Hamilton College
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Class #28: The Limits of Reason
Kant's *Critique of Pure Reason*, Antinomies, On the Ontological Argument (AW 792-800 and 819-823)

I. First Antinomy: On the Infinitude of Space and Time

Kant presents four antinomies, or paradoxes, to supplant his claim that reason has limits, though Ariew and Watkins only present the first three, omitting the fourth on the existence of God.
While Kant believes that some proper metaphysics can be established using synthetic *a priori* reasoning, other topics (e.g. God, free will) are beyond our ken.
Our reason, wanting answers to such questions, speculates.
The problem with such speculation is that we can argue on either side of the debate.
We can establish that the universe is infinite.
We can also establish that it is finite.
Since contradictions (or antinomies) can not hold, Kant sees the existence of such contrary proofs as demonstrating that reason has exceeded its limits.
Again, Kant has the revolutionary Hume as an influence: we can commit such arguments to the flames.

The first antinomy concerns both the temporal and spatial finitude of the universe.
Kant argues that the universe has a beginning from the premise that an infinite series can not be completed.
If the universe existed from infinitely long ago, the present time would be, impossibly, the end of an infinite series.
So, there must have been some beginning.
For the spatial finitude of the universe, Kant claims that the concept of simultaneity presupposes a spatially finite universe.
If the universe were infinitely large, we could not think of all of the universe as existing simultaneously.

On the other side, Kant argues that the universe has no beginning in time from the logical impossibility of creation.
If there were a beginning point, there would have to be something before it.
But, that time would have nothing in it since the universe would not yet have been created.
So the universe would have no way to begin.

Kant's argument that the universe is spatially infinite assumes absolute space.
Imagine you were to go to the end of the universe.
Now, stick out your arm past the edge.
It seems that you could always perform this task.
Thus, the container must be infinite.
If we take the universe to be merely the contained portion, then we have no way to think about the container, the rest of space.
So space itself must be the infinite container.

> If one wants to leave out the void, and hence space as such, as an *a priori* condition for the possibility of appearances, then the entire world of sense drops out (A433/B461, AW 794b).

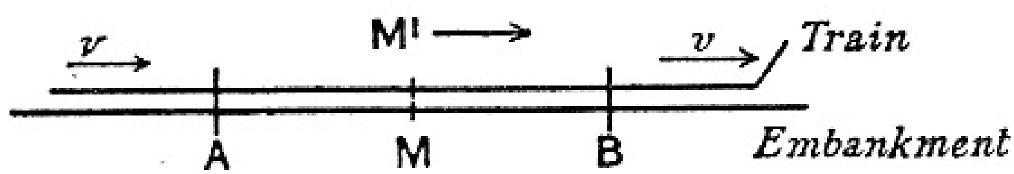
Remember, space is an *a priori* form of intuition, presupposed by all possible experience.

Kant has argued, *a priori*, to both sides of a contradiction.

He concludes that pure reason has exceeded its reach.
There is no knowledge to be had of whether the universe is finite or infinite.
Like the Humean empiricist, Kant concludes that we can not know any facts of the matter.

Notice that, in making this negative argument, Kant assumes that claims about whether the universe is finite or infinite are matters for *a priori* metaphysical reasoning.
In contrast, there are some mathematical and physical facts that undermine his claims.
Kant asserts that the universe must be spatially bound because otherwise we could have no definite concept of simultaneity.
But according to the theory of relativity, simultaneity and time itself are not definite concepts anyway.
They depend on the arbitrary choice of a frame of reference.
Imagine standing on a platform waiting for a train equidistant between two signal posts, A and B.
Imagine further that lights are flashed at A and B in a way that one perceives them as flashing simultaneously.
Now, consider a train traveling in the direction from A to B, passing you at the very moment that the lights are flashed.
To a perceiver on the train, the light at B will appear before the light at A, since the frame of reference (the train) is moving toward B, and away from A.

