Philosophy 405: Knowledge, Truth and Mathematics — Hamilton College
Fall 2014 — Russell Marcus

Class #9: Kant

Our in class discussion of these notes will be lamentably abbreviated, focused on the first three sections.

I. Kantian Analyticity

Kant begins his *Critique of Pure Reason*, indeed his whole metaphysical philosophy, with the claim that mathematical propositions (or judgments) are synthetic *a priori*.
Kant holds that any *a priori* proposition is necessary.
So, his claim is both metaphysical and epistemological.
He is claiming both that mathematical propositions are necessary (metaphysics) and known *a priori* (epistemology).[1]

Kant also claims a semantic thesis, that mathematical propositions are synthetic.
All propositions are either synthetic or analytic.
To determine if a proposition is analytic or synthetic, note that any proposition, in its most rough parsing, affirms a predicate of a subject.
A proposition is synthetic if the predicate adds something to the concept of the subject, if it augments or enlarges the subject.
A proposition is analytic if the predicate is contained within the subject.
Consider the analytic propositions ‘every mother has a child’ and ‘p • q entails p’.
If we unpack the concept of a mother, we find that it entails having a child.
So, predicating ‘has a child’ of the subject ‘every mother’ does not augment it.
The analyticity of ‘p • q entails p’ is even more obvious.

Kant does not distinguish between two related definitions of analyticity, to both of which he appeals.

> A1. A proposition is analytic iff the concept of its predicate is included in the concept of its subject.
> A2. A proposition is analytic iff its denial leads to a contradiction.

Indeed, he seems to believe that A1 and A2 yield the same distinction, that the same propositions are analytic, and the same ones synthetic, on either definition.
A1 is controversial because of its reliance on concepts, which, like mathematical objects, are notoriously obscure entities.
A1 depends on the notion of conceptual containment.
In ‘p • q entails p’, we can actually see the ‘p’ in the ‘p • q’.
In ‘every mother has a child’ we may not see the having of a child in the term ‘mother’.
Still, Kant is just claiming that we can unpack (in some sense) the concept of a mother and find the having of a child in there.

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[1] Discussing the apriority of physical laws, Kant collapses necessity and apriority explicitly. “[Such] propositions are clearly not only necessary, *and hence of a priori origin*, but also synthetic” (*Critique*, B18, p 6, emphasis added). Actually, he here only assumes that a statement is *a priori* if it is necessary, and not the converse. But, he holds both claims. As Hume argued, one can not arrive at a necessary truth from contingent experiences.

One problem with A1 is that not all sentences are best understood in subject-predicate form.
Shapiro attacks Kant's subject-predicate distinction.

> As we know now, not every proposition has a subject-predicate form, and so by contemporary lights Kant's definition of *analyticity* is unnatural and stifling (Shapiro 77).

Shapiro is correct that modern logic and linguistics parse propositions more finely.
Kant was surely aware that his distinction was only rough.
Later, as we will see, Fregean logic provides the tools for a more fine-grained analysis of propositions.
A Fregean analysis can handle the analyticity of statements that are not in subject-predicate form.

> If it is snowing, then it is cold.

Such sentences seem analytic, true in virtue of the conceptual containments of their parts.
Yet, they are not of simple subject-predicate form.

Even if we refine our analysis of propositional structures, we do not know what it means for a concept to contain another.
Does the concept of a cat contain the concept of being a mammal?
Kant believes that the concept of a triangle contains the concept of having three angles.
But, does the concept of a triangle contain the concept of having three sides?

There are at least two different notions of conceptual containment that philosophers have used.
Kant uses what Frege (in the late nineteenth century) called beams-in-the-house analyticity.
When we look at a house, if we want to see if it contains a certain structure, we peel back the walls and literally see the beams.
In contrast, Frege develops a plant-in-the-seeds analyticity.
According to Frege, a statement can be analytic as long as it follows from basic axioms according to analyticity-preserving rules of inference.
Shapiro implies that the Fregean definition makes the Kantian definition obsolete.
The Fregean definition of analyticity, though, remains itself controversial.
The proper notion of conceptual containment, and whether there even is one, remains a problem in contemporary philosophy.
It is an awkward question, especially since the notion of containment is metaphorical.
It's hard to know even what concepts are, let alone how they can contain each other.
Still, if we do not push too hard, we can understand approximately the notion to which Kant is aiming.

A2, which derives perhaps from Leibniz's work in ways in which we have already seen, is less controversial.
But A2 still depends on having a clear notion of contradictions, and how to recognize them.
For now, we will not pursue the differences between A1 and A2.
But, they will become important later, especially when we get to Frege's work.
For now, we need to get clear about the differences among three distinctions: the metaphysical distinction between necessary and contingent propositions; the epistemological distinction between *a priori* and empirical propositions; and the semantic (or conceptual, or even psychological) distinction between analytic and synthetic propositions.

