Logicism

JASON DRISCOLL

Introduction

- Frege was a mathematician, logician, philosopher
  - made important contributions to logic, philosophy of math, and philosophy of language
- Main goal for today's purpose: To create a system of logic with which one could reduce all of mathematics

Leibniz

- Leibniz believed in the use of a universal language through which all human reasoning could be described and mirrored
  - "that all human ideas can be resolved into a few as their primitives" *(On the Universal Science: Characteristic; G VII, 205 (S, 18))*
  - "this language will be the greatest instrument of reason," "when there are disputes among persons, we can simply say: Let us calculate, without further ado, and see who is right" *(The Art of Discovery (1685); C, 176 (W, 51))*
    - It is through this call for a universal language that Frege creates his system of term and predicate logic

Logicism

- For Frege, the synthetic a priori status of mathematical truths given by Kant seemed inadequate. He wanted to search for a more independent, objective justification for mathematics and used his logic to do so.
- "In my *Grundlagen der Arithmetik,* I sought to make it plausible that arithmetic is a branch of logic and need not borrow any ground of proof whatever from either experience or intuition. In the present book, this shall be confirmed, by the derivation of the simplest laws of Numbers by logical means alone." - *Grundgesetze der Arithmetik* §0
- Two notions: 1. That mathematical concepts can be defined in terms of logical concepts; 2. That the principles of mathematics can be demonstrated from logic alone

Context

- For Kant, intuition provided the basis for mathematical objects.
- For Locke, conceptualism allowed for mathematical objects to exist as abstract psychological ideas.
- For Mill, mathematical objects were just imperfect inductions from sense experience.

Discovery versus Justification

- “It can be asked, by what path a proposition was gradually reached, and on the other hand, in what way it is not finally to be most firmly established. The former question possible needs to be answered differently for different people; the latter is more definite, and its answer is connected with the inner nature of the proposition concerned.” -Preface to *Begriffsschrift, III*
- “The firmest proof is obviously the purely logical, which, prescinding from the particularity of things, is based solely on the laws on which all knowledge rests. Accordingly, we divide all truths that require justification into two kinds, those whose proof can be given purely logically and those whose proof must be grounded on empirical fact” -Preface to *Begriffsschrift, III*
- For Frege, the discovery versus justification distinction is paramount, not only it its inspiration for a universal language of logic, but also in its explanation of what mathematical objects are and how they are presented.

Three principles

- 1. Always separate the psychological from the logical, the subjective from the objective; 2. Never ask for the meaning of a word in isolation, but only in the context of a proposition; 3. Never lose sight of the distinction between concept and object (*The Foundations of Arithmetic*, X).
- Frege uses these principles to critique to reasoning of Kant, Locke, and Mill

Mathematical Objects

- Frege holds that mathematical objects are objective and independent of us:
- "Number is no whit more an object of psychology than, let us say the North Sea is…If we say, 'the North Sea is 10,000 square miles in extent' then neither by 'North Sea' nor by '10,000' do we refer to any state of or process in our minds: on the contrary, we assert something quite objective, which is independent of our ideas and everything of the sort" (*Grundlagen* §26)

Non-spatio-temporal objects

- Given than mathematical objects exist independently from us but also objectively, Frege does not say that they are concrete
- “Not every objective object has a place” *Grundlagen* §61
- While mathematics exist objectively due to its independence from us, it also can exist objectively if it can be reduced to logic.

