world.

He argues for the saturation of the plenum from the contradiction he sees as inherent in the concept of empty space.

> Let us fancy a space wholly empty. God could have placed some matter in it without derogating, in any respect, from all other things; therefore, he has actually placed some matter in that space; therefore, there is no space wholly empty; therefore, all is full (Leibniz, Fourth Letter Postscript, AW 303a).

Leibniz believes that the idea of empty space contradicts God's commitment to creating the best of all possible worlds.
His denial of a void implies that there is no space beyond the places of objects.

In contrast to Descartes and Leibniz, atomists like Gassendi argue that the places between objects are empty.
Objects are placed in a transcendent void.
When we move, we change our place relative to the objects around us, and, they argue, we change our location in absolute space.
Newton adopts the view of the atomists and their commitment to a vacuum.

Here is one way to see the difference between Newton's absolutist view and Leibniz's relationalism.
Consider the question, "What exists outside the universe?"
Leibniz, with the Cartesians, answers that the universe extends infinitely, so that there is no outside.
Newton, with the atomists, answers that there is an empty void.
Today, the debate between relationalism and absolutism continues between space-time relationalists, who believe that space-time is an artificial, or nominal, construct out of particular bodies, and substantivalists, who believe in the existence of space-time points or regions.

V.3. Newton's Bucket

Here is a summary of Newton's views on space and time.

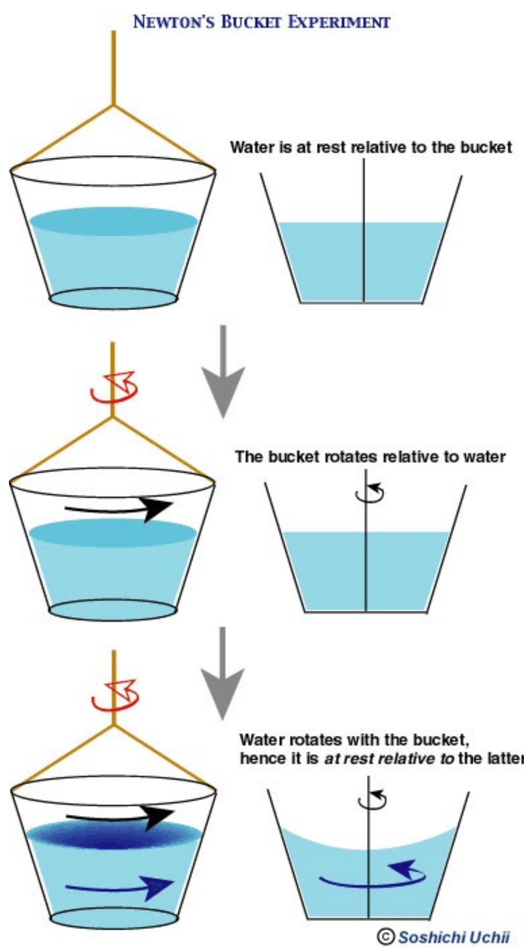
> Absolute time passes steadily without relation to anything external, and thus without reference to any change or way of measuring of time.
> Absolute space remains without relation to anything external.
> Relative spaces are measures of absolute space defined with reference to some system of bodies; a relative space may be in motion.
> The place of a body is the space which it occupies, whether absolute or relative.
> Absolute motion is the translation of a body from one absolute place to another; relative motion is the translation from one relative place to another.

For the absolutist, space is distinct from, and exists independently of, bodies.
It is logically and metaphysically prior to bodies and events among bodies, in that bodies require space but space need not include any bodies.
There is a fact of the matter whether a given body moves and what its true quantity of motion is.
The true motion of a body does not consist of, or cannot be defined in terms of, its motion relative to other bodies.
See the Stanford Encyclopedia of Philosophy article on Newton's views of Space, Time, and Motion.
More speculatively, Newton refers to space as the sensorium of God, and as the seat of divine cognition.

Newton's view can be found in the Scholium, as well as in the other assigned selections.
In the Scholium, Newton starts with definitions of absolute and relative spaces and motions and proceeds to argue for the existence of absolute time and space.
In large part, Newton's arguments are aimed against the Cartesians who defined motion in terms of the translation of a body relative to its surrounding objects in the plenum.
Newton had many reasons to be unhappy with Cartesian physics.
For one, Descartes centered his account of physics around motion, rather than acceleration.
The arguments in paragraphs 8–11, the last of which immediately precedes the discussion of the bucket experiment, are mainly directed at Cartesian physics.
He argues that the definition of motion as translation of a body relative to its surrounding objects will not allow us to arrive at a measurement of absolute motion.

For example, let's assume that bodies that are truly at rest are at rest with respect to one another.
Imagine that there is a distant star which is absolutely at rest.
We might wonder if something in our vicinity, say this table, is at rest, too.
But, if we measure the motion of the table relative to the motions of things around it, we can not know whether it is moving or at rest relative to the distant star.
The table is at rest with respect to its surroundings, but that does not determine whether it is at rest, absolutely.
Thus, true rest cannot be defined simply in terms of position relative to bodies in the vicinity.

Newton discusses other properties of motion that lead to difficulties for Cartesian physics.
The property that if a part of a body maintains a fixed position with respect to the body as a whole, then it participates in the motion of the whole body entails that absolute motion cannot be defined as a translation from the immediately surrounding bodies.
Imagine that I am sleeping in the back of a car.
My femur is at rest with respect to me.
I am at rest with respect to the car.
But, my femur and I are both moving.
The property that a body participates in the motion of its place when it moves away from that place entails that the absolute motion of a body cannot be defined except by means of stationary places.
You can change the relative motion of a body by changing the motion of the bodies to which you are comparing it.
But, you can only change the true, or absolute, motion of a body by applying some force to it.



These arguments from properties and causes are important for characterizing Newton's concept of absolute space and motion.
But, the most influential argument in favor of his thesis that we must posit absolute space in order to make sense of motion is Newton's example of a rotating bucket.
Newton's bucket experiment provides a case in which there are states of a system with different motions, yet which can not be described in terms of changes of place with respect to surrounding objects.
We know that the motions are different in the two states, but we can not differentiate them in terms of local changes of place.

Consider a bucket, suspended by a rope, and filled with water.
Turn the bucket many times, so that the rope twists.
In state 1, hold the bucket still.
The surface of the water inside the bucket is flat.
Now, let go of the bucket.
In state 2, the motion of the bucket is fast, but the motion of the water is slow.
The surface of the water in the bucket remains flat.
The water is moving very rapidly with respect to the bucket, and yet there is no centrifugal force manifested.
After a while, the water begins to turn with the bucket, and centrifugal force pushes the water up the sides of the bucket.

The surface of the water becomes concave.
In state 3, the bucket and the water are at relative rest, and yet the water has a concave surface.

Now, compare state 1 to state 3.
In both states, the water and the bucket are at relative rest.
In state 1, for both the relationalist and the absolutist, there is no motion.
But state 3 is measurably different to state 1, and the relationalist seems unable to describe the difference between the two states.
The water and the bucket are at relative rest in both states.
The absolutist needs merely to point out that in state 3, the system is in absolute motion, while in state 1, the system is at absolute rest.

One problem for the doctrine of absolute motion, a problem which Newton admits, is that, in contrast to rotation, which the bucket experiment measures, it is difficult to measure absolute velocity.
The absolute speed of a body is the rate of change of its position relative to an arbitrary point of absolute space.
According to Newton's account, absolute velocity is a well-defined quantity.
But consider, as Galileo did, riding in a ship at a constant velocity.

