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Class #7: Leibniz and the Empiricists
Sample Introductory Material from
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I. Locke, Leibniz, and Innate Ideas

The portions of Locke's work relevant to the philosophy of mathematics contain both a positive project and a negative project. The positive project, like the related projects of Berkeley and Hume, is an explanation of how minds born as blank slates, or tabulae rasa, can formulate mathematical theories using merely sense experience and our psychological capacities for reflection.

Locke's negative project is his attack on the innate ideas of the rationalists, specifically of Descartes. In the first book of Locke's Essay, 1690, he criticizes arguments for the existence of innate ideas, in order to clear the way for his positive, empiricist project. Leibniz, who was twelve years younger than Locke, wrote an extended commentary on Locke's Essay, called New Essays on Human Understanding, in the years around 1700. In the New Essays, Leibniz defends a theory of innate ideas from Locke's attack. Our interest in the debate over innate ideas comes from the fact that mathematical ideas are supposed to be among the primary candidates for innateness. The questions how and whether we can justify our knowledge of mathematics are on the line.

On the surface, Leibniz and Locke present two very different views of mathematical knowledge. Leibniz claims that we care born with all of mathematical knowledge imprinted on our souls. Locke claims that we are born as blank slates, and derive our mathematical knowledge from sense experience.

> Although the author of the Essay says hundreds of fine things which I applaud, our systems are very different. His is closer to Aristotle and mine to Plato... (Leibniz, New Essays, 47).

Leibniz's comparisons to Plato and Aristotle are evocative, but can be misleading if taken too literally. Leibniz does believe that mathematical theorems are both eternal and necessary, as Plato did. But, he rejects Plato's doctrine of recollection, takes a modern view of causation, and has little use for Plato's more general theory of forms. Similarly, Locke, like Aristotle, presents an account of mathematical knowledge which can be broadly called abstractionist. But, he rejects Aristotle's theories of causation and sensation.

While Locke and Leibniz have significant disagreements, some of them are more semantic than contentful. When we look at the actual arguments, rather than just their labels, we find that the distinction between empiricism and rationalism, between the defenders of innate ideas and their opponents, is not alsways so great.

II. The Argument from Universal Assent

Locke, like most philosophers of the modern period (Berkeley is one significant exception), defended the new science and its method of experimentation. The new science posits a world of material objects, available to sense perception. We think about material objects through the use of our imagination, our capacity to receive sense images. The rationalists derogated those beliefs that were based on sense perception. For Descartes, these images are confused, and the only real properties are those we can understand by pure reason, through innate ideas. An innate idea is one that is implanted in our minds, or souls, rather than learned from sense experience. We are born with innate ideas, according to their proponents, which is why everyone has them, and everyone agrees about them.

The ideas of mathematics are among the most important of the innate ideas. Locke argues that he can avoid positing innate ideas by accounting for all of human knowledge, including mathematical knowledge, on the basis of sense experience.

> Men, barely by the use of their natural faculties, may attain to all the knowledge they have, without the help of any innate impressions, and may arrive at certainty without any such original notions or principles (Locke, Essay, §I.2.1).

Locke focuses on 'What is, is' and 'It is impossible for the same thing to be and not to be'.

> If these "first principles" of knowledge and science are found not to be innate, no other speculative maxims can (I suppose) with better right pretend to be so (Locke, Essay, §I.2.28).

He begins his attack on innate ideas by criticizing a general principle of universal assent that Locke attributes to the defender of innate ideas:

UA: If everyone agrees that p, then p is innate.

Locke calls UA the great argument, but it is unlikely that any defender of innate ideas accepted it. Descartes and Leibniz present no such principle, for example. Still, some defenders of innate ideas may have appealed to universal assent as evidence for innateness. Locke argues that UA is false by presenting some claims that engender widespread agreement while at the same time being tied to sense experience. For example, he argues that the claim that green is not red is self-evident. But, experience of color is not innate.

> I imagine everyone will easily grant that it would be impertinent to suppose the ideas of colors innate in a creature to whom God has given sight and a power to receive them by the eyes from external objects... (Locke, Essay, §I.2.1).

Locke claims that arguments which appeal to UA to establish the existence of innate ideas are both invalid and unsound. An argument is invalid if the conclusion does not follow from the premises. An argument is unsound if the premises are false.