> Events which are simultaneous with reference to the embankment are not simultaneous with respect to the train, and vice versa (relativity of simultaneity). Every reference-body (co- ordinate system) has its own particular time; unless we are told the reference-body to which the statement of time refers, there is no meaning in a statement of the time of an event (Einstein, *Relativity: The Special and General Theory*, Chapter IX).



Similarly, Kant assumes an obsolete concept of infinity.

> The true (transcendental) concept of infinity is this: that the successive synthesis of unit[s] in measuring by means of a quantum can never be completed (A432/B460, AW 793a).

In the nineteenth century, Georg Cantor's work on transfinite numbers established that there are different sizes of infinity.
To count from one size of infinity to the next, we consider the smaller infinity as complete.
Today, we often define a set to be infinite if it can be put into one-one correspondence with a proper subset of itself.
For example, consider the integers and the even integers, which are a proper subset of the integers.
The integers are infinite just because we can match each one with an even integer.
See the appendix on infinity, below, or this longer, fun discussion of infinite arithmetic.

Kant's claim about the certainty of mathematics meet a similar objection.
Kant argued that we have *a priori* knowledge of Euclidean geometry arising from its role as a form of pure intuition assumed in all appearances.

But it turns out that there are various consistent non-Euclidean geometries in addition.
Euclidean geometry, instead of being the sole correct geometry, is just one of a family of mathematical theories, not mathematically favored.
Let's return to the antinomies.

II. The Second Antinomy: On Simples (Monads or Atoms)

The second antinomy concerns whether the world is made of simple objects or is infinitely divisible.
Descartes argues that the material world is, like a geometric object, infinitely divisible.
That claim is at the root of some of his conflict with the atomist Gassendi.
In contrast, both the materialist atomists and the idealist Leibniz argued that the world must be made of some sort of simple objects.
Kant takes the latter claim as the thesis of the second antinomy and the former as the antithesis.

Kant calls the thesis the dialectical principle of monadology.
He can basically adopt Leibniz's arguments for simples: without simples, we can have no composites.
He does point out that Leibniz's monads are not really elements of composites.
Monads are given directly, not as parts of wholes.
So Kant is refocusing the argument on complex material objects.
He puts the argument in the form of a *reductio*, assuming that there are no simples and arguing that there could consequently be no composites.
We must assume their existence to explain the composition of matter.

For the antithesis, Kant emphasizes the impossibility of experiencing a simple.
We have no conception of the noumenal world.
So we can not argue that simples exist there.
We can not experience simples, either.
We can't think about them and they are not objects of possible experience.
Since nature is the set of possible experiences, simples literally can not be part of nature.
The world of objects is a world of appearances given in the forms of intuition: space and time.

> Since, therefore, nothing can ever be given as an absolutely simple object in any possible experience, but since the world of sense must be regarded as the sum of all possible experiences, nothing simple is given in it at all (A437/B465, AW 795b).

Kant does agree that the monadists would be correct if our experiences were of the world of things in themselves.
There would have to be some determinate elements of the objects.
But our experiences are only mere appearances.
And our experiences are in the form of an infinitely divisible space.

III. The Third Antinomy: Freedom and Determinism

The thesis of the third antinomy is that there is Cartesian, libertarian free will.
Kant's argument for the thesis is that the contrary, strict determinism, is impossible.
We might think that every event has a cause.
But that would lead to an infinite regress and the need for some exception to the rule that every event has

a cause.

> If everything occurs according to mere laws of nature, then there is always only a subordinate but never a first beginning, and hence there then is on the side of the causes originating from one another no completeness of the series at all. The law of nature, however, consists precisely in this: that nothing occurs without a cause sufficiently determined *a priori*. Hence the proposition, in its unlimited universality, whereby any causality is possible only according to natural laws contradicts itself... (A444/B472, AW 798a)

On the thesis, every freely chosen act is the absolute beginning of a causal chain.
For the antithesis, strict determinism, Kant argues that freedom is not merely the absence of constraint but the chaotic lack of all rules.
A so-called free act would be utterly inexplicable and unthinkable.