II. Epistemology, Semantics and the Synthetic *A Priori*

The semantic distinction between analytic and synthetic judgments is independent of the epistemological distinction between *a priori* justifications and empirical (or *a posteriori*) ones.
A statement is justified empirically if we appeal in our account of how we know it to sense experience.
Our belief that snow is white is empirical, since we have to see snow to justify knowledge of its whiteness.
In contrast, our belief that 3+2=5 may be justified *a priori*, as prior to, or independent, of sense experience.
We need to see particular snow in order to know that snow is white.
We need experiences with no particular objects in order to know that 2+3=5.
Further, no empirical experiences with undermine that claim.
When we add two cups of water to three cups of salt, and fail to come up with five cups of anything, we do not abandon our claim that two plus three is five.
Instead, we insist on limiting the applications of the mathematical claim to items of the same sort.
Similarly, two chickens added to three foxes does not produce five animals, but three fat foxes and a pile of feathers.
But we do not take the failure of the mathematical claim to apply in this instance as impugning our arithmetic.
The arithmetic claim remains.

We are granting the controversial claim that all *a priori* claims are necessary.
One might further believe that all *a priori* claims must be analytic.
For example, one might think that the only way to reason *a priori* is by analysis of concepts.
Concomitantly, one might align contingency with empirical justification and syntheticity.
On such a view, a claim is contingent when it is justified by appeal to sense experience and brings together concepts that are not necessarily related.
In particular, Hume seems to make these two claims, though he uses a different terminology.
While Hume does not call mathematical propositions necessary, he does claim that they are relations of ideas, following from the principle of contradiction.
It is natural to interpret Hume as claiming that relations of ideas are necessary, justified *a priori* and analytic, following from the principle of contradiction.
Conversely, matters of fact are, for Hume, contingent, justified empirically (by tracing ideas back to initial impressions) and synthetic.

We can depict our interpretation of Hume's claims in the following table:

Hume's Rubric	*A priori*	Empirical
Analytic	Relations of Ideas	--
Synthetic	--	Matters of Fact

Kant agrees with Hume that matters of fact are all synthetic.

> Experiential judgments, as such, are one and all synthetic (*Critique*, A7/B11, p 3).

But, he disagrees that the converse holds.
There are, he says, synthetic claims that are not experiential, or empirical.
In other words, for Kant, the lower-left cell is non-empty.

Kant's Rubric	*A priori*	Empirical
Analytic	Logic	--
Synthetic	Most Mathematics and Metaphysics, and Some Physics	Empirical Judgments

In metaphysics, Kant claims that judgments such as that every effect has a cause are synthetic *a priori*.
We do not find the concept of a cause in the concept of an effect, as Hume argued.
So the claim that every effect has a cause must be synthetic.
Against Hume, since we ordinarily take it to be a necessary truth, it must *a priori*.
In physics, Kant claims that Newton's third law of motion, that for every action there is an equal and opposite reaction, is among the synthetic *a priori* propositions as well.
But, his least contentious examples of synthetic claims that are not empirical are mathematical.
Kant's big claim is that 7+5=12 is not analytic.

> Mathematical propositions, properly so called, are always *a priori* judgments rather than empirical ones; for they carry with them necessity, which we could never glean from experience...It is true that one might at first think that the proposition 7 + 5 = 12 is a merely analytic one that follows, by the principle of contradiction, from the concept of a sum of 7 and 5. Yet if we look more closely, we find that the concept of the sum of 7 and 5 contains nothing more than the union of the two numbers into one; but in [thinking] that union we are not thinking in any way at all what that single number is that unites the two. In thinking merely that union of 7 and 5, I have by no means already thought the concept of 12; and no matter how long I dissect my concept of such a possible sum, still I shall never find in it that 12. We must go beyond these concepts and avail ourselves of the intuition corresponding to one of the two... (*Critique*, B14-5, p 5).

Similarly, for Kant, the concept of a circle does not in itself contain the concept of the tangent meeting the radius at right angles.
That theorem is synthetic, but *a priori*.

Hume agreed that universal physical laws could not be learned from experience.
From that claim, and the insistence that all knowledge comes from experience, Hume inferrd skepticism.
Kant, working in the other direction, starts his reasoning by accepting that there are mathematical, metaphysical, and even physical laws that hold necessarily, that are known *a priori*.
Working backwards, he argues that our cognitive abilities must be such that they allow us to know those principles *a priori*.