Locke's argument that defenses of innate ideas which rely on UA are invalid is that there are better, alternative accounts of any such universal assent. Locke is relying on a principle of parsimony. We should prefer simpler explanations of any phenomenon.

> If it were true in matter of fact, that there were certain truths wherein all mankind agreed, it would not prove them innate, if there can be any other way shown how men may come to that universal agreement, in the things they do consent in, which I presume may be done (Locke, Essay, §I.2.3).

Thus, Locke's claim that UA supports an invalid argument depends on his positive account of our knowledge of innate ideas, which we will examine in our next class. To argue that the use of UA in an argument for the existence of innate ideas is unsound, Locke presents examples of people who do not know the most obvious, purportedly-innate principles, and of principles which should be innate but are not known. I'll call these two arguments the arguments from transparency.

III. The Arguments from Transparency

Locke argues that UA, and thus the doctrine of innate ideas, is easily shown false by considering that we do not know some of the ideas which are alleged to be innate. For example, children do not know lots of them.

It is evident that all children...do not have the least apprehension or thought of them. And the lack of that is enough to destroy that universal assent which must be the necessary concomitant of all innate truths... (Locke, Essay, §I.2.5).

Since we are not born apprehending any ideas, Locke claims, the rationalist's claim leads to a contradiction. The purportedly-innate ideas are known (since they are built-in) and also unknown (since they are not apprehended) at the same time.

We need not appeal to the limitations of children to establish the claim that we lack awareness of some propositions which are supposedly innate. Consider Goldbach's conjecture, that every even number can be written as the sum of two odd primes. Even the best mathematicians do not know if Goldbach's conjecture is true. Since it is supposed to be innate, Locke could argue, it is hard to see why we couldn't find a proof of it.

If the doctrine of innate ideas depended on all innate ideas being conscious, then such examples would surely be decisive. As Descartes argued, consciousness is the primary mark of the mental. Conscious awareness of our thoughts is sufficient for showing that they are mental, though the nature of mental states remains an open question. Locke seems to think that consciousness is also a necessary condition for a thought. That is, Locke presumes that there can be no unconscious thoughts, or thoughts of which we are not aware.

It [seems] to me near a contradiction to say that there are truths imprinted on the soul which it does not perceive or understand (Locke, Essay, §I.2.5).

Let's call this claim the argument from transparency: there can be no innate ideas since all thoughts must be transparently conscious and we are unaware of many ideas which are supposed to be innate. The argument from transparency does not destroy the doctrine of innate ideas, but it forces the defender of innate ideas to adopt a theory of unconscious mental states.

Until the late nineteenth century, the notion of unconscious thought was undeveloped. Still, any defender of innate ideas, in the face of Locke's argument, seems forced to admit that we are unaware of many of our innate ideas. Indeed, when we are just born we are unaware of all of them. Leibniz, perhaps uniquely among the moderns and in contrast to Descartes, flirted with such a view.

Cannot - and should not - a substance like our soul have various properties and states which could not all be thought about straight away or all at once? (Leibniz, New Essays, 78).

Not only did a theory of unconscious mental states deflect the argument from transparency, but it fit neatly into Leibniz's broader metaphysical, epistemological, and semantic views. For example, Leibniz's solution to the problem of free will requires a distinction between finite and infinite analysis. According to the traditional problem of free will, an omniscient God already knows all the future states of the universe, including all of my future actions. Thus, it seems that no action is contingent; everything that happens is already determined to happen, and happens of necessity.

Leibniz agrees that an infinite mind can, by merely analyzing the current state of the universe, know all the future states of the universe. But, our finite minds can only analyze the current state of the universe to a small degree. We do our best to analyze complex ideas into their most simple component parts. As we have seen, the fundamental truths will be what Leibniz calls identities. When we analyze a complex proposition, and decide on its truth value, we are prone to err. The likelihood of error increases with our distance from one of these fundamental identities. In the New Essays, Leibniz recalls an excellent geometer who insisted, falsely, that ellipses, as conic sections, are distinct from oblique sections of cylinders.

> I mentioned him only to indicate how far wrong one can go in denying one idea of another, if the case is one where the ideas need to be explored in depth and this has not been done (Leibniz, New Essays, 408).

