The Transfinites

We've spent much of our time in class talking about what mathematical objects and
knowledge are: existent objects in the world, truths inherent in our minds, relations between our abstracted ideas, or vacuous or even plainly false statements which are just “good enough” that they work.

We haven't spent as much time on what the content of those mathematical objects are. And perhaps rightly so- this isn't a math class.

However, in order to deal with some of the more complex epistemological problems in mathematics, we may have to explore more deeply the content of mathematics.

This brings us to infinity.

- Ἄπειρον = unlimited, indefinite, (infinite?)

- “The belief that there is something apeiron stems from the idea that only then genesis and decay will never stop, when that from which is taken what is generated is apeiron.” - *Physics* Aristotle

- Zeno's Achilles and the Hare, Dichotomy Paradoxes

Anaxagoras

ἕτερον δὲ οὐδέν ἐστιν ὅμοιον οὐδένι, ἀλλ' ὅτῳ πλεῖστα ἔνι, ταῦτα ἐνδηλότητα ἓν ἕκαστόν ἐστι καὶ ἦν.

7. καὶ ἐπεὶ ἤρξατο ὁ νοῦς κινεῖν, ἀπὸ τοῦ κινουμένου παντὸς ἀπεκρίνετο, καὶ ὅσον ἐκίνησεν ὁ νοῦς, πᾶν τοῦτο διεκρίθη. κινουμένων δὲ καὶ διακρινομένων ἡ περιχώρησις πολλῷ μᾶλλον ἐποίει διακρίνεσθαι.

8. τὸ μὲν πυκνὸν καὶ διερὸν καὶ ψυχρὸν καὶ τὸ ζοφερὸν ἐνθάδε συνεχώρησεν ἔνθα νῦν <ἡ γῆ>· τὸ δὲ ἀραιὸν καὶ τὸ θερμὸν καὶ τὸ ξηρὸν <καὶ τὸ λαμπρὸν> ἐξεχώρησεν εἰς τὸ πρόσω τοῦ αἰθέρος.

9. ἀπὸ τουτέων ἀποκρινομένων συμπήγνυται γῆ· ἐκ μὲν γὰρ τῶν νεφελῶν ὕδωρ ἀποκρίνεται, ἐκ δὲ τοῦ ὕδατος γῆ, ἐκ δὲ τῆς γῆς λίθοι συμπήγνυνται ὑπὸ τοῦ ψυχροῦ, οὗτοι δὲ ἐκχωρέουσι μᾶλλον τοῦ ὕδατος.

12. ὁ δὲ νοῦς, ὡς ἀεί ποτε, κάρτα καὶ νῦν ἐστιν, ἵνα καὶ τὰ ἄλλα πάντα, ἐν τῷ πολλῷ περιέχοντι καὶ ἐν τοῖς ἀποκριθεῖσι καὶ ἐν τοῖς ἀποκρινομένοις.

13. οὐ κεχώρισται ἀλλήλων τὰ ἐν τῷ ἑνὶ κόσμῳ οὐδὲ ἀποκέκοπται πελέκει οὔτε τὸ θερμὸν ἀπὸ τοῦ ψυχροῦ οὔτε τὸ ψυχρὸν ἀπὸ τοῦ θερμοῦ.

15. οὔτε γὰρ τοῦ σμικροῦ ἐστι τό γε ἐλάχιστον, ἀλλ' ἔλασσον ἀεί. τὸ γὰρ ἐὸν οὐκ ἔστι τὸ μὴ οὐκ εἶναι.

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7. Simpl. *Phys.* 66 r; 300, 31. 33. *DE* καὶ, a*F* omit.
8. Simpl. *Phys.* 38 r; 179, 3. Cf. *Dox.* 562, 3.
4. 179, 4 Diels would supply τὸ before διερὸν and ψυχρὸν. 5. From *Dox.* 562 add ἡ γῆ . . . τὸ λαμπρὸν.

9. Simpl. *Phys.* 38 r 179, 8. In part 33 r 155, 21. Cf. 106 v 460, 13–14. 155, 22. λίθοι συμπήγνυνται.

12. Simpl. *Phys.* 33 r 157, 7. Simpl. ὅσα ἐστί τε, corr. Diels: πολλὰ περιέχοντι, corr. Diels; cf. p. 155, 31: προσκριθεῖσι . . . ἀποκρινομένοις, corr. Diels; cf. 156, 28.

13. Simpl. *Phys.* 37 r 175, 12 beginning with οὐδέ. To πελέκει, 38 v 176, 29.

15. Simpl. *Phys.* 35 v 164, 17. Cf. 35 r 166, 15.
164, 17. MS. τὸ μή, Zeller, *Phil. Gr.* i.[4], 884 n. 3 τομῇ. After

ἀλλὰ καὶ τοῦ μεγάλου ἀεί ἐστι μεῖζον. καὶ ἴσον ἐστὶ τῷ σμικρῷ πλῆθος, πρὸς ἑαυτὸ δὲ ἕκαστόν ἐστι καὶ μέγα καὶ σμικρόν.