> Shut yourself up with some friend in the main cabin below decks on some large ship, and have with you there some flies, butterflies, and other small flying animals. Have a large bowl of water with some fish in it; hang up a bottle that empties drop by drop into a wide vessel beneath it.
>
> With the ship standing still, observe carefully how the little animals fly with equal speed to all sides of the cabin. The fish swim indifferently in all directions; the drops fall into the vessel beneath; and, in throwing something to your friend, you need to throw it no more strongly in one direction than another, the distances being equal; jumping with your feet together, you pass equal spaces in every direction.
>
> When you have observed all of these things carefully (though there is no doubt that when the ship is standing still everything must happen this way), have the ship proceed with any speed you like, so long as the motion is uniform and not fluctuating this way and that. You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still. In jumping, you will pass on the floor the same spaces as before, nor will you make larger jumps toward the stern than towards the prow even though the ship is moving quite rapidly, despite the fact that during the time that you are in the air the floor under you will be going in a direction opposite to your jump. In throwing something to your companion, you will need no more force to get it to him whether he is in the direction of the bow or the stern, with yourself situated opposite.
>
> The droplets will fall as before into the vessel beneath without dropping towards the stern, although while the drops are in the air the ship runs many spans. The fish in the water will swim towards the front of their bowl with no more effort than toward the back, and will go with equal ease to bait placed anywhere around the edges of the bowl. Finally the butterflies and flies will continue their flights indifferently toward every side, nor will it ever happen that they are concentrated toward the stern, as if tired out from keeping up with the course of the ship, from which they will have been separated during long intervals by keeping themselves in the air... (Galileo Galilei, *Dialogues Concerning the Two Chief World Systems*)

We cannot determine from observations inside the cabin whether the boat is at rest in harbor or sailing smoothly.
The point of the ship example, in this context, is to show that Newton's absolute velocity cannot be experimentally determined, unlike absolute rotation.

Newton recognized the problem.

> Yet the thing is not altogether desperate; for we have some arguments to guide us, partly from the apparent motions, which are the differences of the true motions, partly from the forces, which are the causes and effects of the true motions (Newton, Scholium to Definitions in *Principia*, AW 288a).

I will not pursue the details of Newton's solutions, which are really the elements of his mechanics.

V.4. Leibniz's Relationalism

Newton did not correspond directly with Leibniz.
There were hard feelings between the two over credit for developing the calculus.
Leibniz discusses many conflicts in his correspondence with Newton's secretary, Samuel Clarke.
Newton, it seems, did participate in constructing some of the correspondence, though some of it appears to be written by Clarke alone.

> Our dispute consists in many other things. The question is whether God does not act in the most regular and most perfect manner; whether his machine is liable to disorder, which he is obliged to mend by extraordinary means; whether the will of God can act without reason; whether space is an absolute being; also concerning the nature of miracles; and many such things, which make a wide difference between us (Leibniz, Third Letter §16, AW 299a).

We are focusing only on the question of whether space is relational or absolute.

One problem with Newton's claim is that space seems difficult to classify as a substance or an attribute.
Newton does not take space to be a substance, for it lacks causal powers.
But, it is also not an attribute, since its existence transcends the existence of any things.
Unlike, say, redness, it doesn't need a thing to be predicated of.

> If space is a property or attribute, it must be the property of some substance. But of what substance will that bounded empty space be an affection or property, which the persons I am arguing with suppose to be between two bodies? (Leibniz, Fourth Letter §8, AW 300a).

So, space is real, but hovers in between substance and attribute.
We could, for Newton, call it a pseudo-substance.
Leibniz seems to think that this argument is important.
He derives consequences from it that seem to impugn the perfections of God.
But, it is not clear that the argument has the ramifications that Leibniz takes it to have.
Perhaps the classification of all objects into substances and attributes is incomplete.

Leibniz's more influential arguments derive from general principles which he claims rescue science from nonsense.

> Those great principles of sufficient reason and of the identity of indiscernibles change the state of metaphysics. That science becomes real and demonstrative by means of these principles, whereas before it did generally consist in empty words (Leibniz, Fourth Letter §5, AW 299b).

Leibniz says that the doctrine of absolute space and time lead to absurdities.
Could the universe, for example, have been created at a different time?

Could it be moved three inches to the left?
There would be no way to distinguish two universes that were identical in all their relations among objects, but put into a different space, or reoriented.

> Those two states, the one such as it is now, the other supposed to be the quite contrary way, would not at all differ from one another. Their difference therefore is only to be found in our chimerical supposition of the reality of space in itself. But in truth, the one would exactly be the same thing as the other, they being absolutely indiscernible, and consequently there is no room to inquire after a reason for the preference of the one to the other (Leibniz, Third Letter §5, AW 297b-298a; see also Fourth Letter §13, AW 300a-b).

Instead, Leibniz argues, space is a set of relations among bodies.
Time is an abstract relation among events (or perceptions).
Those systems of relations might be thought of as abstract, but they should not be reified.
Elsewhere, in the Fifth Letter, Leibniz refers to the structure of space time as analogous to a family tree, which is just set of organizing relations, and not a thing in itself.
The infinite divisibility of space and time are further arguments against their reality; no really existing thing could be infinitely divisible.
We must take space and time to be ideal, or imaginary constructs derived from the appearances of bodies.

V.5. Kant's First Antinomy: On the Infinitude of Space and Time

Kant's first antinomy concerns both the temporal and spatial finitude of the universe.
He argues that the universe has a beginning from the premise that an infinite series can not be completed.
If the universe existed from infinitely long ago, the present would be, impossibly, the end of an infinite series.
So, there must have been some beginning.
For the spatial finitude of the universe, Kant claims that the concept of simultaneity presupposes a spatially finite universe.
If the universe were infinitely large, we could not think of all of the universe as existing simultaneously.

On the other side, Kant argues that the universe has no beginning in time from the impossibility of creation.
If there were a beginning point, there would have to be something before it.
But, that time would have nothing in it since the universe would not yet have been created.
So the universe would have no way to begin.

Kant's argument that the universe is spatially infinite assumes absolute space.
Imagine you were to go to the end of the universe.
Now, stick out your arm past the edge.
It seems that you could always perform this task.
Thus, the container must be infinite.
If we take the universe to be merely the contained portion, then we have no way to think about the container, the rest of space.
So space itself must be the infinite container.

> If one wants to leave out the void, and hence space as such, as an *a priori* condition for the possibility of appearances, then the entire world of sense drops out (Kant, *Critique of Pure Reason* A433/B461, AW 794b).

Remember, space is, for Kant, an *a priori* form of intuition, presupposed by all possible experience.

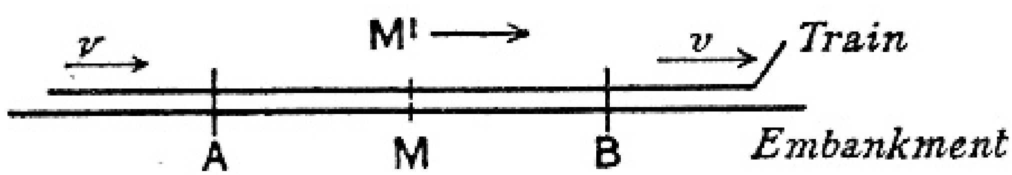
Kant has argued to both sides of a contradiction.
The universe is both infinite and finite, in both space and time.
Since the arguments are both unassailable individually and impossible, he concludes that pure reason has exceeded its reach in this matter.
There is no knowledge to be had of whether the universe is finite or infinite.
Like Hume, in this case, Kant concludes skeptically.

