Philosophy 203: History of Modern Western Philosophy Hamilton College
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Appendix to Kant Notes on Infinity

I. The infinite hotel

Consider the infinite hotel: a hotel with infinitely many rooms.
The hotel is fully booked.
A new guest arrives.
We can add the new guest, by shifting every current guest from Room n to Room n+1.
Then, Room 1 will be available for the arriving guests.
We can perform the same procedure repeatedly, adding single guests.
We can generalize the procedure to add any finite number of guests, m, by shifting all current guests from Room n to Room n+m.

Next, an infinite bus with an infinite number of guests arrives.
We can still accommodate them, but we need a new procedure.
We can add the infinitely many new guests by shifting every current guest from Room n to Room 2n.
Now, all the even rooms are filled, but the odd rooms are vacant.
We can put the infinite number of new guests in the odd-numbered rooms.

Next, an infinite number of infinite busloads of guests arrives.
We can still accommodate them.
Shift all current guests from Room n to Room 2^n.
Now, all the rooms that are powers of two are filled, leaving lots of empty rooms.
We can place the people on the first bus in room numbers 3^n (for n people on the bus), the people in the second bus in rooms 5^n, the people in the third bus to rooms 7^n, and so on for each (prime number)^n.
Since there are an infinite number of prime numbers, there will be an infinite number of infinite such sequences.
And, there will be lots of empty rooms!

II. Cardinality, size, and correspondence

The splitting headache which may arise from thinking about infinite numbers may correspond to a split between two ways to think about cardinal numbers.
We use them to measure size.
But, we also use one-one correspondence to characterize cardinal numbers.
With finite numbers, these two approaches converge.
The size of a group is the same as the correspondence between the objects in the group and some initial segment of the natural numbers.
That is, if we have five hedgehogs, we can line them up and give them each a number from one to five.

With transfinite numbers, as with the infinite hotel, the two concepts diverge.
The size of the integers seems to be bigger than the size of the even numbers.
But, they can be put into one-one correspondence with each other.

Georg Cantor, in the mid-nineteenth century, relied on the one-one correspondence notion to generate different kinds of infinite, or transfinite, numbers.
When we list the members of something, we are putting them into one-one correspondence with the natural numbers.
Cardinal numbers are the sizes of sets, the number we count to when we put the set in one-one correspondence with the natural numbers.
But, it turns out that we can not make certain lists.

For example, we can not list the real numbers.
The real numbers may be represented as their decimal expansions: non-repeating, non-terminating.
Imagine that we have such a list.
Let's represent that list abstractly:

$a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7...$
$b_1\ b_2\ b_3\ b_4\ b_5\ b_6\ b_7...$
$c_1\ c_2\ c_3\ c_4\ c_5\ c_6\ c_7...$
$d_1\ d_2\ d_3\ d_4\ d_5\ d_6\ d_7...$
...

By hypothesis, the list contains all real numbers.
But, we can, for any list, demonstrate a number which does not appear on the list.
Consider the following number

$a_1\ b_2\ c_3\ d_4\ e_5\ f_6\ g_7...$

That number could be on the list.
Now, take each digit in that number and change it: add one to each number other than nine, and replace all nines with zeroes.
The following number is certainly not on the list.
For, it is different from the first number on the list in the first digit, different from the second number on the list in the second digit, and so on, for all numbers on the list.
If we add this new number to the list, we can form a new number that's not on the resulting list, by the same process.
Thus, all possible lists of real numbers are necessarily incomplete.
This proof is called Cantor's diagonalization argument.
It shows that the ordinary concept of size is not precisely the same as the concept of one-one correspondence.
Mathematicians tend to think of size now as one-one correspondence.
But, their use of 'size' differs from that of ordinary people, once we get to transfinites.

Numbers actually have two different functions.
Cardinal numbers measure size.
Ordinal numbers measure rank.
Let's look a bit at both, starting with the cardinals.

III. Cardinal arithmetic

Cardinal numbers are sets which we use to measure the sizes of sets, by one-one correspondence.
We are all familiar with many properties of cardinal numbers.
For all cardinal numbers a, b, and c, whether finite or transfinite, the following relations hold:

1. $a+b=b+a$
2. $ab=ba$
3. $a + (b + c) = (a + b) + c$
4. $a \bullet (b \bullet c) = (a \bullet b) \bullet c$
5. $a \bullet (b + c) = ab + ac$
6. $a^{(b+c)} = a^b \bullet a^c$
7. $(ab)^c = a^c \bullet b^c$
8. $(a^b)^c = a^{bc}$

But some properties of finite cardinal numbers do not hold for transfinite numbers.
Notice that $a+1=a$, when a is transfinite.
And $2a=a$ holds as well.
Even $a \bullet a=a$
We can show these all by considering a bijective mapping from one set to the other.
We showed all of these facts in the infinite hotel.