> For experience would provide neither strict universality nor apodeictic certainty... (*Critique*, A31/B47).

Kant does not argue that innate ideas about objects outside of us are built into our minds, in the way that

Descartes and Leibniz alleged.
Instead, he argues that there are certain cognitive structures that impose an order to our possible experience.
The mind has templates for judgments, which are imposed and can be known *a priori*.
But, against those who defend innate ideas, it does not contain judgments themselves.
If we look at our cognitive structures, turning our reasoning on itself, we can find the necessary structure of our reasoning, and grounds for synthetic *a priori* claims.
That process, which Kant calls transcendental reasoning, is the essence of Kant's work.
Kant's transcendental arguments lead to a description of our subjective conceptual framework, which nevertheless holds necessarily for all possible experience.

III. Brief Aside on Kant's Rubric

One way to question Kant's work on the philosophy of mathematics is to wonder whether his analysis of the four types of proposition is correct.
Consider that Kant claims that there are no analytic *a posteriori* claims.

> All analytic judgments are *a priori* even when the concepts are empirical, as, for example, "Gold is a yellow metal"; for to know this I require no experience beyond my concept of gold, which contained the thought that this body is yellow and metal. It is, in fact, this thought that constituted my concept; and I need only analyze it, without looking beyond it elsewhere (*Prolegomena* 267, p 12).

In contrast, some philosophers have argued that there are analytic *a posteriori* claims.
Consider 'All tokens of C-sharp have volume,' or 'All material objects have weight.'
Those sentences seem to be analytic, either on the principle of contradiction criterion (A1) or the conceptual containment criterion (A2).
They also seem empirical, since we have to have sense experience to learn about volume and weight.

The claims that there are analytic *a posteriori* claims and that Kant has provided an accurate analysis of the nature of all propositions are both controversial.

Furthermore, philosophical work in the past half-century has also called the identification of *a priori* judgments with necessary ones, and of empirical judgments with contingent ones.
Saul Kripke has argued both that there are necessary yet empirical judgments, like the claim that water is H_2O, and that there are contingent yet *a priori* judgments, like the claim that the standard meter is one meter long.
For necessary empirical propositions, the identification of water with H_2O is a theoretical identity.
If we found a substance that looked and functioned like water, but that turned out to be have a different chemical structure, we would say that it was not water.
Still, it is clearly discovered empirically that water has the chemical structure that it has.

In the latter case, contingent *a priori* propositions, it is a contingent fact that the standard meter is the length that it is.
With changes in temperature or humidity, it expands or contracts.
But, because it is the way that we (once) measure(d) the meter, it is *a priori* that it be one meter.
I shall put all of these questions aside, in order to get a better grasp on Kant's philosophy of mathematics.

IV. Kant’s Mathematics and Metaphysics

Much of Kant’s work depends on the claim that there are synthetic *a priori* judgments in mathematics.
Kant claims that philosophy, more specifically metaphysics, consists mainly of synthetic *a priori* judgments.
If mathematical judgments were all analytic, then they would, as Hume claimed, all follow from the principle of contradiction; or they would be the result of mere conceptual analysis.
If mathematical propositions were analytic, the scope of reason would narrow, and the possibility of metaphysics would diminish.

Indeed, without synthetic *a priori* propositions, we are inexorably led to Humean skepticism.
Kant credits Hume with awakening him from his dogmatic slumbers, with showing that there must be a category of judgments besides the analytic-*a priori*-necessary (i.e. relations of ideas) and the synthetic-empirical-contingent (i.e. matters of fact).
If all our empirical beliefs are matters of fact, and trace back to initial impressions, then we get the problem of induction.
We can never know scientific laws.
And, as Hume’s arguments entail, we can never know to go out through the door rather than the window.
We have no reason to believe that the future will be like the past, that the laws of nature are uniform.
The physical and metaphysical laws are not analytic.
We can not find the effects in the causes.
And, they are not learned by instances.
They can not be learned at all.

Hume showed that presuppositions about language and its relation to the world, like those in the work of Locke and Berkeley, led to a deep skepticism.
But, Hume left mathematics alone.

Kant’s solution to the skeptical problem is embodied in his transcendental method.
We determine what the nature of our minds must be in order to support the most central, important claims: what are the necessary conditions of the structure of our minds such that we get the necessary truths of mathematics, and laws of nature?
We impose a conceptual structure, built into the very fabric of our minds, which determines, *a priori*, the framework.
We are given a noumenal world, and we structure this world by imposing our concepts upon it.
In order to solve the skeptical problem, Kant opens up the synthetic *a priori* category, and mathematics falls in.