V.6. Beyond the First Antinomy

Kant's skeptical conclusion about our abilities to know whether space and time are finite or infinite may be too hasty.
Notice that Kant assumes that claims about whether the universe is finite or infinite are matters for *a priori* metaphysical reasoning.
Today, physicists believe that there are empirical answers to the question.
There are some mathematical and physical facts that undermine his claims.

Kant asserts that the universe must be spatially bound because otherwise we could have no definite concept of simultaneity.
But according to the theory of relativity, simultaneity and time itself are not definite concepts anyway.
They depend on the arbitrary choice of a frame of reference.
Imagine standing on a platform waiting for a train equidistant between two signal posts, A and B.
Imagine further that lights are flashed at A and B in a way that one perceives them as flashing simultaneously.
Now, consider a train traveling in the direction from A to B, passing you at the very moment that the lights are flashed.
To a perceiver on the train, the light at B will appear before the light at A, since the frame of reference (the train) is moving toward B, and away from A.

> Events which are simultaneous with reference to the embankment are not simultaneous with respect to the train, and vice versa (relativity of simultaneity). Every reference-body (co- ordinate system) has its own particular time; unless we are told the reference-body to which the statement of time refers, there is no meaning in a statement of the time of an event (Einstein, *Relativity: The Special and General Theory*, Chapter IX).



Similarly, Kant assumes an obsolete concept of infinity.

> The true (transcendental) concept of infinity is this: that the successive synthesis of unit[s] in measuring by means of a quantum can never be completed (Kant, *Critique of Pure Reason* A432/B460, AW 793a).

In the nineteenth century, Georg Cantor's work on transfinite numbers established that there are different sizes of infinity.
To count from one size of infinity to the next, we consider the smaller infinity as complete.

Today, we often define a set to be infinite if it can be put into one-one correspondence with a proper subset of itself.
For example, consider the integers and the even integers, which are a proper subset of the integers.
The integers are infinite just because we can match each one with an even integer.
See the appendix on infinity, at the end of these notes, or this longer, fun discussion of infinite arithmetic.

Kant's claims about the certainty of mathematics meet similar objections.
Kant argues that we have *a priori* knowledge of Euclidean geometry arising from its role as a form of pure intuition assumed in all appearances.
In the early nineteenth century, it became clear that there are various consistent non-Euclidean geometries.
Euclidean geometry, instead of being the sole correct geometry, as Kant argued, is just one of a family of mathematical theories, not mathematically favored.
Moreover, it turns out that physical space is non-Euclidean, hypeorbolic rather than flat.

Kant's arguments that we cannot know whether space and time are finite or infinite and his arguments that we know Euclidean geometry to be the *a priori* structure of space thus both seem overly dogmatic and false.
Indeed, one dominant theme of Kantian scholarship is that these claims are devastating failures of Kant's metaphysical system.
Still, the form of Kant's work was and is enormously influential, not least in his claim, perhaps derived from Hume's work, that certain kinds of philosophical assertions are beyond the limits of reason.
The point of Kant's discussion of the antinomies is precisely to demonstrate the bounds of reason.
Kant believes that the theses and the antitheses of each antinomy are equally defensible.
There is thus no claim that we can establish about questions of the infinitude of space, whether there are simples, or whether we are free.
Such claims are beyond our ability to know.
We see attempts to circumscribe the limits of reason especially in the work of the twentieth-century philosopher Ludwig Wittgenstein, and the followers of Wittgenstein's earlier work, the logical empiricists.

§IV. The Third Antinomy: Freedom and Determinism

[Note: We're skipping this section in Spring 2016.]

The thesis of the third antinomy is that there is Cartesian, libertarian free will.
Kant's argument for the thesis is that the contrary, strict determinism, is impossible.
We might think that every event has a cause.
That would lead to an infinite regress and the need for an exception to the rule that every event has a cause.

> If everything occurs according to mere laws of nature, then there is always only a subordinate but never a first beginning, and hence there then is on the side of the causes originating from one another no completeness of the series at all. The law of nature, however, consists precisely in this: that nothing occurs without a cause sufficiently determined *a priori*. Hence the proposition, in its unlimited universality, whereby any causality is possible only according to natural laws contradicts itself... (Kant, *Critique of Pure Reason* A444/B472, AW 798a)

On the thesis, every freely chosen act is the absolute beginning of a causal chain.
For the antithesis, strict determinism, Kant argues that freedom is not merely the absence of constraint but the chaotic lack of all rules.
A so-called free act would be utterly inexplicable and unthinkable.

> The coherence of appearances determining one another necessarily according to universal laws - which is called nature - would for the most part vanish, and along with it so would the mark of empirical truth which distinguishes experience from a dream (Kant, *Critique of Pure Reason* A451/B479, AW 800b)

A libertarian act would not be a possible experience.

§VII. Kant on the Ontological Argument

Finally, let's turn to Kant's discussion of the argument for the existence of God in Descartes's Fifth Meditation, which Kant named the ontological argument.
If you ask philosophers what they believe is wrong with the ontological argument, they will most likely point to Kant's rejection of the argument
As we have seen, Hume's influence on Kant was profound.
His psychological reinterpretation of the concept of causation was a precedent for Kant's transcendental idealism.
Kant's claims about the limits of pure reason have Humean roots, too.
Kant's reason for rejecting the ontological argument is derived from Hume's claims about the nature of existence claims as well as from Gassendi's claim that existence is not a perfection.
In his Objections to Descartes's *Meditations*, Gassendi complains that the ontological argument is invalid because existence is not a perfection.
One can not conclude the existence of God from the claim that existence is a perfection because existence is not a property.
It is a precondition for having properties.
Descartes disagreed with Gassendi, though the argument was left without a resolution.
Hume and Kant revive Gassendi's claim by adding a supporting argument.
Hume claims that the idea of existence, since it does not come from a distinct impression, adds nothing to the idea of an object.

> Though certain sensations may at one time be united, we quickly find they admit of a separation, and may be presented apart. And thus, though every impression and idea we remember be considered as existent, the idea of existence is not derived from any particular impression. The idea of existence, then, is the very same with the idea of what we conceive to be existent. To reflect on any thing simply, and to reflect on it as existent, are nothing different from each other. That idea, when conjoined with the idea of any object, makes no addition to it. Whatever we conceive, we conceive to be existent. Any idea we please to form is the idea of a being; and the idea of a being is any idea we please to form (Hume, *A Treatise on Human Nature* §I.II.VI).

Kant, following Hume, claims that existence is not a property the way that the perfections are properties.
Existence can not be part of an essence because it is not a property.
Whether we think of a thing as existing or not, as necessarily existing or not, we are thinking of the same thing.

> A hundred real thalers do not contain the least coin more than a hundred possible thalers (Kant, *Critique of Pure Reason* A599/B627, AW 822a).

Kant distinguishes between real (or determining) predicates and logical predicates.
A logical predicate is just something that serves as a predicate in grammar.
In 'the Statue of Liberty exists', we are predicating (grammatically) existence of the statue.

We are not saying anything substantive about the statue.
In 'the Statue of Liberty is over 150 feet tall', we use a real predicate.
Any property can be predicated of any object, grammatically.
'Seventeen loves its mother' is a grammatical sentence, even if it is nonsensical.
'Loves one's mother' is a real predicate.
Kant's point is that one can not do metaphysics through grammar alone.
Existence is a grammatical predicate, but not a real predicate.