16. καὶ ὅτε δὲ ἴσαι μοῖραί εἰσι τοῦ τε μεγάλου καὶ τοῦ σμικροῦ πλῆθος, καὶ οὕτως ἂν εἴη ἐν παντὶ πάντα. οὐδὲ χωρὶς ἔστιν εἶναι, ἀλλὰ πάντα παντὸς μοῖραν μετέχει. ὅτε τοὐλάχιστον μὴ ἔστιν εἶναι, οὐκ ἂν δύναιτο χωρισθῆναι, οὐδ' ἂν ἐφ' ἑαυτοῦ γενέσθαι· ἀλλ' ὅπωσπερ ἀρχὴν εἶναι καὶ νῦν, πάντα ὁμοῦ. ἐν πᾶσι δὲ πολλὰ ἔνεστι, καὶ τῶν ἀποκρινομένων ἴσα πλῆθος ἐν τοῖς μείζοσί τε καὶ ἐλάσσοσι.

17. τὸ δὲ γίνεσθαι καὶ ἀπόλλυσθαι οὐκ ὀρθῶς νομίζουσιν οἱ Ἕλληνες· οὐδὲν γὰρ χρῆμα γίνεται οὐδὲ ἀπόλλυται, ἀλλ' ἀπὸ ἐόντων χρημάτων συμμίσγεταί τε καὶ διακρίνεται. καὶ οὕτως ἂν ὀρθῶς καλοῖεν τό τε γίνεσθαι συμμίσγεσθαι καὶ τὸ ἀπόλλυσθαι διακρίνεσθαι.

(18.) πῶς γὰρ ἂν ἐκ μὴ τριχὸς γίνοιτο θρὶξ καὶ σὰρξ ἐκ μὴ σαρκός;

εἶναι Schorn inserts οὔτε τὸ μέγιστον, comparing previous line and 166, 16.

16. Simpl. *Phys.* 35 v 164, 24.
17. Simpl. *Phys.* 34 v 163, 20.
18. Schol. in Gregor. Naz. Migne 36, 911. (Cf. *Hermes* xiii. 4, Diels.)

- 15. “For neither is there a least of what is small, but there is always a less. For being is not non-being.” - Anaxagoras (trans. Arthur Fairbanks, *The First Philosophers of Greece*)

- 16. “**And since the portions of the great and the small are equal in number**, thus also all things would be in everything ... And there are many things in all things, and of those that are separated there are things equal in number in the greater and the lesser.” - Anaxagoras (trans. Arthur Fairbanks, *The First Philosophers of Greece*)

- But suddenly, Aristotle.

Aristotle

- “It is plain, too, that the infinite cannot be an actual thing and a substance and principle. For any part of it that is taken will be infinite, if it has parts: for 'to be infinite' and 'the infinite' are the same, if it is a substance and not predicated of a subject. Hence it will be either indivisible or divisible into infinites. But the same thing cannot be many infinites. (Yet just as part of air is air, so a part of the infinite would be infinite, if it is supposed to be a substance and principle.) Therefore the infinite must be without parts and indivisible.” - *Physics,* Aristotle (trans. R. P. Hardie and R. K. Gaye)

- Forms a new distinction, potential and actual, to which infinity can be applied:

Aristotle

- Potential: In general, the possibility whereby an object can be made actual
  - The numbers may be actualized as we count them, but infinity, or apeiron, is never actualized. We never reach infinity.
- Actual: The fulfillment of a potential, the existence of object in question brought into reality
  - And so, is infinity a potential which cannot be actualized? What defines it as a potential, then?

Aristotle 'n Friends

- The potential/actual distinction remains the primary way in which to think about infinity throughout the Middle Ages up until the departure from Scholasticism and the development of infinitesimal calculus and, in some ways, even after- Leibniz himself does not speak a great deal on the properties of the infinitesimals he uses.

- Locke, Hume, Berkeley (especially Berkeley), Kant, and nearly all other contemporary mathematicians dismissed the use of the actual infinite for a variety of reasons, and sought the development of limit calculus

Cantor

- Georg Cantor (1845-1918)
  - Developed first form of set theory
  - Made the distinction between cardinal and ordinal numbers
  - Defined the infinite in terms of set theory
  - Demonstrated that not all infinite sets are equal in size; some are larger than others

Sets

- Naive set theory: Anything can be contained within a set:
  - {my hair, Greenland, Mars}
  - Ø
  - {1,2,3,4}
  - {1,2,3,4,...}
  - And so on

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The Grand Hotel (Hilbert)

- A paradox in the sense that the conclusion seems absurd, not in the sense that it leads to an actual contradiction.
- I have a hotel, it has an infinite number of rooms.
  - It is quite a popular hotel, and so all of the rooms have guests in them.
  - Some poor schmuck comes in; he wants a room. Can I get him one?
  - Sure! I'll send out a message for all guests in room n to move to room n+1. The first room is then free- he can have it.
  - I can repeat this process for any finite number of guests who want in.