> The coherence of appearances determining one another necessarily according to universal laws - which is called nature - would for the most part vanish, and along with it so would the mark of empirical truth which distinguishes experience from a dream (A451/B479, AW 800b)

A libertarian act would not be a possible experience.

The point of Kant's discussion of the antinomies is to demonstrate the bounds of reason.
Kant believes that all of the theses and antitheses are equally defensible.
There is thus no claim that we can establish about questions of the infinitude of space, whether there are simples, or whether we are free.
Such claims are beyond our ability to know.

I mentioned above that Ariew and Watkins omit the fourth antinomy on the existence of God.
We will look instead at Kant's rejection of the ontological argument, which appears later in the *Critique*.

IV. The Ontological Argument

If you ask philosophers what they believe is wrong with the ontological argument, they will most likely point to Kant's rejection of the argument
Kant's opposition to the argument has roots in the work of both Gassendi and Hume.
Hume's influence on Kant was profound.
His psychological reinterpretation of the concept of causation was a precedent for Kant's transcendental idealism.
Kant's claims about the limits of pure reason have Humean roots, too.
Kant's reason for rejecting the ontological argument is derived from Hume's claims about the nature of existence claims as well as from Gassendi's claim that existence is not a perfection.
In the Objections and Replies to Descartes's *Meditations*, Gassendi complains that the ontological argument is invalid because existence is not a perfection.
One can not conclude the existence of God from the claim that existence is a perfection because existence is not a property.
It is a precondition for having properties.
Descartes disagreed with Gassendi but the argument was left without a resolution.
Hume and Kant revive Gassendi's claim by adding a supporting argument.
Hume claims that the idea of existence, since it does not come from a distinct impression, adds nothing to

the idea of an object.

> Though certain sensations may at one time be united, we quickly find they admit of a separation, and may be presented apart. And thus, though every impression and idea we remember be considered as existent, the idea of existence is not derived from any particular impression. The idea of existence, then, is the very same with the idea of what we conceive to be existent. To reflect on any thing simply, and to reflect on it as existent, are nothing different from each other. That idea, when conjoined with the idea of any object, makes no addition to it. Whatever we conceive, we conceive to be existent. Any idea we please to form is the idea of a being; and the idea of a being is any idea we please to form (Hume, *A Treatise on Human Nature* §I.II.VI).

Kant, following Hume, claims that existence is not a property the way that the perfections are properties.
Existence can not be part of an essence because it is not a property.
Whether we think of a thing as existing or not, as necessarily existing or not, we are thinking of the same thing.

> A hundred real thalers do not contain the least coin more than a hundred possible thalers (A599/B627, AW 822a).

Kant distinguishes between real (or determining) predicates and logical predicates.
A logical predicate is just something that serves as a predicate in grammar.
In 'the Statue of Liberty exists', we are predicating (grammatically) existence of the statue.
But, we are not saying anything substantive about the statue.
In 'the Statue of Liberty is over 150 feet tall', we use a real predicate.
Any property can be predicated of any object, grammatically.
'Seventeen loves its mother' is a grammatical sentence, even if it is nonsensical.
'Loves one's mother' is a real predicate.
Kant's point is that one can not do metaphysics through grammar alone.
Existence is a grammatical predicate, but not a real predicate.

Kant's objection support's Gassendi's criticism of Descartes's version of the argument.
It also accounts for earlier objections from Gaunilo and Caterus.
Gaunilo, responding to Anselm's version of the ontological argument, wondered whether having the concept of the most perfect island entails its existence.
Caterus wondered if the concept of the necessarily existing lion entails the actual existence of a lion.
Kant says that in predicating existence of a concept, we are just restating the concept, and not saying anything about the object.
When we say that 'God exists', we are not making a real assertion.
We are just restating the concept of God.
Hume claimed that the contrary of any matter of fact is possible.
Kant agrees.