Kant's objection support's Gassendi's criticism of Descartes's version of the argument.
It also accounts for earlier objections from Gaunilo and Caterus.
Gaunilo, responding to Anselm's version of the ontological argument, wondered whether having the concept of the most perfect island entails its existence.
Caterus wondered if the concept of the necessarily existing lion entails the actual existence of a lion.
Kant says that in predicating existence of a concept, we are just restating the concept, and not saying anything about the object.
When we say that 'God exists', we are not making a real assertion.
We are just restating the concept of God.
Hume claimed that the contrary of any matter of fact is possible.
Kant agrees.

> If you admit—as any reasonable person must—that any existential proposition is synthetic, then how can you assert that the predicate of existence cannot be annulled without contradiction? For this superiority belongs only to analytic propositions as their peculiarity, since their character rests precisely on this [necessity] (Kant, *Critique of Pure Reason* A598/B626, AW 821b)

Part of Kant's support for his assertion that existence is not a predicate is that existence is too thin.
We do not add anything to a concept by claiming that it exists.
The real and possible thalers must have the same number of thalers in order that the concept attach to the object.
If there are more thalers in the real thalers, then the concept and the object would not match.
So, we do not add thalers when we mention that the thalers exist.
But, do we add something?
When my daughter and I discuss the existence of the tooth fairy, we are debating something substantive.
If we are going to debate the existence of something, whether it be the tooth fairy or black holes, we seem to consider an object and wonder whether it has the property of existing.
We thus have to consider objects which may or may not exist.
There may be many such objects, e.g. James Brown and Tony Soprano.
Some philosophers, like Meinong, attribute subsistence to dead folks and fictional objects.
One might say that James Brown has the property of subsisting without having the property of existing.
That is, Kant's claim that existence is not a real predicate, while influential, may not solve the problem.

In ordinary cases, Hume and Kant certainly are correct that logic, or reason, can not make existence claims.
The question is whether logic can make this one existence claim.
Kant's claim that existence is not a real predicate, while influential, may not solve the problem.

Many contemporary philosophers are swayed in Kant's direction by their familiarity with first-order logic's distinction between predication and quantification, and by the distinction between grammatical form and logical form.
In Fregean logic, properties like being a god, or a person, or being mortal or vain, get translated as predicates.
Existence is taken care of by quantifiers, rather than predicates.

To say that God exists, we say '$(\exists x)Gx$' or '$(\exists x)\ x=g$'
Note that the concept of God is represented independently of the claim of God's existence.
First-order logic is supposed to be our most austere, canonical language.
As Frege says, it puts a microscope to our linguistic usage.
Thus, there does seem to be a real difference between existence and predication and between the grammar of natural language and the true logical form of our claims.

Still, formal systems can be constructed with all sorts of properties.
We can turn any predicate into a quantifier, or a functor, even turn all of them into functors.
Is first-order logic the best framework for metaphysics?
Is Kant's linguistic solution to the ontological argument decisive?

These questions get discussed in courses on logic, philosophy of science, philosophy of language, and philosophy of mathematics.

Appendix to Kant Notes on Infinity
Excerpted from *What Follows*, Draft of Spring 2016

Logic, Infinity, and Paradox

The calculus developed by Leibniz and Newton in the seventeenth and eighteenth centuries was both wildly successful and oddly unnerving. Its central technique involves finding the area under a curve by dividing the curve into infinitely many, infinitely small areas, called infinitesimals by Leibniz and fluxions by Newton, then adding these infinitely many areas together. On the one hand, the results were precise, perfect, and widely applied in science and mathematics. On the other hand, the idea of adding an infinite number of infinitely small areas seemed preposterous to some mathematicians and philosophers, more so when the results often turned out to be small finite numbers. How can the sum of an infinite number of infinitely small quantities be $\sqrt{7}/2$ or –3? Infinity was supposed to be the realm of God and paradoxes, not productive mathematical methods.

Among the most unsettling results which led Frege and others to seek more secure systems of inference were those of Georg Cantor which showed that there are different sizes of infinity, indeed infinitely many different sizes of infinity. Until the mid-nineteenth century, the infinite was a concept perhaps more of interest to philosophers than to mathematicians. Earlier mathematicians certainly knew about a variety of concepts of mathematical infinity. There were large infinities of addition: the infinity of the counting numbers, the dense infinity of the rational numbers and the continuous infinity of the real numbers, and the infinity of space. There were small infinities: the number of points in a finite line segment, say, or the infinitesimals or fluxions used in the calculus of Leibniz and Newton.

But infinities led to paradoxes. Among the oldest and most influential of the problems are known as Zeno's paradoxes. Little is known of the eponymous Zeno of Elea, who lived in the fifth century B. C. E., beyond that his paradoxes are intended to support the claims of the philosopher Parmenides that reality is one, uniform, and unchanging. While we seem to experience a complex, variegated, and changing world, Parmenides claimed that the real world is stable and constant, unlike the world we perceive. Zeno's paradoxes, invoking the infinite divisibility of space and time, seem to show an error in our beliefs about a changing world.

For example, consider the famous paradox of the racing Achilles and the Tortoise. Achilles gives the tortoise a head-start, let's say one hundred feet. But now, Achilles can never catch the tortoise. For while he runs the hundred feet initially separating the pair, the tortoise is also in motion, though more slowly than Achilles. When Achilles gets to the tortoise's starting point, the tortoise will have moved, say, ten feet further. And when Achilles reaches one hundred and ten feet, the tortoise again has moved further, another foot, say. As Achilles moves the further foot toward the tortoise, the tortoise is once again a little bit further along. This goes on infinitely: no matter how many times Achilles reaches a given point formerly occupied by the tortoise, the tortoise will have moved a little further. Achilles can never catch the tortoise.

Or consider the paradox of the arrow, which assumes that time is composed of atomic instants, ones which cannot be further subdivided. The arc traced by a flying arrow consists of some number of these instants. Consider any one of these instants, and ask whether the arrow is moving at that instant. If the arrow is in motion at the instant, then it must be at one place at the beginning of the instant and at another distinct place at the end of that instant. But then there seem to be parts of an instant, its beginning and its end, contrary to our assumption that time consists of atomic instants. Hence the arrow cannot be in motion at any instant. But the flight of the arrow consists of the sum of its motions at each instant. Since it does not move at any instant, the sum of these instants is zero: the arrow does not move.

Mathematicians and philosophers dealt with the paradoxes largely by constructing some distinctions, between actual and potential infinities, for example, and between categorematic and syncategorematic uses of the infinite. The distinction between actual and potential infinity is found in Aristotle's work from the fourth century B. C. E. Aristotle claims that the Achilles paradox, for example, is solved by the observation that Achilles need not traverse an actual infinite series of distances, which would be impossible. Instead, the infinite number of distances is only potentially infinite. We don't actually divide space in the way that Zeno presumes. Thus, the paradox is merely potential and unproblematic. Similarly, if time is not actually infinitely divisible, but only potentially so, the arrow can fly.

In the thirteenth century, the terms 'categorematic' and 'syncategorematic' were introduced to distinguish ways of speaking about the infinite. When one speaks of an infinite quantity as if it actually existed, as when one says that there are infinitely many points on a line, one speaks categorematically and dangerously. But when one says that a line can be extended, potentially, indefinitely, one speaks syncategorematically. Again, uses of this distinction were invoked to avoid accidentally saying something paradoxical or unacceptable about actual infinities.

So mathematicians and philosophers mainly avoided invoking infinities as much as possible, relinquishing their resistance in the case of the successful calculus but often with guilty consciences, and in the face of severe criticism, For example, the philosopher Bishop George Berkeley wrote a treatise in the eighteenth century, *The Analyst*, in which he accused the proponents of the calculus of basing their work on fundamental errors about the nature of space. It wasn't until the nineteenth century, when Dedekind, Weierstrass, and others arithmetized analysis by showing how to define limits more precisely, that the calculus was seen to be put on firm footing.