The apparent contingency of my future actions, the appearance of free will, is grounded in such ignorance of infinite analysis. Still, there is an infinite analysis of the current state of the universe, which could be performed by an infinite mind. That analysis would reveal all the contents of my mind, including my unconscious mental states. Unconscious mental states are the kinds of things that an omniscient God could know, and which would support the absolute metaphysical determinism of the universe.

Let's take a moment to review. Locke argued that the defender of innate ideas relies on an unsound argument from UA. He claimed that the argument from UA is unsound because we are unaware of many of our supposedly-innate ideas. Locke's argument presumes that an innate idea must be transparent to our consciousness. The defender of innate ideas, Leibniz in this case, can deflect Locke's criticism by rejecting transparency, and adopting a more-plausible theory of mental states on which we can have innate ideas of which we are unaware.

In fact, we do not need a substantial theory of unconscious mental processes to see that Locke's transparency claim is odd. Even memory seems to refute the claim. We are only thinking of a small portion of our beliefs at any one time. We sometimes have to work to recall our experiences, or the capital of Kentucky, or the value of seventeen squared. Recollection seems to be precisely the kind of process that shows that our beliefs are not transparent.

IV. Reason

Given that every one doesn't know some of their innate ideas and some people don't know any of them, the defender of innate ideas might claim that such ideas require development.

> The whole of arithmetic and of geometry should be regarded as innate, and contained within us in a potential way, so that we can find them within ourselves by attending carefully and methodically to what is already in our minds, without employing any truth learned through experience or through being handed on by other people (Leibniz, New Essays, 77).

The proponent of a mature theory of innate ideas, then, can claim that we should expect our innate ideas not to be immediately apparent. We have to reason to them, or unfold them from within. Leibniz says that they are in us potentially. Here is a passage which recalls both Leibniz's earlier work in "Meditation on Knowledge, Truth and Ideas," on the primary truths, as well as foreshadowing Frege's nineteenth-century work on logic.

> In a larger sense, which is a good one to use if one is to have notions which are more comprehensive and determinate, any truths which are derivable from primary innate knowledge may also be called innate, because the mind can draw them from its own depths, though often only with difficulty (Leibniz, New Essays, 78)

Locke takes recourse to reason or development on the part of the rationalist to be a concession, and no help in avoiding the accusation of contradiction.

> To make reason discover those truths thus imprinted, is to say, that the use of reason discovers to a man what he knew before: and if men have those innate impressed truths originally, and before the use of reason, and yet are always ignorant of them till they come to the use of reason, it is in effect to say, that men know and know them not at the same time (Locke, Essay, §I.2.9).

Locke's claim that the defenders of innate ideas must claim that we are aware of all of our ideas is implausible, turning the rationalist into a straw person. As Leibniz argues, Locke is relying on a weak principle.

> I cannot accept the proposition that whatever is learned is not innate. The truths about numbers are in us; but we still learn them... (Leibniz, New Essays, 85).

For Leibniz, we have both primary innate ideas, and derivative ones. We have to learn them all, through the use of reason, and we will never learn some of them. Even if the Lockean accepts that the transparency thesis is indefensible, she might remain uncomfortable with Leibniz's appeal to reason. To think that there are innate ideas that are forever inaccessible to us seems, to the empiricist, to push the claim of innateness too far. But, we are finding ways in which the rhetoric of innateness obscures some points of agreement between the empiricists and the rationalists. Most importantly, Leibniz's appeal to our capacity for reasoning may be compatible, to some degree, with Locke's appeal to our psychological capacities for reflection. The question of how compatible the two view are will have to be approached by examining the differences between innate maxims and capacities.

V. Maxims and Capacities

Locke considers a possible rationalist response to the problematic contradiction of both knowing and not knowing the supposedly-innate ideas. The rationalist can distinguish between innate maxims and innate capacities.

> The capacity, they say, is innate; the knowledge acquired. But then to what end such contest for certain innate maxims? (Locke, Essay, §I.2.5).

If only the capacity for acquiring innate ideas is built in, then the rationalist can avoid the problematic contradiction. The rhetoric used by Locke and Leibniz may obscure the fundamental similarities in their work. Locke admits that we have psychological capacities to reflect on our ideas, to compare and contrast and recombine ideas as we will. Leibniz's claim that we have dispositions to discover innate ideas may be not so different. Leibniz welcomes the view that our capacities are innate.