> If you admit - as any reasonable person must - that any existential proposition is synthetic, then how can you assert that the predicate of existence cannot be annulled without contradiction? For this superiority belongs only to analytic propositions as their peculiarity, since their character rests precisely on this [necessity] (A598/B626, AW 821b)

Part of Kant's support for his assertion that existence is not a predicate is that existence is too thin.
We do not add anything to a concept by claiming that it exists.

The real and possible thalers must have the same number of thalers in order that the concept attach to the object.
If there are more thalers in the real thalers, then the concept and the object would not match.
So, we do not add thalers when we mention that the thalers exist.

V. Evaluating Kant's Solution

Kant says that we don't add any thalers when we shift from discussing possible thalers to discussing actual thalers.
But, do we add something?
When my daughter and I discuss the existence of the tooth fairy, we are debating something substantive.
If we are going to debate the existence of something, whether it be the tooth fairy or black holes, we seem to consider an object and wonder whether it has the property of existing.
We thus have to consider objects which may or may not exist.
There may be many such objects, e.g. James Brown and Tony Soprano.
Some philosophers, like Meinong, attribute subsistence to dead folks and fictional objects.
One might say that James Brown has the property of subsisting without having the property of existing.
That is, Kant's claim that existence is not a real predicate, while influential, may not solve the problem.

In ordinary cases, Hume and Kant certainly are correct that logic, or reason, can not make existence claims.
The question is whether logic can make this one existence claim.
Kant's claim that existence is not a real predicate, while influential, may not solve the problem.

Many contemporary philosophers are swayed in Kant's direction by their familiarity with first-order logic's distinction between predication and quantification, and by the distinction between grammatical form and logical form.
In Fregean logic, properties like being a god, or a person, or being mortal or vain, get translated as predicates.
Existence is taken care of by quantifiers, rather than predicates.
To say that God exists, we say '$(\exists x)Gx$' or '$(\exists x)\ x=g$'
Note that the concept of God is represented independently of the claim of God's existence.
First-order logic is supposed to be our most austere, canonical language.
As Frege says, it puts a microscope to our linguistic usage.
Thus, there does seem to be a real difference between existence and predication and between the grammar of natural language and the true logical form of our claims.

Still, formal systems can be constructed with all sorts of properties.
We can turn any predicate into a quantifier, or a functor, even turn all of them into functors.
Is first-order logic the best framework for metaphysics?
Is Kant's linguistic solution to the ontological argument decisive?

These questions get discussed in courses on logic, philosophy of science, philosophy of language, and philosophy of mathematics.
I'll hope to see some of you in my language course in the fall.

Appendix to Kant Notes on Infinity

AI. The infinite hotel

Consider the infinite hotel: a hotel with infinitely many rooms.
The hotel is fully booked.
A new guest arrives.
We can add the new guest, by shifting every current guest from Room n to Room n+1.
Then, Room 1 will be available for the arriving guests.
We can perform the same procedure repeatedly, adding single guests.
We can generalize the procedure to add any finite number of guests, m, by shifting all current guests from Room n to Room n+m.

Next, an infinite bus with an infinite number of guests arrives.
We can still accommodate them, but we need a new procedure.
We can add the infinitely many new guests by shifting every current guest from Room n to Room 2n.
Now, all the even rooms are filled, but the odd rooms are vacant.
We can put the infinite number of new guests in the odd-numbered rooms.

Next, an infinite number of infinite busloads of guests arrives.
We can still accommodate them.
Shift all current guests from Room n to Room 2^n.
Now, all the rooms that are powers of two are filled, leaving lots of empty rooms.
We can place the people on the first bus in room numbers 3^n (for n people on the bus), the people in the second bus in rooms 5^n, the people in the third bus to rooms 7^n, and so on for each (prime number)^n.
Since there are an infinite number of primes, there will be an infinite number of infinite such sequences.
And, there will be lots of empty rooms!