In the mid-nineteenth century, though, the mathematician Georg Cantor constructed a startling and influential proof that there are different sizes of infinity. This proof changed the way philosophers and mathematicians thought about and worked with infinity, introducing us to what the late nineteenth-century mathematician David Hilbert called Cantor's paradise of infinitary mathematics.

To get a feel for the different sizes of infinity, we will consider a now-classic concept which is sometimes called the infinite hotel.

The Infinite Hotel

You are the desk clerk at an infinite hotel which has, let us suppose, infinitely many rooms. The hotel is fully booked when a new guest arrives. In a finite hotel, you would have to turn away the potential new guest. But in an infinite hotel, you can add the new guest. To do so, shift every current guest from Room n to Room n+1: the guest in Room 2 moves to Room 3; the guest who was in Room 3 moves to Room 4; the guest from Room 4 moves to Room 5; and so on. Now Room 1 is available for the arriving guest.

If a further finite group of guests arrives, you can perform the same procedure to free up any finite number of rooms. Just add any finite number of guests, m, by shifting all current guests from their current Room n to Room n+m and putting the new guests in the first m rooms. If seven guests arrive, for example, move all the current guests to rooms with numbers exactly seven greater than their current rooms. If a billion guests arrive, just move them to rooms with numbers a billion greater than the ones they are in currently. Then slot the new guests into the newly vacant rooms with numbers at the beginning of the natural number sequence.

Next, a bus with an infinite number of guests arrives. If you try to shift all guests from Room n to Room (n + the number of guests on the bus), you have to move the guest in Room 2, say, to Room 2 + infinity. But since there is no number 'infinity' (or so one might think) you do not know where to put the current guests.

You can still accommodate an infinite number of new guests, but you have to use a new procedure. Shift every current guest from Room n to Room 2n. The guest in Room 2 moves to Room 4, the guest who was in Room 3 moves to Room 6, and so on. Now, all the even-numbered rooms are filled, and the odd-numbered rooms are vacant. We can put the new guests in the odd-numbered rooms: Room 1, Room 3, Room 5, and so on.

Next, an infinite number of infinite busloads of guests arrives. You can still accommodate them, but again you need a different procedure. Shift all current guests from Room n to Room 2^n. So the person in Room 2 stays in Room 2^1 (i.e. Room 2); the person who was in Room 2 moves to Room 2^2 (i.e. Room 4); the person who was in Room 3 moves to Room 2^3 (i.e. Room 8); the person who was in Room 4 moves to Room 2^4 (i.e. Room 16); and so on. All of the present guests can be accommodated in the infinite number of rooms that are powers of two, leaving lots of empty rooms. We can place the people on the first bus in room numbers 3^n (for n people on the bus), the people in the second bus in rooms 5^n, the people in the third bus to rooms 7^n, and so on for each (prime number)^n. Since there are an infinite number of prime numbers, there will be an infinite number of infinite such sequences. And, there will still be lots of empty rooms left over!

A natural question to arise is whether there are any sets of guests that the infinite hotel could not accommodate. This question is precisely a question about the fine structure of the numbers and whether there are different sizes of infinity.

Two Concepts of Size

Numbers have at least two different functions: measuring the size of a collection of things; and ordering, or ranking, a series. When we use numbers to measure size, we use the property of the numbers called cardinality. When we use them to measure rank (first, second, third...), we use the property called ordinality. Mathematicians sometimes consider the numbers in their different uses as different objects altogether. Thus we have cardinal numbers and ordinal numbers.

One way to characterize cardinal numbers is to invoke one-one correspondence. Consider a basket of apples and a classroom of hungry kindergartners. We can determine whether there are the same number of apples and kindergartners by giving each student exactly one apple. If there are no extra apples or children, there are (or were, since the kids were hungry) the same number of each. The view that we can define numbers in terms of one-one correspondence has become known as Hume's principle, though the use of one-one correspondence to measure size precedes Hume.

Another way to think about numbers, perhaps more-closely related to their ordinal properties, is in terms of wholes and parts: a > b if and only if there is some positive number c such that b + c = a.

With finite numbers, the characterization in terms of one-one correspondence converges with the characterization in terms of parts and wholes. The size of a group is the same as the correspondence between the objects in the group and some initial segment of the natural numbers. If we have five hungry students, we can line them up (ordinally) and give them each a number from one to five (cardinally).

But these two concepts diverge with transfinite numbers. The size of the integers seems to be bigger than the size of the even numbers since the size of a whole seems to be greater than the size of its proper part and the even numbers are a proper part of the integers. But, the even numbers (E) and the integers (N) can be put into one-one correspondence with each other.

E: 2, 4, 6, 8...
↕ ↕ ↕ ↕
N: 1, 2, 3, 4...

Let's give names to these different concepts of size. Two sets have the same *size*$_h$ (for Hume) if they can be put in one-one correspondence with each other. Two sets have the same *size*$_w$ (for the whole is greater than the sum of its parts) if it is not possible to put either in one-one correspondence with a proper part of itself. So, N and E have the same size$_h$ but different size$_w$s.

You might think, and before Cantor's work in the mid-nineteenth century it was widely believed, that there is just one size of infinity, that all infinities have the same size$_h$. That claim turns out to be false. Moreover, size$_h$ has come to be recognized as the central notion of cardinality. Parallel conclusions can be drawn about infinite ordinal numbers. There are many, indeed infinitely many, different infinite numbers. These are the conclusions of Cantor's influential diagonal argument, one of the most important intellectual discoveries of all time.

Cantor's Diagonal Argument

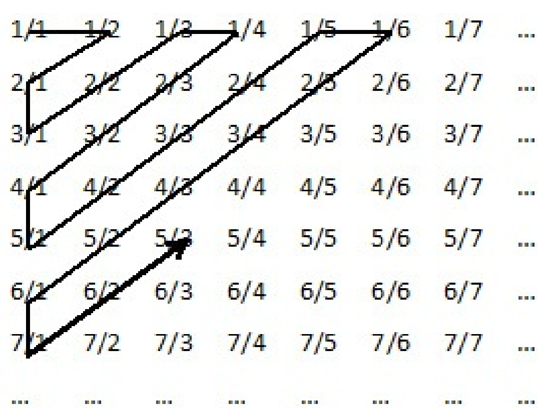
When we make a list, we put objects into one-one correspondence with the natural numbers: item 1, item 2, item 3, and so on. Any infinite list will thus be the same size$_h$ as the natural numbers. For example, we can show how to list the even numbers, as we showed in the previous section; the set of even numbers is the same size$_h$ as the set of integers.

For another example, we can list the prime numbers. The set of primes, P, like the set of even numbers, is a proper subset of the set of natural numbers. But again, it is an infinite set the same size$_h$ as the natural numbers, which we can show by putting the primes in a list.

P: 2, 3, 5, 7, 11, 13...
↕ ↕ ↕ ↕ ↕ ↕
N: 1, 2, 3, 4, 5, 6, ...

By listing the even numbers, or the odd numbers, or the multiples of seven, or the prime numbers, we are showing that such sets of numb4ers have the same infinite cardinality as the set of natural numbers, despite being proper subsets of the natural numbers. It is characteristic of an infinite set that it can be put into one-one correspondence with a proper subset of itself.