> The actual knowledge of [propositions of arithmetic and geometry] is not innate. What is innate is what might be called the potential knowledge of them... (Leibniz, New Essays, 86).

But, Leibniz does not thus believe that his views are compatible with Locke's.

> I have also used the analogy of a veined block of marble, as opposed to an entirely homogeneous block of marble, or to a blank tablet - what the philosophers call a tabula rasa. For if the soul were like

such a blank tablet then truths would be in us as the shape of Hercules is in a piece of marble when the marble is entirely neutral as to whether it assumes this shape or some other. However, if there were veins in the block which marked out the shape of Hercules rather than other shapes, then that block would be more determined to that shape and Hercules would be innate in it, in a way, even though labour would be required to expose the veins and to polish them into clarity, removing everything that prevents their being seen. This is how ideas and truths are innate in us - as inclinations, dispositions, tendencies, or natural potentialities, and not as actions... (Leibniz, New Essays, 52).

Leibniz, taking thought to be an action, is willing to give up any claim to innate thoughts.

Locke thinks that by admitting that only our capacities are innate, the rationalist must cede the argument. If the rationalist is only committed to innate capacities, then everything we know, including empirical propositions, would be innate. Consider the obviously empirical proposition that, say, my copy of Locke's Essay has a white cover. In order to learn that proposition, we have to have built-in capacities for perception of colors and shapes. All learning depends on innate capacities. But reliance on innate capacities is no evidence that the content a belief is really innate. Thus, Locke believes that he has a reductio argument against the rationalist's view.

LR

LR1. The doctrine of innate ideas amounts to no more than the claim that we have an innate capacity for receiving ideas.

LR2. But an innate capacity is just a potential presupposed by all acquisition of ideas, indeed all acquisition of knowledge. In other words, if logic and mathematics are innate, then so are all the beliefs that we ordinarily take to be empirical.

LR3. It is not the case that all knowledge is innate.

LRC. So, no knowledge is innate.

One way for the rationalist to avoid Locke's reductio argument is to deny LR3, and accept that all knowledge is innate. In fact, I think that Descartes, like Plato, is best interpreted as holding the claim that everything we know is innate. His view is more similar to Plato than is ordinarily recognized. Leibniz, too, is best interpreted as denying LR3. But, denying LR3 is really implausible. It would be prudent for Leibniz to be able to resist LR in a more palatable way. To that end, Leibniz also denies LR2, arguing that there is a difference between our uses of innate capacities to acquire ideas using the senses and our uses of innate capacities to reveal the eternal truths.

The question of whether our innate capacity allows us to find what is inside us, or whether it allows us to learn merely from experience, separates the rationalist from the empiricist. Descartes and Leibniz are committed not merely to having innate capacities, but to the claim that significant content is built into the soul.

It is not a bare faculty, consisting in a mere possibility of understanding those truths: it is rather a disposition, an aptitude, a preformation, which determines our soul and brings it about that they are derivable from it (Leibniz, New Essays, 80).

It is difficult to see where exactly Locke and Leibniz part ways. Leibniz provides an evocative metaphor, of the statue of Hercules in the marble. But, to quote Berkeley, philosophers should not use metaphors. We want a literal understanding of the difference.

Locke believes that we have substantial psychological capacities, as we will see. Leibniz believes that our innate ideas are predispositions. We are not going to be able to get Locke to admit innate ideas. We will not get Leibniz to give them up. But, it is worth remembering that these philosophers are not as far apart on this issue as they seem.

VI. The Temporal Order and the Order of Justification

In contrast, Locke and Leibniz do have a serious disagreement about philosophical method. Whether the rationalist is ceding the claim that what is innate are certain capacities, or demanding that innateness is more than a mere capacity, she must distinguish between the kinds of mental processes that are acceptable to the empiricist, like Locke's psychological capacities to reflect and remember, and those which go beyond the tabula rasa. Locke and Leibniz can agree that our minds are born with the capacity to receive sense images. (Leibniz, who denies transeunt causation, does not in fact agree with this point, but his reasons for opposing it are not relevant here.) Leibniz, in the most important exchange of the debate, argues that we acquire certain maxims, the innate ideas, in ways that sense experience coupled with Lockean capacities could not explain.