The Grand Hotel (Hilbert)

- Now, a coach arrives with an infinite number of guests, all of whom want rooms.
  - I can't just do the same procedure; I'd be doing it forever!
  - Instead, I'll move the guest in room *n* to room 2*n*. Guest in room 1 moves to 2, guest in room 2 moves to 4, and so on.
  - All of the odd-numbered rooms are now free. There are an infinite number of odd rooms; everyone gets a room!
- My hotel is getting more popular; an infinite number of coaches each carrying an infinite number of guests arrive, wanting rooms.
  - We can still do this- provided that we can label each passenger on each bus.
  - First, we can empty the odd-numbered rooms as we did previously, using 2*n*. Next, we take the passengers in the first coach and place them in the room 3n for passengers *n* = 1,2,3,... (going in rooms 3, 9, 27, and so on)Next, we take the passengers in the second coach and place them in room 5n for passengers *n* = 1,2,3,... (going in rooms 5, 25, 125, and so on). The pattern will be the following: for each coach *I* and passenger *n*, they will go into *p*n , where p is the prime number in the *I* +1 spot in the well-ordered set of prime numbers (essentially, the [*I*+1]-st prime number).

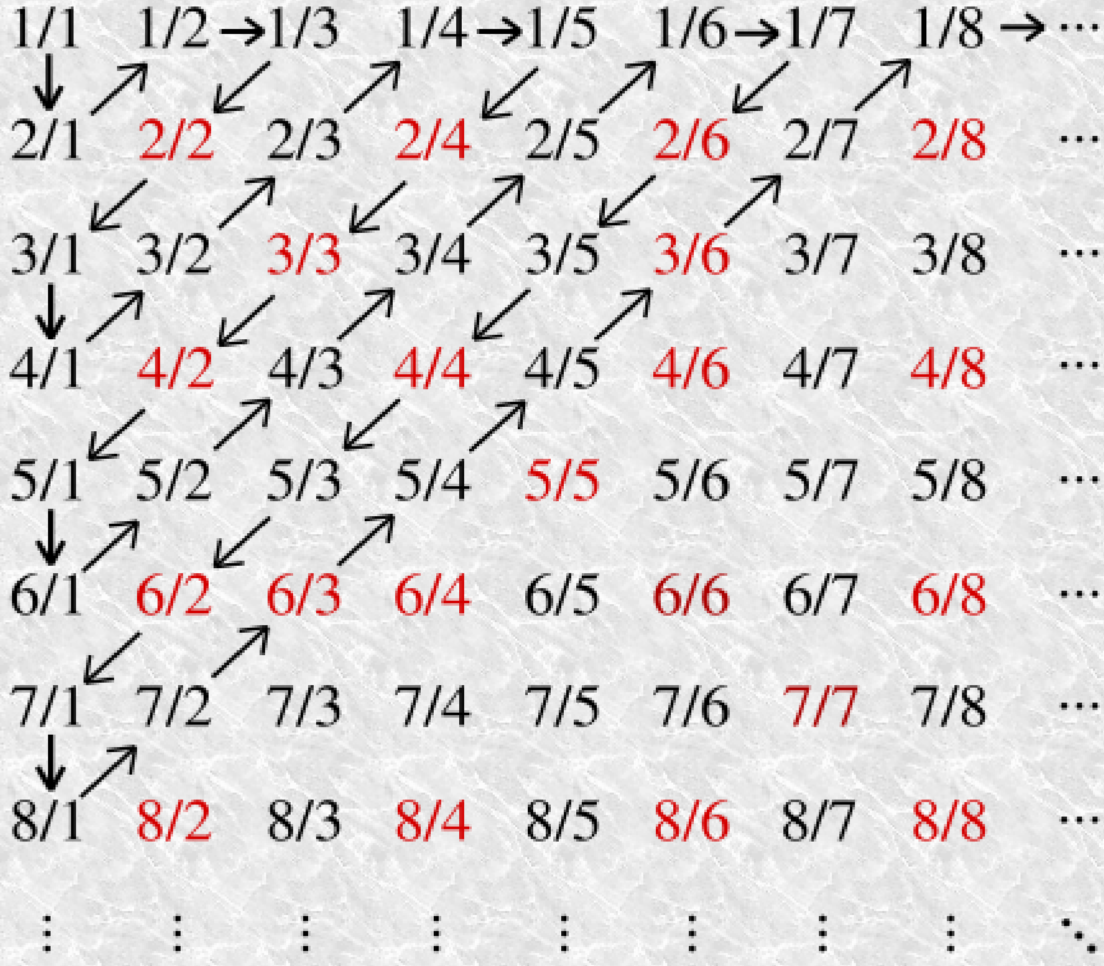
The Grand Hotel (Hilbert)

- The important idea is this: In a finite hotel, saying “every room has a guest in it” and “No more guests can check in is the same information. For an infinite hotel, this is not the case!