So both the set of prime numbers and the set of even numbers are the same size$_h$ as the natural numbers while being a smaller size$_w$. Other sets have a larger size$_w$ but the same size$_h$ as the natural numbers. Consider the rational numbers, all ordered pairs of natural numbers. We often call the rational numbers fractions, taking the first of the ordered pair as the numerator and the second as the denominator. There seem to be more rationals than natural numbers. Between every two natural numbers there are many rational numbers, though the reverse is not true. But using a neat trick, we can make a list of the rationals too, showing that they can by put into one-one correspondence with the natural numbers and thus that the natural numbers and the rational numbers have the same size$_h$. Just follow the path indicated by the arrows in the diagram below to construct a complete (if sometimes redundant) list. [2]



Such constructions may tempt us to think that any set of numbers can be listed and thus that all sets of numbers have the same size$_h$. But if there were some kinds of sets whose members could not be put into a

[2] While this neat technique traces a path through the diagram which is sometimes diagonal, it is *not* what we call the diagonal argument.

list, then that set would be strictly larger than the set of natural numbers, in both $size_h$ and $size_w$. There would be different sizes of infinity, however we measure size.

Cantor shows that we can not make certain lists. In terms of the infinite hotel, he shows that there are sets of guests which could not be accommodated. In general, Cantor shows how to construct sets of larger and larger $size_h$s. In particular, his diagonal argument proves that we can not list the real numbers.

There are different versions of the diagonal argument, and it can be applied to both numbers and, perhaps more generally, to sets. Let's take a look at the argument as it applies to the real numbers. The real numbers may be represented as their decimal expansions, many of which are non-repeating and non-terminating. The structure of the argument is a *reductio ad absurdum*, or indirect proof. We start by supposing, contrary to our desired conclusion, that we can make a list of all of the real numbers. For simplicity's sake, let's imagine that we can list all of the real numbers between zero and one; it turns out that we can't even list these reals. Each such real, in its decimal representation, will consist of a 0, a decimal point, and a sequence (perhaps terminating or repeating, and perhaps not) of natural numbers. We can ignore the zero and the decimal point and just look at the sequence of digits past the decimal point.

So we can write, we are supposing for *reductio*, a complete list of such sequences. Let's represent that hypothetical list, L, abstractly, using a concatenation of variables.

L $\quad a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7...$
$b_1\ b_2\ b_3\ b_4\ b_5\ b_6\ b_7...$
$c_1\ c_2\ c_3\ c_4\ c_5\ c_6\ c_7...$
$d_1\ d_2\ d_3\ d_4\ d_5\ d_6\ d_7...$
...

So, for example, '$a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7$..' could represent '3756920...', which would stand for the real number whose decimal expansion starts 0.3756920...

By hypothesis, L contains the decimal extensions of all real numbers. Cantor's diagonal technique allows us to find a number which does not, in principle, appear in L, contradicting our assumption that L is a complete list.

Consider the number C, defined by concatenating one term from each of the numbers in the list L. We select the number by looking the diagonal of L, taking the first term from the first number in the list, the second term from the second number, and so on.

$$C = a_1\ b_2\ c_3\ d_4\ e_5\ f_6\ g_7...$$

C could be in L. It has the same first number as the first number in the list, the same second number as the second number in the list, and so on.

But, given C, we can construct a new number which can not be on the list, showing the list to be incomplete. Just change each digit in C to create a new number C*. For instance, to construct C*, we can add one to each digit of C other than nine, and replace all nines in C with zeroes.

Now, C* is certainly not in L. For, C* is different from the first number in L in its first digit, different from the second number in L in its second digit, and so on, for all numbers on the list.

In a quixotic attempt to complete the list, we could add C* to L, to make a new list, L*. But the same procedure allows us to form a new number, say C**, that's not in L*. However complete we make our list, we can always find a number that is not in it.

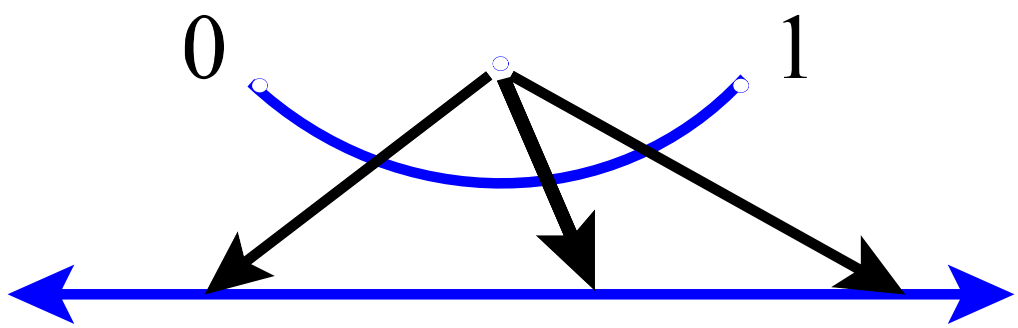
Thus, all possible lists of real numbers are necessarily incomplete. We are in principle prevented from establishing a one-one correspondence between the natural numbers and the real numbers. There are strictly more real numbers than natural numbers.

The preceding proof is called a diagonal argument, due to its method of producing C* along the diagonal of the list.

Into the Transfinite and Cantor's Theorem

With Cantor, let's call the size of the natural numbers $\aleph_0$. Since the real numbers between zero and one have a strictly larger size than $\aleph_0$, we can say that the set of reals between zero and one has a size greater than $\aleph_0$. Just as the set of natural numbers contains many proper subsets with the size $\aleph_0$, the set of real numbers has many subsets of its greater size. To see this, first remember that we use the real numbers as the representations of all of the points on a line. We can show that there are the same number of real numbers overall as there are real numbers between zero and one by providing a mapping between the real numbers (points on a line) between zero and one and all the real numbers (points).

Here is a geometric demonstration.



For each point on the curved line between zero and one, we can find a point on the infinite line, and vice-versa. Since there is a one-one mapping between the two lines, there are the same number of points in each line.

If you prefer an analytic proof, take f(x) = tan (2x-1)/2. The tangent curve in that domain goes between negative infinity and positive infinity, over a domain between zero and one. Thus we can correlate each point on the x-axis with a unique real number on the y-axis, and vice-versa.

We have seen now that infinity is at least more complicated than was thought, prior to the nineteenth century. There are at least two different sizes of infinity, even in terms of $size_h$. It turns out that there are in fact infinitely many different sizes of infinity, since a generalized version of the diagonal argument can be run on any set of any size. For any set, there is another set of greater size.

To get a feel for the properties of infinite numbers, let's take a look at some properties of numbers and whether they hold just for finite numbers or for infinite numbers as well. I won't prove that these properties hold, here. The familiar properties FP hold for all cardinal numbers, finite or transfinite.

FP For all cardinal numbers a, b, and c:

$a + b = b + a$

$ab = ba$

$a + (b + c) = (a + b) + c$

$a \bullet (b \bullet c) = (a \bullet b) \bullet c$

$a \bullet (b + c) = ab + ac$

$a^{(b + c)} = a^b \bullet a^c$

$(ab)^c = a^c \bullet b^c$

$(a^b)^c = a^{bc}$

Not all properties of finite numbers hold for infinite numbers. We already saw one property which holds only for infinite sets, the property of having a proper subset which is the same $size_h$ as itself. The properties at NI hold for infinite numbers, but not for finite numbers.

NI For infinite cardinals a:

$a + 1 = a$

$2a = a$

$a \bullet a = a$

We can show the three claims at NI by producing bijective (one-one) mappings between sets of each size. That's what we did in the discussion of the infinite hotel.

Consider one final important property, NI*, which holds both of finite and transfinite numbers.

NI* $2^a > a$

Whether a is finite or infinite, 2^a will always be a number with a larger cardinality. NI* has an analog in set theory: the power set (or set of all subsets) of a set is always strictly larger than the given set. The set-theoretic claim is called Cantor's Theorem and its proof is a set-theoretic version of the diagonal argument, a proof of which I'll leave for the end of this appendix.