AII. Cardinality, size, and correspondence

The splitting headache which may arise from thinking about infinite numbers may correspond to a split between two ways to think about cardinal numbers.
We use them to measure size.
But, we also use one-one correspondence to characterize cardinal numbers.
With finite numbers, these two approaches converge.
The size of a group is the same as the correspondence between the objects in the group and some initial segment of the natural numbers.
That is, if we have five hedgehogs, we can line them up and give them each a number from one to five.

With transfinite numbers, as with the infinite hotel, the two concepts diverge.
The size of the integers seems to be bigger than the size of the even numbers.
But, they can be put into one-one correspondence with each other.

Georg Cantor, in the mid-nineteenth century, relied on the one-one correspondence notion to generate different kinds of infinite, or transfinite, numbers.
When we list the members of something, we are putting them into one-one correspondence with the natural numbers.
Cardinal numbers are the sizes of sets, the number we count to when we put the set in one-one

correspondence with the natural numbers.
But, it turns out that we can not make certain lists.

For example, we can not list the real numbers.
The real numbers may be represented as their decimal expansions: non-repeating, non-terminating.
Imagine that we have such a list.
Let's represent that list abstractly:

$a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7$...
$b_1\ b_2\ b_3\ b_4\ b_5\ b_6\ b_7$...
$c_1\ c_2\ c_3\ c_4\ c_5\ c_6\ c_7$...
$d_1\ d_2\ d_3\ d_4\ d_5\ d_6\ d_7$...
...

By hypothesis, the list contains all real numbers.
But, we can, for any list, demonstrate a number which does not appear on the list.
Consider the following number

$a_1\ b_2\ c_3\ d_4\ e_5\ f_6\ g_7$...

That number could be on the list.
Now, take each digit in that number and change it: add one to each number other than nine, and replace all nines with zeroes.
The following number is certainly not on the list.
For, it is different from the first number on the list in the first digit, different from the second number on the list in the second digit, and so on, for all numbers on the list.
If we add this new number to the list, we can form a new number that is not on the resulting list, by the same process.
Thus all possible lists of real numbers are necessarily incomplete.
This proof is called Cantor's diagonalization argument.
It shows that the ordinary concept of size is not precisely the same as the concept of one-one correspondence.
Mathematicians tend to think of size now as one-one correspondence.
But their use of 'size' differs from that of ordinary people once we get to transfinites.

Numbers actually have two different functions.
Cardinal numbers measure size.
Ordinal numbers measure rank.
Let's look a bit at both, starting with the cardinals.

AIII. Cardinal arithmetic

Cardinal numbers are sets which we use to measure the sizes of sets, by one-one correspondence.
We are all familiar with many properties of cardinal numbers.
For all cardinal numbers a, b, and c, whether finite or transfinite, the following relations hold:

1. $a+b=b+a$
2. $ab=ba$

3. $a + (b + c) = (a + b) + c$
4. $a \bullet (b \bullet c) = (a \bullet b) \bullet c$
5. $a \bullet (b + c) = ab + ac$
6. $a^{(b+c)} = a^b \bullet a^c$
7. $(ab)^c = a^c \bullet b^c$
8. $(a^b)^c = a^{bc}$

But some properties of finite cardinal numbers do not hold for transfinite numbers.
Notice that a+1=a, when a is transfinite.
And 2a=a holds as well.
Even a•a=a
We can show these all by considering a bijective mapping from one set to the other.
We showed all of these facts in the infinite hotel.

Consider one final important property which holds both of finite and transfinite numbers.

