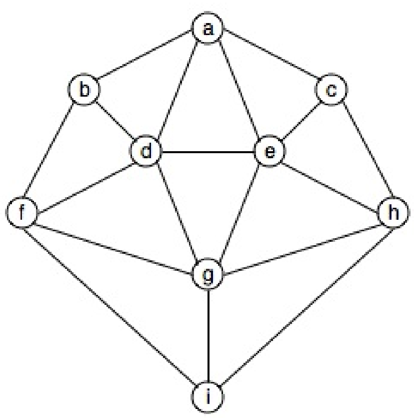
Philosophy 405: Knowledge, Truth and Mathematics — Hamilton College
Fall 2010 — Russell Marcus

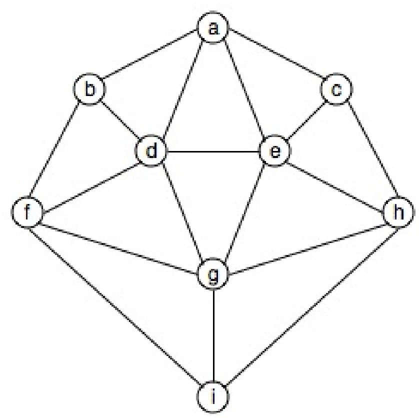
Constructive and Non-Constructive Proofs

A Constructive Proof:

Definition: A coloring of a graph is an assignment of a color to each node of the graph.
Definition: A graph is 3-colorable if any coloring which uses only three colors does not assign the same color to any two nodes which share a branch.
Definition: A graph is 4-colorable if any coloring which uses only four colors does not assign the same color to any two nodes which share a branch.

Theorem: There are graphs which are 4-colorable but which are not 3-colorable.
Proof: In two stages. Present a graph which is not 3-colorable but which is 4-colorable. (See below.
Stage 1: Prove that the graph is not 3-colorable. Stage 2: Show that the graph is 4-colorable.





A Non-Constructive Proof

Claim: There exist irrational numbers x and y such that x^y is rational.

Proof:

Let $z = \sqrt{2}^{\sqrt{2}}$.
Either z is rational or z is irrational, though we do not know which.
If z is rational then z is our desired number with $x = y = \sqrt{2}$.
If z is irrational, then let $x = z$ and $y = \sqrt{2}$.

$$x^y = {\sqrt{2}^{\sqrt{2}}}^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^{2} = 2.$$

On these different assignments of irrational values to x and y, x^y is again rational.
Whether z is rational or irrational, there exist irrational numbers x and y such that x^y is rational.

QED