Given Cantor's theorem, which shows that there are infinitely many different sizes of infinity, we can start naming the infinite numbers of differing cardinalities, proceeding beyond the sequence of natural numbers. We can define a sequence of infinite cardinalities:

$\aleph_0, \aleph_1, \aleph_2, \aleph_3, \aleph_4 \ldots$

While there are an infinite number of infinite cardinalities on this list, set theorists, by various ingenious methods including the addition of axioms which neither follow from nor contradict the standard axioms, generate even larger cardinals than these. Cardinal counting gets pretty wild. There are ethereal cardinals, subtle cardinals, almost ineffable cardinals, totally ineffable cardinals, remarkable cardinals, superstrong cardinals, and superhuge cardinals, among many others. All of these cardinal numbers are transfinite and larger than any of the sequence of alephs.

Returning to just the alephs, the natural numbers have the size $\aleph_0$. We saw that the rational numbers have the same cardinality. There are more real numbers than natural numbers, as we saw in the diagonal argument. But how many reals are there? Are the reals the next largest size of infinity, $\aleph_1$, or are there other sizes of infinity between the size of the natural numbers and the size of the real numbers? Cantor's continuum hypothesis is the claim that the reals are of size $\aleph_1$, but the size of the real numbers is one of the most interesting open questions in mathematics.

The Continuum Hypothesis

Certain questions in the history of mathematics have proven difficult to answer. Fermat's theorem, that there are no n for which there are a, b, and c, such that $a^n + b^n = c^n$, was conjectured in 1637, written in the margin of Fermat's copy of Diophantus's *Arithmetica*. It was proven in 1994. Goldbach's conjecture, that every even number greater than four can be written as the sum of two odd primes, remains unproven, though most mathematicians believe that it is also true.

The parallel postulate is more interesting. It can fail, as it does in non-Euclidean spaces. But it can also hold, as it does in Euclidean space. Thus, we have decided that the question whether the parallel postulate is true is, strictly speaking, ill-formed. There is no one true answer. There are different kinds of spaces, each defined by a different version of the parallel postulate.

Cantor's continuum hypothesis is interesting in part because we do not know if it is like Fermat's theorem or Goldbach's conjecture, in having a solution, or whether it is ill-formed, like the parallel postulate. Cantor provided a method for generating larger and larger transfinite numbers. He shows that the cardinal number of the reals is equal to $2^{\aleph_0}$. He also shows $2^{\aleph_0}$ t is greater than $\aleph_0$. Cantor's theorem does not show, however, that it is the next greater transfinite number. Let's take $\aleph_1$ to be the name we give to the next transfinite cardinal after $\aleph_0$. The continuum hypothesis is that $\aleph_1 = 2^{\aleph_0}$. The generalized continuum hypothesis is that $\aleph_{n+1} = 2^{\aleph_n}$.

To understand how the continuum hypothesis might be false, remember that certain operations on finite numbers which generate larger numbers, like exponentiation, skip numbers. When we multiply a finite cardinal by two, or seventeen, or add six, or raise to the π power, we generate cardinals that are not merely one larger. Only the successor function yields the next natural number. So it seems possible that 2^n, for transfinite n, is more like ordinary exponentiation in skipping some transfinite numbers, rather than like succession, which gives the next largest number. In other words, we do not know that $2^{\aleph_0}$ is $\aleph_1$. Indeed, we do not even know that transfinite cardinal numbers can be ordered linearly.

At the 1900 Paris Congress, David Hilbert cited the continuum hypothesis as one of the ten most important unsolved problems in mathematics. Cantor believed the continuum hypothesis, but he could not prove it. Two results in the twentieth century further entrenched the problem. In 1940, Kurt Gödel showed that the continuum hypothesis is consistent with the standard axioms of set theory. But in 1963, Paul Cohen showed that its negation is consistent with set theory. Thus, the continuum hypothesis is independent of the standard axioms. We can consistently consider the continuum to be of all different sizes: $\aleph_1$, $\aleph_2$, $\aleph_3$, etc. Even the additions of many large cardinal axioms to the standard axioms of set theory fail to settle the question.

But we can settle the question of the size of the continuum by adopting some stronger axioms for set theory. Some mathematicians believe that the continuum hypothesis, even the generalized version, is so intuitively true that we should just adopt it, or an equivalent, as part of set theory. As we will see, Gödel favored this approach. Alternatively, we could take the question to be ill-formed, like the question of whether the parallel postulate is true. Perhaps there are different set theories, with different sizes of the continuum. Mathematicians are divided on whether the continuum hypothesis is true, though opinion has generally turned against it.

Ordinals and Counting

As we saw above, we use cardinal numbers to measure size, while we use ordinal numbers for ranking; first, second, third, etc. Cantor defined cardinal numbers in terms of ordinal numbers, making the ordinals more fundamental. Frege's attempt to define the numbers followed Cantor's work; Frege sought independent definitions of the ordinals and cardinals. Remember, the development of modern logic, like the logic in our first three chapters, was in the service of Frege's project in the foundations of arithmetic. To finish this section, let's look briefly at the ordinal numbers, and at how we can define arithmetic by using the more general set theory.

Cantor developed set theory in order to generate his theory of transfinites. Frege assumed a similar set theory in his work. Despite some differences, Cantor and Frege both used inconsistent set theories, which we now call naive set theory for its assumption that any property determines a set. The inconsistency was discovered by Bertrand Russell; it is called Russell's paradox. Russell's paradox shows that some properties taken to define sets lead to contradictions.

To understand Russell's paradox, one needs a little understanding of set theory. Sets are collections which have members, the items in the set. They can include other sets as members. Let's say that there are seventy million pet dogs in the United States and ninety million pet cats. So there's a set of pet dogs, D, with seventy million members, and a set of pet cats, C, with ninety million members. There's also a set of pet dogs or cats, B, with one hundred and seventy million members. And there's a set, let's call it P, with just two members, both sets: the set of dogs and the set of cats. FS lists all four of these sets.

FS
D: {all the pet dogs in the U.S.}
C: {all the pet cats in the U.S.}
B: {anything which is either a pet dog or a pet cat in the U.S.}
P: { {the pet dogs}, {the pet cats}}

The dogs and cats are not members of the set P. They are members of the two sets which are members of this set.

Since sets can have other sets as members, as P does, they might even have themselves as members. Consider a possible set of all sets. Since it is itself a set, it should be a member of itself. It turns out that this possible set is too large actually to be a set, a fact which is a consequence of Russell's paradox.

Russell, presenting the paradox to Frege, inquired about the property of not including oneself. Consider the set of all and only sets which do not include themselves. If it includes itself, then it shouldn't include itself, since it is the set only of sets which do not include themselves. But if it doesn't include itself, then it should include itself since it is the set of all sets which do include themselves. The set of all sets which do not include themselves both includes itself and does not include itself, a contradiction. Naive set theory is inconsistent. Some properties cannot consistently determine sets.

To avoid Russell's paradox, set theory is ordinary presented axiomatically, rather than naively, these days. Instead of assuming that every property determines a set, we start with some simple sets, perhaps just the empty set, and rules for constructing sets, like the axioms of Zermelo-Fraenkel set theory, or ZF.