So, it is essential to evaluate Kant’s claim that mathematical propositions like ‘7+5=12’ are synthetic.
If they were analytic, they would have to follow from conceptual analysis, or from the principle of contradiction.
The argument that it is not a product of conceptual analysis is more compelling than the argument that it does not follow from the principle of contradiction.
The claim that we have to go beyond the seven and the five, and the concept of their sum, in order to think of the twelve, seems plausible, insofar as we can make sense of conceptual analysis.
But, since Kant identifies A1 and A2, i.e. since he appeals to both criteria for analyticity, Kant must claim that no contradiction follows from the denial of some mathematical propositions.

> There is no contradiction in the concept of a figure which is enclosed within two straight lines, since the concepts of two straight lines and of their coming together contain no negation of a figure. The impossibility arises not from the concept in itself, but in connection with its construction in space, that is, from the conditions of space and of its determination (*Critique* A220/B268, p 28).

The denials of some claims in geometry do seem not to be contradictory.
Indeed, it turns out that we can deny the parallel postulate without contradiction.
But, the denials of other mathematical theorems do seem contradictory.
For example, '$7+5\neq 12$' does seem like a contradiction.
If the claim is synthetic, then it has to be logically possible for $7+5\neq 12$.
That is a harder claim to accept.
Kant is clearly relying on a very weak notion of possibility.
It is only in the broadest sense which the denials of mathematical propositions are possible.
Again, we need a better criterion for identifying contradictions than Kant provides, one which Frege will construct a century later.

V. An Aside on Foundations

Nowadays, mathematicians generally think of set theory as the foundational theory of mathematics.
Some mathematicians and philosophers are working on an even more fundamental theory, called category theory.
For Kant, who lived at a time before set theory was developed, there were really two different fundamental mathematical theories: arithmetic (including algebra) and geometry.
Calculus, and analytic geometry, bridged the two disciplines, but at a higher level.
For classical mathematicians, of course, geometry was the more fundamental discipline.
Even for Newton, geometry was more basic.
With the development of analytic geometry in the seventeenth century, and the development of abstract algebra, many philosophers and mathematicians began to think of arithmetic as more fundamental.
By Kant's time, it was more standard to think of geometry as derivative, in some sense, from arithmetic.
But, Kant takes the two mathematical theories as essentially independent.
Geometry arises out of spatial intuition.
Arithmetic comes from the comibination of our temporal and spatial intuitions.

VI. Intuition

For Hume, the truths of mathematics were relations of ideas because they followed from definitions according to the principle of contradiction.
Kant disagrees that mathematical statements are true by definition.
They have to be present in my thought.
We have to work at them, and not merely by analyzing concepts.
Let's look a little more carefully at the construction of mathematical concepts, at mathematical methodology, in order to see how mathematics is supposed, by Kant, to be synthetic.

> Let the geometrician take up [the questions what relation the sum of a triangle's angles bears to a right angle]. He at once begins by constructing a triangle. Since he knows that the sum of two

> right angles is exactly equal to the sum of all the adjacent angles which can be constructed from a single point on a straight line, he prolongs one side of his triangle and obtains two adjacent angles, which together are equal to two right angles. He then divides the external angle by drawing a line parallel to the opposite side of the triangle, and observes that he has thus obtained an external adjacent angle which is equal to an internal angle - and so on. In this fashion, though a chain of inference *guided throughout by intuition*, he arrives at a fully evident and universally valid solution of the problem (*Critique* A716-7/B744-5, p 15; emphasis added).

At the beginning of this process of mathematical thought, the geometer is given some definitions.
As we work through the construction, though, we extend those definitions.
We extend a line, we draw new lines.

> I must not restrict my attention to what I am actually thinking in my concept of a triangle (this is nothing more than the mere definition); I must pass beyond it to properties which are not contained in this concept, but yet belong to it (*Critique* A718/B746).

Notice that Kant's claim that mathematics is synthetic *a priori* is based on his claim about what we actually think.
Kant's account of mathematics is essentially psychological.

> The assertion that 7+5 is equal to 12 is not an analytic proposition. For neither in the representation of 7, nor in that of 5, nor in the representation of the combination of both, do I think the number 12. (That I must do so in the *addition* of the two numbers is not to the point, since in the analytic proposition the question is only whether *I actually think* the predicate in the representation of the subject (*Critique*, A164/B205, p 23; final emphasis added).

The heart of Kant's philosophy of mathematics thus involves the construction of objects of mathematics in intuition.
Kant is using 'intuition' as a technical term.
In our sense experience, we are given something, which we might call the passing show.

> The effect of an object on our capacity for representation, insofar as we are affected by the object, is *sensation*. Intuition that refers to the object through sensation is called *empirical* intuition. The undetermined object of an empirical intuition is called *appearance* (A19-20/B34, p 8).