Predicate Logic

- Formal language and a method of proof for representing inferences among predications
- Allows you to map the relationship between objects and properties
- If we can create a system that adequately mirrors thought, then we can analyze philosophical arguments objectively

Predicate Logic

- Names of object: “2” or “Mary”
- Complex terms: “2^2”
- Sentences: “Mary loves John”
- Functions: $(\ )^2$, love
- Binary truth-value: True or False
- With a system of logic, expressions can be converted into logical form, allowing us to analyze objects and their relation to predicates objectively

Predicate Logic

- Mary is happy = H_m
- Mary loves John = L_{mj}
- Ever person is mortal = $\forall x(Px \rightarrow Mx)$
- Some person is mortal = $\exists x(Px \,\&\, Mx)$
- Once we add rules for how to manipulate these name and complex terms, we can utilize proofs to mirror thought objectively

Method

- Frege must demonstrate that we can reduce the theory of natural numbers to pure logic, and that natural numbers theory is the basis for all of mathematics.
- Frege begins his work by asking about the nature of numbers and how they are represented. The first step is to redefine natural numbers using truths in logic in order to analytically derive basic laws of number theory and the foundations of arithmetic generally.

The Nature of Numbers

- Recall that Mill held that numbers were numbers of something, properties of physical objects
- Frege reconceptualizes numbers as applying to concepts, not objects
- “While looking at one and the same external phenomenon, I can say with equal truth both ‘it is a copse’ and ‘it is five trees,’ or both ‘here are four companies’ and ‘here are 500 men.’ Now what changes here from one judgements to the other is neither any individual object, nor the whole, the agglomeration of them, but rather my terminology. But that is itself only a sign that one concept has been substituted for another”
- Instead of saying “There are 500 men,” Frege would like to say “the number of men is 500”
- Notice how numbers, in this manner, apply to the concepts and not the obects

Defining Numbers

- Recall that Frege's first step is to redefine natural numbers using logic
- He begins by assigning definitions for numbers that designate them as objects having specific identity conditions
- Numbers are extensions of the extensions of concepts: "The number which belongs to the concept *F* is the extension of the concept 'equal to the concept *F*'" (*Grundlagen* §68).
- On one-one correspondence: "We have to define the sense of the proposition 'the number which belongs to he concept *F* is the same as that which belongs to the concept *G*'…In doing this, we shall be giving a general criterion for the identity of numbers. When we have thus acquired a means of arriving at a determinate number and of recognizing it again as the same, we can assign it a number word as its proper name. Hume long ago mentioned such a means: 'when two numbers are so combined as that the one has always an unit answering to every unit of the other, we pronounce them equal" *Grundlagen* §62-3

Defining Numbers

- For instance, the number 0 is defined as the number that applies to the extension of the concept “not identical with itself.” (§74).
- Zero belongs to a concept is nothing falls under that concept
- 1 belongs to the concept of all 1-membered sets
- 2 belongs to the concept of all 2-membered sets

Successor Definition

- "There exists a concept of *F*, and an object falling under it *x* such that the number which belongs to the concept *F* is *n* and the number which belongs to the concept 'falling under *F* but not identical with *x*' is *m*" is to mean the same as '*n* follows in a series of natural numbers directly after *m*" (§76).

Russel's Paradox

- Russel's paradox is a paradox resulting from one of the axioms that Frege employs. This is Basic Law V:

$$\{x|Fx\} = \{x|Gx\} \equiv (\forall x)(Fx \equiv Gx)$$

- This states that the extensions of two concepts are equal if and only if the same objects fall under the two concepts
- This leads to Proposition 91: A predicate F holds of a term if and only if the object to which the term refers is an element of the set of Fs.

The Paradox

- The paradox: What if we assume the predicate F is "is not an element itself." Or as Russel states in his letter: "to be a predicate that cannot be predicated of itself."
- If a predicate holds of a term if and only if it refers to an element in its set (prop 91), then Russel's predicate causes a contradiction
- The predicate "to be a predicate that cannot be predicated of itself" leads to a paradox

Conclusions

- While Russel will eventually clean up some of the problems caused by the paradox, Frege's logicist project failed
- Math's analyticity remains as a topic of debate
- Frege still posits relevant arguments against Kant, Mill, and Locke
  - reemphasizes the importance of discovery versus justification and Frege's three principles
  - the tidy use of proof and definition