9. $2^a > a$

In set-theoretic terms, this ninth claim is that $\wp(a) > a$.
'$\wp(a)$' refers to the power set of a, the set of all subsets of a set a.
Consider a set A = {2, 4, 6}
Then $\wp(A) = \{\{2\}, \{4\}, \{6\}, \{2, 4\}, \{2, 6\}, \{4, 6\}, \{2, 4, 6\}, \varnothing\}$
In general the power set of a set with n elements will have 2^n elements.
Since sets with n members are the same size as sets with n+1 members, or with 2n members, or with n^2 members, for infinite n, we might think that sets with n members are the same size as sets with 2^n members.
For, with infinite numbers, it is not always clear that what we think of as a larger set is in fact larger.
But that turns out not to so.
The power set of a set is always strictly larger than the set and two raised to any number is larger than that number.

The claim that $\wp(a) > a$ has been called Cantor's paradox.
$\wp(a)>a$ is now taken to be Cantor's theorem.
The proof of the theorem is a set-theoretic version of the diagonalization argument.
We want to show that the cardinal number C of the power set of a set is strictly larger than the cardinal number of the set itself (i.e. $C(\wp(A)) > C(A)$).
To show that fact, it suffices to show that there is no function which maps A one-one and onto its power set.
A function is called one-one if it every element of the domain maps to a different element of the range.
A function maps a set A onto another set B if the range of the function is the entire set B, i.e. if no elements of B are left out of the mapping.

Proof of Cantor's Theorem

Assume that there is a function $f: A \Rightarrow \wp(A)$
Consider the set $B = \{x \mid x \in A \bullet x \notin f(x)\}$
B is a subset of A, since it consists only of members of A.
So, B is an element of $\wp(A)$, by definition of the power set.
That means that B itself is in the range of f.

Since, by assumption, f is one-one and onto, there must be an element of A, b, such that f(b) is B itself.
Is $b \in B$?
If it is, then there is a contradiction, since B is defined only to include sets which are not members of their images.
If it is not, then there is a contradiction, since B should include all elements which are not members of their images.
Either way, we have a contradiction.
So, our assumption fails, and there must be no such function.
$\wp(A) > A$

QED

Let's call the size of the natural numbers $\aleph_0$.
Then the real numbers, and the real plane, are the size of the power set of the natural numbers, $2^{\aleph_0}$.
We can proceed to generate larger and larger cardinals through the power set process.
Moreover, set theorists, by various ingenious methods, including addition of axioms which do not contradict the given axioms, generate even larger cardinals.

AIV. Ordinal numbers

Let's start counting.
By adding one, here, we normally mean taking the successor of 1.
So, $\omega+1$ will be the successor of ω.
Ordinals generated in this way are called successor ordinals.
In transfinite set theory, there are also sets which are called limit elements.
We get them by taking the union of all the members of a set.
Ordinal numbers, set-theoretically, are just special kinds of sets, well-ordered sets.
A set is well-ordered if, basically, we can find an ordering relation on the set, and it has a first element.
If we consider all the sets that correspond to the finite ordinals, and combine them into a whole, we can get another well-ordered set.
This will be a new ordinal, and it will be larger than all of the ordinals in it.
So, there are two kinds of ordinals: successor ordinals and limit ordinals.

$1, 2, 3, \ldots \omega$
$\omega+1, \omega+2, \omega+3 \ldots 2\omega$
$2\omega+1, 2\omega+2, 2\omega+3 \ldots 3\omega$
$4\omega, 5\omega, 6\omega \ldots \omega^2$
$\omega^2, \omega^3, \omega^4 \ldots \omega^\omega$
$\omega^\omega, (\omega^\omega)^\omega, ((\omega^\omega)^\omega)^\omega, \ldots \varepsilon^0$

The limit ordinals are the ones found after the ellipses on each line.
Large ordinals correspond to the large cardinals.

Notice that limit ordinals are taken as the completions of an infinite series.
Kant, in the antinomies, denied that there can be any completion of an infinite series.
But, Cantor's diagonal argument shows that there are different levels of infinity.
And, we form ordinals to represent the ranks of these different levels of infinity precisely by taking certain series to completion.