Not all intuitions must be empirical.
Some intuitions are pure, about the structure (or form) of space and/or time themselves.
But, all intuitions are particular, or singular.
They represent particular objects, rather than general rules.

In empirical intuitions we can divide the matter from the form.
The matter is what corresponds to sensation.
If I am holding a pen and looking at it, I am given some appearance in intuition.
Additionally, this appearance has certain abstract properties, a form.
The particulars of the form of this appearance are unique to my experience of the pen.
But the general properties of the form of appearances are properties of all such experiences.
All experiences take place in space and in time.

My experience of the pen is thus necessarily given in intuition in both space and time.

Some intuitions contain no empirical matter.
These are pure intuitions.
We can consider pure intuitions by performing what might be thought of as Lockean abstraction.
It is the kind of abstraction that Berkeley did not disallow, the consideration of some properties of an idea, rather than others.
We can consider pure intuitions by thinking about intuitions without any matter.

> If from the representation of a body I separate what the understanding thinks in it, such as substance, force, divisibility, etc., and if I similarly separate from it what belongs to sensation in it, such as impenetrability, hardness, color, etc., I am still left with something from this empirical intuition, namely, extension and shape. These belong to pure intuition, which, even if there is no actual object of the senses or of sensation, has its place in the mind *a priori*, as a mere form of sensibility (A20-1/B15, p 9).

Note Kant's method here
We arrive at our consideration of pure forms of intuition by a method something like abstraction.
But Kant does not claim that our knowledge of space (and time) is derived from abstraction.
We are discovering that knowledge of space and time is necessarily presupposed in any empirical intuition.
Kant is presenting a transcendental argument, not a deductive one.
He does not move forward, as Locke does, from experiences of two objects to beliefs about the number two.
He moves backward, from the claim that two and two are four to the forms of intuition which support such claims.

In the Transcendental Aesthetic, Kant claims that there are two underlying forms of all intuitions: space and time.
We represent objects as outside of us using our outer sense.
All objects outside of us are represented as extended in space; space is the form of outer sense.
We represent objects according to our inner sense as in time.
The idea of a possible experience occurring outside of space or time is nonsense.
Instead of despairing of learning of space and time from experiences which presuppose it, as Hume tried and failed, Kant inverts his account to make space and time subjective forms of intuition.
They are ways in which we structure the world of things in themselves, not ways in which the world exists in itself.
They are properties of appearances, which are the objects of our empirical intuition.
Kant's transcendental exposition of space and time explains how we can have certainty of mathematics.
Geometry is the study of the form of outer sense, of pure, *a priori* intuitions of space.
Arithmetic, depends essentially on construing addition as successions in time.

So, there are two pure forms of intuition, space and time, which are not things in themselves, nor properties of things in themselves, but presuppositions we must impose on all our possible experience.
The faculty of intuition is what gives us appearances.
But, in order to go beyond those mere intuitions and develop mathematics, we have to think about them.
We have to reason, or impose concepts, on the intuitions.

VII. Imposition of Concepts

Intuitions are just the nearly-raw data, the content, of experience.
Our intuitions are passive, and what is given in intuition is not structured by the understanding.
What we are given lacks conceptual structure.
In order to think about those intuitions, we have to cognize them.
Appearances are structured; so the structure must come from somewhere.
We cognize using whatever conceptual apparatus we have.
The raw data of intuition must thus be processed in the understanding by the imposition of concepts.
This act of arranging what is given in intuition is what Kant calls synthesis of the manifold.
This synthesis is cognized by the structured application of concepts in the understanding.
If the synthesis is empirical, then we have an ordinary empirical cognition.
If the synthesis is pure, then we can arrive at pure concepts of the understanding.
Intuition and understanding thus work together to produce experience.

> "Thoughts without content are empty; intuitions without concepts are blind" (A51/B76).

The Transcendental Aesthetic consists of Kant's explications of the pure intuitions of space and time; we read just a little bit of the Aesthetic.
The Transcendental Analytic is the much longer explication of the categories of the understanding, how we impose our conceptual apparatus on what is given in intuition.
I put three parts of the analytic (the axioms of intuition, the anticipations of perception, and the postulates of thought) in the reading, at the end of the selection.
In those sections, Kant discusses how our cognition organizes that which is given in intuition.
The understanding takes a particular intuition and considers it under general, conceptual rules.