Zermelo-Fraenkel Set Theory[3]

Substitutivity: $(\forall x)(\forall y)(\forall z)[y=z \supset (y \in x \equiv z \in x)]$
Pairing: $(\forall x)(\forall y)(\exists z)(\forall w)[w \in z \equiv (w = x \vee w = y)]$
Null Set: $(\exists x)(\forall y)\sim y \in x$
Sum Set: $(\forall x)(\exists y)(\forall z)[z \in y \equiv (\exists w)(z \in w \bullet w \in x)]$
Power Set: $(\forall x)(\exists y)(\forall z)[z \in y \equiv (\forall w)(w \in z \supset w \in x)]$
Selection: $(\forall x)(\exists y)(\forall z)[z \in y \equiv (z \in x \bullet \mathscr{F}u)]$
for any formula $\mathscr{F}$ not containing y as a free variable
Infinity: $(\exists x)(a \in x \bullet (\forall y)(y \in x \supset Sy \in x)$

Ordinal numbers, set-theoretically, are just special kinds of sets, ones which are well ordered.

WO
A set A is well ordered by a relation <, if for all x, y, and z in A
1. $\sim x < x$
2. $(x < y \bullet y < z) \supset x < z$
3. Ether $x < y$, $x = y$, or $y < x$
4. Every nonempty subset of A has a smallest element.

[3] Many mathematicians adopt a further axiom, the axoim of Choice, yielding a theory known as ZFC. Choice says that given any set of sets, there is a set which contains precisely one member of each of the subsets of the original set.

For convenience, we standardly pick a particular ordinal to represent each particular number. We choose one example of a well-ordering for each number and use it as the definition of that number.

To move through the ordinals from smaller to larger, we most often look for the successor of a number, the set which stands for the next ordinal number. Ordinals which can be constructed in this way are called successor ordinals. In transfinite set theory, there are also sets which are called limit elements. We get to them not by finding a successor of a set, but by collecting all the sets we have counted so far into one further set. This operation of collecting several sets into one is called union. If we combine all the sets that correspond to the finite ordinals into a single set, we get another well-ordered set. This new set will be another ordinal: there will be a well-ordering on it, and it will have a minimal element. This limit ordinal will be larger than all of the ordinals in it.

So, there are two kinds of ordinals: successor ordinals and limit ordinals. Limit ordinals are the way in which we jump from considering successors to the next infinite ordinal number. It is like getting to the end of an infinite sequence and jumping to the next level of infinity.

Here is a list of ordinal numbers in order of their sizes.

ON $1, 2, 3, ... \omega$
$\omega+1, \omega+2, \omega+3...2\omega$
$2\omega+1, 2\omega+2, 2\omega+3...3\omega$
$4\omega, 5\omega, 6\omega...\omega^2$
$\omega^2, \omega^3, \omega^4...\omega^\omega$
$\omega^\omega, (\omega^\omega)^\omega, ((\omega^\omega)^\omega)^\omega,...\varepsilon^0$

The list ON is of ordinals, so by '1', I mean the first ordinal, rather than the cardinal '1'. ω is the first transfinite ordinal, corresponding to the set of natural numbers, the cardinal number $\aleph_0$. The limit ordinals are the ones found after the ellipses on each line, the completions of an infinite series.

Cantor's Theorem

I mentioned that Cantor's theorem is the set-theoretic analogue of the arithmetic claim that $2^a > a$. In set-theoretic terms, this claim is that $\mathbb{C}(\wp(A)) > \mathbb{C}(A)$. '$\mathbb{C}(A)$' refers to the cardinality of a set A; $\mathbb{C}$ is the measure of the size of a set. For finite sets, $\mathbb{C}(A)$ is just the number of elements of A. '$\wp(A)$' refers to the power set of a, the set of all subsets of a set a. PS shows two finite sets and their power sets.

PS $S_1=\{a, b\}$ $\wp(S_1)=\{\{a\}, \{b\}, \{a, b\}, \varnothing\}$
$S_2=\{2, 4, 6\}$ $\wp(S_2)=\{\{2\}, \{4\}, \{6\}, \{2, 4\}, \{2, 6\}, \{4, 6\}, \{2, 4, 6\}, \varnothing\}$

In general the power set of a set with n elements will have 2^n elements, which is why the number-theoretic claim that $2^n > a$ is the arithmetic correlate of the set-theoretic claim that $\mathbb{C}(\wp(A)) > \mathbb{C}(A)$. For infinite n, sets with n members are the same size as sets with n+1 members, or with 2n members, or with n^2 members. With infinite numbers, it is not always clear that what we think of as a larger set is in fact larger. We might believe that sets with n members are the same size as sets with 2^n members. This conclusion would be erroneous. $\mathbb{C}(\wp(A)) > \mathbb{C}(A)$.

The claim that $\mathbb{C}(\wp(A)) > \mathbb{C}(A)$ used to be called Cantor's paradox; it is now called Cantor's theorem. The proof of the theorem is a set-theoretic version of the diagonalization argument. Understanding it requires some familiarity with set theory. Most basically, a set is a collection of objects, a plurality considered as a unit. We can define sets either by listing their elements, or by stating a rule for inclusion in the set. A is defined in the first way; B is defined in the second way.

A = {Alvin, Simon, Theodore}
B = {x | x is one of the three most popular singing chipmunks}

An element, ∈, of a set is just one of its members. Here we see two true claims about elements of the sets defined above.

Alvin ∈ A
Theodore ∈ B

A subset S of a set A is a set which includes only members of A. If S omits at least one member of A, it is called a proper subset. The set C is a subset of A.

C = {Alvin, Simon}

We can express the subset relation of C to A as 'C ⊆ A'. C is also a proper subset of A, which means that it is a subset of A while not being identical to A, and which we can write as C ⊂ A.

To prove Cantor's theorem, we need two more set-theoretic definitions.

> A function is called *one-one* if it every element of the domain maps to a different element of the range: f(a)=f(b) ⇒ a=b
>
> A *function maps a set A onto another set B* if the range of the function is the entire set B, i.e. if no elements of B are left out of the mapping.

To prove Cantor's theorem, we want to show that the cardinality of the power set of a set is strictly larger than the cardinality of the set itself (i.e. ℂ(℘(A)) > ℂ(A)). It suffices to show that there is no function which maps A one-one and onto its power set.

Proof of Cantor's Theorem

Assume that there is a function f: A ⇒ ℘(A)
Consider the set B = {x | x ∈ A • x ∉ f(x)}
B is a subset of A, since it consists only of members of A.
So, B is an element of ℘(A), by definition of the power set.
That means that B itself is in the range of f.
Since, by assumption, f is one-one and onto, there must be an element of A, b, such that f(b) is B itself.
Is b ∈ B?

> If it is, then there is a contradiction, since B is defined only to include sets which are not members of their images.
> If it is not, then there is a contradiction, since B should include all elements which are not members of their images.

Either way, we have a contradiction.
So, our assumption fails.
There is no such function f: A ⇒ 𝒫(A).
𝒫(A) is strictly larger than A.
ℂ(℘(A)) > ℂ(A)).

QED

Summary

Cantor's theory of transfinites transformed the way we think of infinity. His diagonal argument shows that there are different levels of infinity. We form ordinals to represent the ranks of these different levels of infinity by taking certain series to completion. Completing an infinite series violates the restriction on actual infinity and syncategorematic infinities that blocked Zeno's paradoxes (and others). That such a completion is mathematically consistent and fecund entails that new responses to Zeno are necessary.

The mathematics of infinity have developed robustly in the last century and a half. While some philosophers and mathematicians initially resisted the surprising results, set theorists today work productively on higher transfinites, seeking proper and more full axiomatizations of set theory and even asking whether there are multiple set-theoretic universes. Axiomatic set theory is a vibrant area of contemporary research, at the intersection of logic, mathematics, and philosophy.