Nobody questions whether experience is necessary for us to have knowledge. The question is whether experience is sufficient to account for what we know. Descartes argued that the information that we get from the senses is just not good enough to support clear and distinct judgments about the physical world. That is the point of the wax argument in Meditation Two. Leibniz, foreshadowing Hume, argues that some ideas could not be acquired without positing innate ideas beyond sense experience and Locke's psychological capacities.

Although the senses are necessary for all our actual knowledge, they are not sufficient to provide it all, since they never give us anything but instances, that is particular or singular truths. But however many instances confirm a general truth, they do not suffice to establish its universal necessity; for it does not follow that what has happened will always happen in the same way (Leibniz, New Essays, 49).

Leibniz's argument evokes Chomsky's twentieth-century poverty-of-the-stimulus argument for linguistic nativism. Linguistic nativism is the claim that the most fundamental rules of language are innate in our minds. Chomsky argues that language is grown, like an appendage, rather than learned, and that what he calls universal grammar (UG) is innate. His evidence for these claims is that children learn too much grammar too quickly to be explained by exposure to (or experience with) language. The central argument for nativism is called a poverty of the evidence, or poverty of the stimulus, argument (POTS).

The POTS argument also relies on the claim that children learn the lexicon (vocabulary) of their first language too quickly to be explained purely behaviorally. While they learn the specific words behaviorally, these words must hook onto pre-existing concepts.

It is a very difficult matter to describe the meaning of a word, and such meanings have great intricacy and involve the most remarkable assumptions, even in the case of very simple concepts, such as what counts as a possible "thing." At peak periods of language acquisition, children are "learning" many words a day, meaning that they are in effect learning words on a single exposure. This can only mean that the concepts are already available, with all or much of their intricacy and structure predetermined, and the child's task is to assign labels to concepts, as might be done with very simple evidence (Chomsky, "Language and Problems of Knowledge," 689).

Thus, Chomsky concluded, our abilities to use language must be built into our brains. Poverty of the evidence arguments, from Leibniz and Chomsky, are important because they undermine arguments, like those of Locke, which arise from the temporal order of learning.

A child knows not that three and four are equal to seven, till he comes to be able to count seven, and has got the name and idea of equality; and then, upon explaining those words, he presently assents to, or rather perceives the truth of that proposition (Locke, Essay, §I.2.16).

Locke is arguing that our knowledge of mathematical ideas, and logical maxims, is just like our knowledge of empirical claims. We don't know that, say, a whale is a mammal until we have knowledge of what those terms mean. Once we learn the meanings of the terms, then we can see that 'whales are mammals' is true. Similarly, according to Locke, we learn that 3+4=7 when we learn the meanings of 3, 4, 7, +, and =. Locke says that empiricism accounts for the temporal difference in learning 3+4=7 and 18+19=37. We learn the terms of the latter sentence later.

Poverty of evidence arguments show that the temporal order of our belief acquisition is irrelevant. Indeed, reliance on the temporal order of learning to reflect the order of justification is a species of the logical error called the genetic fallacy. It confuses the origin of one's ideas with their justification. I may learn that 2+2=4 by counting apples, but the truth of that claim is independent of how I learned it. There is an empirical element in the learning of terms, whether mathematical terms or empirical ones, and of associating terms with ideas. But, justification is independent of the temporal order. We are aware of particular claims before we know general ones. But, the general truths are more fundamental, in the order of justification, than the particular ones. We seek to reduce our knowledge to knowledge of axioms, which are simple and most perspicuous. Leibniz puts the point in terms of intellectual ideas, which are the innate ones.

Intellectual ideas, from which necessary truths arise, do not come from the senses...It is true that explicit knowledge of truths is subsequent (in temporal or natural order) to the explicit knowledge of ideas; as the nature of truths depends upon the nature of ideas, before either are explicitly formed, and truths involving ideas which come from the senses are themselves at least partly dependent on the senses. But the ideas that come from the senses are confused; and so too, at least in part, are the truths which depend on them; whereas intellectual ideas, and the truths depending on them, are distinct, and neither [the ideas nor the truths] originate in the senses; though it is true that without the senses we would never think of them (Leibniz, New Essays, 81).

Leibniz's rejection of the importance of temporal order in learning is probably the most important lesson we can take from the debate between Locke and Leibniz over innate ideas. The question whether mathematical truths are innate can not be decided by observing how we learn them. We have to look at the way in which we justify them, and their character. We have to see whether sense experience is sufficient for their justification, or whether we have to posit an innate capacity, or disposition, to learn them.