Kant develops twelve categories of concepts in four classes:

Quantity		Relation	
	Unity		Inherence and Subsistence (substance)
	Plurality		Causality
	Totality		Community (Interaction)
Quality		Modality	
	Reality		Possibility and Impossibility
	Negation		Existence and Non-Existence
	Limitation		Necessity and Contingency

The development of these categories, in what is called the metaphysical deduction of the categories, proceeds transcendentally rather than empirically.
Prior philosophers (e.g. Locke, Hume) proceeded empirically, looking at our psychological processes and generalizing.
Kant insists that such empirical deductions could never yield the necessity that underlies synthetic *a priori* reasoning.
Instead, Kant sets out to show that the categories necessarily apply to the manifold given in intuition, in the Transcendental Deduction.
He shows how the sensible and intellectual functions of our cognitive capacities align, in two stages.

In the first stage, Kant argues that the categories apply to any being with sensible intuition.

We proceed from a diverse manifold given in intuition to a single thought of a single, conscious person.
When we do so, we combine (either by synthesis or otherwise) the manifold.
This combination is an active function of our cognition, in contrast to the passivity of intuition.
We act on the manifold in intuition, unifying it.
Since our action is subjective, the application of the conditions of unity are subjective.
When we determine an intuition, we make it ours.
Further, an empirical unity is subjective, in that everyone's individual experiences are independent.
But the unity is also objective, since it determines objects for us.
Kant contrasts 'if I support this body, then I feel a pressure of heaviness' with 'this body is heavy'.
Since we have some knowledge of physical laws, we are able to make the latter claim.
But, unless the subjective unity of apperception were also objective, we could only make the former claim.
The relation among appearances is not merely arbitrary or accidental.
We know of causal relations.
Thus, we must be able to make objective claims about objects, not merely subjective claims.
Intuitions become objects for an individual, but they are still objects.
We can distinguish between fantasies and appearances.

In the second stage of the Transcendental Deduction, Kant shows that the categories necessarily apply to human sensibility.
We intuit the world in space and time.
But space and time, besides being forms of intuition, are also intuited themselves.
Since space and time are pure forms of intuition, they are presupposed in all experience.
Since any experience is already structured, or determined, space and time, as we experience them, are deeply embedded in those experiences.
Since any experience also presupposes the application of the categories, space and time themselves must be subject to the categories.
Not only do the categories apply to any intellect which receives appearances in intuition.
They apply specifically to our intuition which is sensible in the forms of outer sense (space) and inner sense (time).
Abstracting space and time, we find that the categories were presupposed.
The forms of intuition meet up with the categories of the understanding in large part because they are both *a priori* impositions of the subject.
We don't know about the conditions on objects in the noumenal world.
We do know that for us, experiences (i.e. appearances of objects in nature) must have certain abstract features.

VIII. Mathematics and Concepts

Some of the concepts which we impose on the unstructured manifold are less clearly relevant to our course than others.
The categories of quantity are directly relevant to our understanding of mathematics, especially arithmetic.

> The pure image of all magnitudes for outer sense is space; that of all objects of the senses in general is time. But the pure *schema* of magnitude, as a concept of the understanding, *is number*, a representation which comprises the successive addition of homogeneous units. Number is

> therefore simply the unity of the synthesis of the manifold of a homogeneous intuition in general, a unity due to my generating time itself in the apprehension of the intuition (*Critique* A142-3/B182, pp 11-12).

We construct, in pure intuition, a figure.
For arithmetic, we construct stroke-symbols, and add them up.
Then, we have rules for adding symbols which depend on those symbols.

> [Mathematics] abstracts completely from the properties of the object that is to be thought in terms of...a concept of magnitude. It then chooses a certain notation for all constructions of magnitude as such (numbers), that is, for addition, subtraction, extraction of roots, etc. Once it has adopted a notation for the general concept of magnitudes so far as their different relations are concerned, it exhibits in intuition, in accordance with certain universal rules, all the carious operations through which the magnitudes are produced and modified (*Critique* A717/B745, p 15).

Using, for example, our fingers, we can construct empirical intuitions.
But, these only correspond to the pure intuitions, and do not constitute them.
The pure intuition need not be like a picture.

> If five points be set alongside one another, thus, ..., I have an image of the number five. But if, on the other hand, I think only a number in general, whether it be five or a hundred, this thought is rather the representation of a method whereby a multiplicity, for instance a thousand, may be represented in an image in conformity with a certain concept, than the image itself. For with such a number as a thousand the image can hardly be surveyed and compared with the concept (*Critique* A140/B179, p 11).

Notice the return to a Cartesian view about mathematical ideas.
Descartes had separated thought from sensation.