Consider one final important property which holds both of finite and transfinite numbers.

9. $2^a > a$

In set-theoretic terms, this ninth claim is that $\wp(a) > a$.
'$\wp(a)$' refers to the power set of a, the set of all subsets of a set a.
Consider a set A = {2, 4, 6}
Then $\wp(A)$ = {{2}, {4}, {6}, {2, 4}, {2, 6}, {4, 6}, {2, 4, 6}, ∅}
In general the power set of a set with n elements will have 2^n elements.
Since sets with n members are the same size as sets with n+1 members, or with 2n members, or with n^2 members, for infinite n, we might think that sets with n members are the same size as sets with 2^n members.
For, with infinite numbers, it is not always clear that what we think of as a larger set is in fact larger.

The claim that $\wp(a) > a$ has been called Cantor's paradox.
$\wp(a)>a$ is now taken to be Cantor's theorem.
The proof of the theorem is a set-theoretic version of the diagonalization argument.
We want to show that the cardinal number C of the power set of a set is strictly larger than the cardinal number of the set itself (i.e. $C(\wp(A)) > C(A)$).
To show that fact, it suffices to show that there is no function which maps A one-one and onto its power set.
A function is called one-one if it every element of the domain maps to a different element of the range.
A function maps a set A onto another set B if the range of the function is the entire set B, i.e. if no elements of B are left out of the mapping.

Proof of Cantor's Theorem

> Assume that there is a function f: A $\Rightarrow \wp(A)$
> Consider the set B = $\{x \mid x \in A \bullet x \notin f(x)\}$
> B is a subset of A, since it consists only of members of A.
> So, B is an element of $\wp(A)$, by definition of the power set.
> That means that B itself is in the range of f.
> Since, by assumption, f is one-one and onto, there must be an element of A, b, such that f(b) is B itself.
> Is $b \in B$?
> If it is, then there is a contradiction, since B is defined only to include sets which are not members of their images.
> If it is not, then there is a contradiction, since B should include all elements which are not members of their images.
> Either way, we have a contradiction.
> So, our assumption fails, and there must be no such function.
> $\wp(A) > A$

QED

Let's call the size of the natural numbers $\aleph_0$.
Then the real numbers, and the real plane, are the size of the power set of the natural numbers, $2^{\aleph_0}$.
We can proceed to generate larger and larger cardinals through the power set process.
Moreover, set theorists, by various ingenious methods, including addition of axioms which do not contradict the given axioms, generate even larger cardinals.

IV. Ordinal numbers

Let's start counting.
By adding one, here, we normally mean taking the successor of 1.
So, ω+1 will be the successor of ω.
Ordinals generated in this way are called successor ordinals.
In transfinite set theory, there are also sets which are called limit elements.
We get them by taking the union of all the members of a set.
Ordinal numbers, set-theoretically, are just special kinds of sets, well-ordered sets.
A set is well-ordered if, basically, we can find an ordering relation on the set, and it has a first element.
If we consider all the sets that correspond to the finite ordinals, and combine them into a whole, we can get another well-ordered set.
This will be a new ordinal, and it will be larger than all of the ordinals in it.
So, there are two kinds of ordinals: successor ordinals and limit ordinals.

> $1, 2, 3, \ldots \omega$
> $\omega+1, \omega+2, \omega+3 \ldots 2\omega$
> $2\omega+1, 2\omega+2, 2\omega+3 \ldots 3\omega$
> $4\omega, 5\omega, 6\omega \ldots \omega^2$
> $\omega^2, \omega^3, \omega^4 \ldots \omega^\omega$
> $\omega^\omega, (\omega^\omega)^\omega, ((\omega^\omega)^\omega)^\omega, \ldots \varepsilon^0$

The limit ordinals are the ones found after the ellipses on each line.
Large ordinals correspond to the large cardinals.

Notice that limit ordinals are taken as the completions of an infinite series.
Kant, in the antinomies, denied that there can be any completion of an infinite series.
But, Cantor's diagonal argument shows that there are different levels of infinity.
And, we form ordinals to represent the ranks of these different levels of infinity precisely by taking certain series to completion.