The rationalist accepts, in general, all the psychological capacities that the empiricist accepts. Thus, from a purely methodological perspective, the burden of proof is on the rationalists to establish the existence of innate ideas. Still, if the empiricist wants us to believe that there are no innate ideas, she must present a plausible positive account of our knowledge of mathematics. That's Locke's task.

VII. Restriction and Reclamation

As a rule, the empiricist has difficulty explaining our knowledge of mathematics. The empiricist claims that all knowledge arises from sense experience. Mathematical objects are not sensible. It is difficult to see how sense experience can support claims about mathematical objects. Furthermore, many mathematical claims are universal in nature. But, our experience is limited, and finite. Again, it is difficult to see how sense experience can support universal mathematical claims.

The rationalists, like Descartes and Leibniz, appealed to innate ideas to explain how we can have knowledge of universal claims about mathematical objects. Descartes appealed to our knowledge of infinitely many geometric objects with more characteristics than we could possibly sense. He used the example of a chiliagon. We have certain knowledge of an object that we could not have gained from the senses. Leibniz claimed that we have intuitive knowledge of the basic claims of mathematics, and intuitive knowledge of all of the steps by which we justify our beliefs in more complex statements. Again, such intuition could not, Leibniz claimed, be merely sensible, since we have universal knowledge of abstract objects.

Any empiricist account of our knowledge of mathematics can not rely on innate ideas. There are thus two obvious empiricist options. First, the empiricist can deny that we have mathematical knowledge. Berkeley chooses this first option, claiming that mathematics rests on a fundamental error of assuming the existence of non-sensible objects. In contemporary philosophy, fictionalists like Hartry Field also choose this first option, claiming that existential mathematical claims are false and universal mathematical claims are only vacuously true.

Alternately, the empiricist can try to account for mathematical knowledge using only our sense experience. Following Leibniz, Locke and Hume rely on demonstration and a limited form of intuition. Starting with ideas of sensation, they argue, we can use reason to discover relations among them.

> I do not doubt but it will be easily granted that the knowledge we have of mathematical truths is not only certain, but real knowledge, and not the bare empty vision of vain insignificant chimeras of the brain. And yet, if we will consider, we shall find that it is only of our own ideas (Locke, Essay, IV.4.6, p 8).

This latter empiricist strategy, the reclamation project, has two fronts. In one direction, it gives up some of the general principles supposedly known innately. Let's call this their restriction strategy. In the other direction, it attempts to reclaim some of the knowledge that was formerly thought to rely on innate ideas.

The empiricists have two sets of tools for their reclamation project. First, they have sensation, and any ideas which can be attributed to our sense experience. Second, they have the psychological capacities of our minds, including memory and the ability to reflect on our ideas. While they reject innate principles, they do not deny our natural capacity to reason and intuit.

VIII. Relations of Ideas

Hume and Locke are mainly in agreement in their development of the empiricist's reclamation project. Hume, as usual, has a nicer statement of the position.

> All of the objects of human reason or inquiry may naturally be divided into two kinds, namely, relations of ideas and matters of fact. Of the first kind are the sciences of geometry, algebra, and arithmetic, and, in short, every affirmation which is either intuitively or demonstratively certain. That the square of the hypotenuse is equal to the square of the two sides is a proposition which expresses a relation between these figures. That three times five is equal to the half of thirty expresses a relation between these numbers. Propositions of this kind are discoverable by the mere operation of thought, without dependence on what is anywhere existent in the universe. Though there never were a circle or triangle in nature, the truths demonstrated by Euclid would forever retain their certainty and evidence (Hume, Enquiry IV.1, p 3).

Hume claims that the basic tool for discovering whether a given statement is a relation of ideas is the principle of contradiction. The principle of contradiction says that if a statement entails a contradiction, then it is necessarily false. We use the principle of contradiction in proofs by reductio ad absurdum, or indirect proof. We know the mathematical claims that Hume cites because their negations are self-contradictory. Further, Hume believes that a statements can be known to be necessarily true only if its negation entails a contradiction. Hume proceeds to argue, in the Inquiry, and elsewhere, that many claims that have been accepted as certainly true, like statements of the laws of nature or of the existence and goodness of God, can not be so, since their negations are not contradictory.