> Many properties can be very clearly and very distinctly demonstrated of [a chiliagon], which could certainly not happen if we perceived it only in a confused manner or...only in a verbal way. In fact, we have a clear understanding of the whole figure, even though we cannot imagine it in its entirety all at once. And it is clear from this that the powers of understanding and imagining do not differ merely in degree but are two quite different kinds of mental operation. For in understanding the mind employs only itself, while in imagination it contemplates a corporeal form (Descartes, *Fifth Replies* AT 384-5).

Locke and Berkeley and Hume, in their attempts to avoid innate ideas, had re-instated Aristotle's essential connection between sensation and thought, in the representational theory of mind.
Ideas had to be like pictures, derived from some sensations.
Kant is again cleaving thought and sensation.
We have pure intuitions, which may be not be pictures.
Pure intuitions, as mathematical ones, are formal, rather than material.

> In respect to this material element, which can never be given in any determinate fashion otherwise than empirically, we can have nothing *a priori* except indeterminate concepts of the synthesis of possible sensations...As regards the formal element, we can determine our concepts

> in *a priori* intuition, inasmuch as we create for ourselves, in space and time, through a homogeneous synthesis, the objects themselves - these objects being viewed simply as *quanta*... The determination of an intuition *a priori* in space (figure), the division of time (duration), or even just the knowledge of the universal element in the synthesis of one and the same thing in time and space, and the magnitude of an intuition that is thereby generated (number), - all this is the work of reason through construction of concepts, and is called *mathematical* (*Critique* A723-4/B751-2, p 17).

What is important, for mathematicians, about a pure intuition is the rule it dictates, not the picture in the mind that accompanies it.
The matter of the intuition is empirical.
But the form of the intuition is known *a priori*, and is independent of its matter.
Thus, general arithmetical concepts can be applied to particular spatial and temporal intuitions (to magnitudes, considered abstractly), and geometrical concepts can be applied to spatial intuitions, without taking those intuitions to be empirical.

Here is another look at Kant's argument for the syntheticity of mathematics in the appeals to intuition.
Consider how we use signs or symbols or diagrams when we are engaged in mathematical reasoning.
Consider drawing a diagram (say, of an isosceles triangle with an altitude) to accompany a proof (say, that the altitude is also an angle bisector).
We use such pictures to evoke concepts.
But we do not take those pictures to be the subjects of our mathematical theorems.
We take the drawing to be an example, and instance of a more general rule.
The empirical intuition does not exhaust the reference of the proof.
But, the proof remains about intuitions.
If we were just analyzing concepts, definitions, we would have no need for such intuitions.
Mathematical work is synthetic, but it remains *a priori*.
Again, the inferences are conceptual, but guided by intuition (A717/B745, p 15).

IX. The Berkeley Problem, Revisited and Extended

Kant's appeals to the conceptualizing of our pure forms of intuition seem a lot like Locke's appeals to abstraction.
For Locke, mathematical theorems applied to abstract ideas which might be triggered by empirical experiences, but which were mental constructs, abstractions from the particulars.
Berkeley attacked Locke on the possibility of having an abstract idea of a triangle.
For, the abstract idea is supposed to apply to all kinds of triangles, and some triangles have properties which contradict the properties of other triangles.

> If any man has the faculty of framing in his mind such an idea of a triangle as is...described [by Locke], it is in vain to pretend to dispute him out of it, nor would I go about it. All I desire is that the reader would fully and certainly inform himself whether he has such an idea or not. And this, methinks, can be no hard task for anyone to perform. What is more easy than for anyone to look a little into his own thoughts, and there try whether he has, or can attain to have, an idea that shall correspond with the description that is... given [by Locke] of the general idea of a triangle, which is *neither oblique nor rectangle, equilateral, equicrural nor scalenon, but all and none of these at once*? (Berkeley, *Principles* Introduction §13, p 466).

For Berkeley, who took ideas to be mental pictures, it seems contradictory to claim that one could have an idea of a triangle that is scalene, isosceles, and equilateral.

Kant avoids some of Berkeley's criticism by claiming that our intuitions are not pictures.
He claims that we can only use the parts of the triangle that we construct for the proof.
We rely on the rule which guides the construction in intuition.
If we are constructing a triangle, we have to construct three angles.
We need not construct three angles the measures of all of which are less than ninety degrees.
Depending on which kind of triangle we are considering, we appeal to different rules.
We must distinguish among the properties which belong to all triangles, the concepts of which will accompany any construction of a triangle, and those which belong only to some triangles, and which we are free to add to our construction or not.