- What this illustrates is the difference between **ordinal numbers** and **cardinal numbers** when dealing with infinite sets.
- Ordinal = first, second, third, fourth,... (rank, order)
- Cardinal = 1, 2, 3, 4,... (size, magnitude)
- An easy example is the set of all odd numbers and the set of all natural numbers. The set of odd numbers is a proper subset of the set of natural numbers, but if we set them in one-to-one correspondence, each odd number can be attached to a natural number.

Transfinite Cardinals

- We can use these two different forms of number to begin an examination of infinite sets
  - What does it mean for a set to be infinite?
  - If a set can be placed into one-to-one correspondence with one of its proper subsets (i.e., set A contains all of the members of set B, but there are members of set A which are not contained in set B), it is an infinite set.
    - Sets such as the set of even numbers, the set of prime numbers, and even the set of rational numbers can all be placed into one-to-one correspondence with each other and the set of natural numbers; they all have cardinality of aleph-null.
      - They are also called countably infinite, since we can place them in an exaustive, ordered list (which is just another way of placing a set in one-to-one correspondence with the set of natural numbers


1/1 1/2 1/3 1/4 1/5 1/6 1/7 1/8 ···
2/1 2/2 2/3 2/4 2/5 2/6 2/7 2/8 ···
3/1 3/2 3/3 3/4 3/5 3/6 3/7 3/8 ···
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7/1 7/2 7/3 7/4 7/5 7/6 7/7 7/8 ···
8/1 8/2 8/3 8/4 8/5 8/6 8/7 8/8 ···

Transfinite Cardinals

We have seen how Cantor developed the foundations of set theory, made a distinction between cardinal and ordinal numbers, and defined infinity within set theory. However, we have not yet seen how he demonstrated that some infinite sets are larger than others.

In short, Cantor was able to demonstrate that there are sets with a cardinality larger than aleph-null.

There are a number of ways to demonstrate this, but we will stick to the famous Diagonal Argument.

The Diagonal Argument

We've seen how the set of even numbers, the set of prime numbers, and the set of rational numbers can all be placed into one-to-one correspondence with the set of natural numbers- they can be counted.

Now consider the set of real numbers.
Real numbers are the quantities that exist on a continuous number line. They include the rational numbers and the irrational numbers.

Let's take only a section of the set real numbers, say the reals between 0 and 1 on a number line. Now, let's try to place those real numbers in an arbitrary list, like the following:

s1 = 0.19382048...
s2 = 0.95839238...
s3 = 0.34958393...
s4 = 0.29584394...
s5 = 0.58930285...

The Diagonal Argument

Now, let's find a real number on our list, st . St will be between 0 and 1, and have the following digital expansion:for the nth place in its digital expansion, it will have a digit equal to that in the nth place in number sn plus one, unless that digit is a 9. If it is 9, then we shall subtract one from the digit in the nth place in number sn.

s1 = 0.**1**9382048...
s2 = 0.9**5**839238...
s3 = 0.34**9**58393...
s4 = 0.295**8**4394...
s5 = 0.5893**0**285...

St = 0.26891...

If we do this, we will find that st differs from every other number on the list we have created by at least one digit; namely, the nth digit in the expansion of every number sn. We have failed to create an exhaustive list of the real numbers from 0 to 1 and, correspondingly, the set of real numbers as a whole. The set of real numbers has a cardinality larger than that of the set of real numbers.

The Continuum Hypothesis

If we were to work out the actual set theory behind the diagonal argument, we will find that the cardinality of the set of real numbers (also known as the continuum) is actually two raised to the power of aleph-null.

While we know that two raised to the power of aleph-null > aleph null, we do not know if the cardinality of the continuum is the next largest cardinality after aleph-null.

The hypothesis that this is the case, that two to the power of aleph-null = aleph-one, is known as the continuum hypothesis.

While Cantor himself believed that the hypothesis was true (and provable), Kurt Gödel and Paul Cohen later demonstrated that the truth or falsity of CH was independent (undecidable) from the axioms of ZFC (Zermelo-Fraenkel with the axiom of choice).