Philip Kitcher, explaining Kant's view, distinguishes three types of properties:

> R-properties, which all triangles have, analytically, e.g. having three angles;
> S-properties, which are synthetic, arising from the construction in intuition for any triangle, e.g. the sum of two sides is greater than the third;
> A-properties, like being scalene, which need not apply to every triangle.
>
> R-properties are just the properties which the schema alone determines; for the triangle an example of an R-property would be the property of having three sides. S-properties are those properties which the schema and the structure of space together determine; the side-sum property and the property of having the internal angle-sum equal to 180 degrees are both supposed to be S-properties. Finally, there are the A-properties, peculiarities of the particular figure drawn, such as the scaleneness of the triangle (Kitcher, "Kant on Mathematics" 44).

A similar analysis can perhaps apply to arithmetic examples.

> R-properties, like 12=12, or commutativity.
> S-properties, like 5+7=12.
> A-properties, like being prime.

We are free to construct objects with any A-properties we like.
They do not follow from the nature of, say, the triangle.
Their denials are logically possible.
We can construct a scalene triangle, or an equilateral triangle, as we wish.

S-properties do not follow from the nature of the triangle, and their denials are also logically possible.
But, we are only free to construct objects with or without these properties insofar as we are free to construct different spatial structures.

There are different kinds of space: Euclidean and non-Euclidean.
Consider an interstellar triangle.
The sum of its angles will not be 180°, due to the curvatures of space-time corresponding to the gravitational pull of the stars, and other large objects.
Space-time is not Euclidean, but hyperbolic.
The difference between hyperbolic space-time and Euclidean space-time involves the parallel postulate.

Euclid's parallel postulate states that if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.
Playfair produced an equivalent, and perhaps more intuitive, formulation: Given a line and a point outside that line, there is exactly one line going through the point that is parallel to the given line.
In hyperbolic geometry, instead of there being one line that we can draw parallel to the given line, there are an infinite number of lines.

Kant assures us that all space is necessarily Euclidean.

> Our exposition...establishes the *reality*, that is, the objective validity, of space in respect of whatever can be presented to us outwardly as object (*Crituque* B44/A28).

We construct our intuitions in Euclidean space.
Mathematics is just the study of the pure forms of intuition.
Our knowledge of geometry is *a priori* knowledge of the necessary structure of space.
Our knowledge of arithmetic is *a priori* knowledge of the necessary structure of "combinatorial" aspects of space and time.

> The determination of an intuition *a priori* in space (figure), the division of time (duration), or even just the knowledge of the universal element in the synthesis of one and the same thing in time and space, and the magnitude of an intuition that is thereby generated (number), - all this is the work of reason through construction of concepts, and is called *mathematical* (*Critique* A724/B752, p 17).

Thus, Kant might have believed that we need not worry about the difference between S-properties and A-properties.
But, since we know that there is a difference between them, Kant is faced with a descendent of the Berkeley problem.
If we need not draw triangles in intuition in order to derive theorems about them, then mathematics could proceed analytically, by definitions.
The fundamental claim that mathematics is synthetic *a priori* entails that we construct mathematical objects in intuition in order to generate appropriate concepts.
Given the different structures of space, Kant would have to argue that we can know, *a priori*, which space we are using in our intuition.
For, we have to know whether we are constructing an A-property or an S-property.
Kitcher argues that Kant has no way to do that without already knowing the properties of space which underlie my intuitions.

> Let G be a basic geometrical truth. G is supposed to be synthetic *a priori*. Its synthetic status arises because its truth value is, partially, determined by the structure of space. It is logically possible that space have a structure such that G be false. (Otherwise, G would be analytic.) Further, it is logically possible that G might have been false in such a way that many figures actually had the property ascribed to them by G. How could we have determined from inspection of such a figure that the property was only an A-property and that we should not therefore generalize over it? We can answer this question only if we can decide what counts as the application of a rule on the structure of space and what was our free decision in drawing the figure. Yet to distinguish S-properties from A-properties is just to recognize the structure of

> space. We could not therefore come to know G in the way which Kant describes (Kitcher, "Kant on Mathematics", 45-6).

Kant's claims that Euclidean geometry is known *a priori* and that *a priori* knowledge is infallible thus can not both hold.

X. Looking Forward

In the historical sources we have explored, there is an interplay between metaphysics and epistemology.
Those philosophers who provided a substantial mathematical ontology (e.g. Plato, Descartes) seemed committed to an unacceptable epistemology.
Those philosophers who started with commonsense epistemologies (e.g. Aristotle, Locke) seemed committed to an unacceptably anemic mathematical ontology.
Where Kant denied that the empirical intuition constituted or sufficed for mathematical knowledge, Mill will deny that we can get anything more than that.
Mill severely restricts mathematical necessity and apriority.
But, he provides a much less controversial epistemology.

A different response to Kant, that of Frege, argues that Kant is wrong about mathematics being synthetic. Frege will argue for a different notion of analyticity.