The only objects of the abstract sciences or of demonstration are quantity and number...All other inquiries of men regard only matter of fact and existence and these are evidently incapable of demonstration. Whatever is may not be. No negation of a fact can involve a contradiction (Hume, Enquiry XII.3, p 4).

In other words, the principle of contradiction is both sufficient and necessary for justifying our knowledge of all necessary truths, including those of mathematics.

We are possessed of a precise standard by which we can judge of the equality and proportion of numbers and, according as they correspond or not to that standard, we determine their relations without any possibility of error (Hume, Treatise I.3.1, p 8).

Unfortunately, the principle of contradiction, by itself, can not do all the work. We need auxiliary tools to frame an hypothesis, and to determine whether a statement is in fact a contradiction. In the nineteenth and twentieth centuries, logicians following Frege developed a syntactic test for contradiction, by developing a formal language in which contradictions could be represented. A contradiction is any statement of the form . While Hume and the other moderns did not have this criterion, they of course understood that to assert any sentence and its negation was a contradiction. But, the account of how to know whether one sentence was a negation of another had yet to be developed. Both Locke and Hume thus appeal to our psychological ability to recognize contradictions. They also appeal to our ability to recognize identities, statements whose negations are contradictions.

Thus, there are actually two tools for determining whether a statement is a relation of ideas.

RI1. The principle of contradiction.
RI2. The imagination's ability to recognize similarity and difference.

IX. Intuition and Proof

Leibniz also appealed to these abilities in order to explain our knowledge of mathematics. Leibniz called an ability to recognize identities intuitive knowledge. As I mentioned earlier, the differences between Leibniz and Locke can seem small. Leibniz's account of our knowledge of mathematics appeals to either intuitive or symbolic knowledge of the axioms, along with a weaker class, adequate knowledge, of how theorems are derived from axioms. Locke appeals to what he calls intuitive and demonstrative knowledge. Intuitive knowledge is RI2.

If we will reflect on our own ways of thinking, we shall find that sometimes the mind perceives the agreement or disagreement of two ideas immediately by themselves, without the intervention of any other. And this, I think, we may call intuitive knowledge (Locke, Essay IV.2. 1, p 4).

Hume makes similar claims.

> Only four [philosophical relations], depending solely upon ideas, can be the objects of knowledge and certainty. These four are resemblance, contrariety, degrees in quality, and proportions in quantity or number. Three of these relations are discoverable at first sight and fall more properly under the province of intuition than demonstration (Hume, Treatise I.III.1, p 7).

Demonstrative knowledge uses RI1, and, more broadly, proofs.

> When the mind cannot so bring its ideas together, as by their immediate comparison and as it were juxtaposition or application one to another, to perceive their agreement or disagreement, it is inclined, by the intervention of other ideas (one or more, as it happens) to discover the agreement or disagreement which it searches; and this is that which we call reasoning (Locke, Essay IV.2.2, p 4).

In other words, for both Leibniz and the empiricists engaged in the reclamation project, we have both intuitive knowledge or immediate apprehension of some basic principles, and derivative knowledge of more complex statements. Leibniz claimed that intuitive knowledge could not be explained by sense experience. Locke and Hume, believing it to be just the result of a natural psychological ability to recognize similarities, differences, and contradictions, argue that this ability is acceptable to empiricists, and includes no appeal to innate ideas.

Moreover, both Locke and Leibniz believe that our beliefs based on demonstration are weaker than those which are immediately, or intuitively, apprehended. Leibniz classifies them as adequate, but neither symbolic nor intuitive, which are both more secure categories. Locke thinks that the certainty of our claims diminishes the longer our demonstrations extend.

> It is true the perception produced by demonstration is also very clear, yet it is often with a great abatement of that evident luster and full assurance that always accompany that which I call intuitive; like a face reflected by several mirrors one to another, where as long as it retains the similitude and agreement with the object, it produces a knowledge, but it is still, in every successive reflection, with a lessening of that perfect clearness and distinctiveness which is in the first; until at last, after many removes, it has a great mixture of dimness, and is not at first sight so knowable, especially to weak eyes. Thus it is with knowledge made out by a long train of proof (Locke, Essay IV.2.6, p 5).

Locke's mirror metaphor may be interpreted in two ways. First, he may be saying that demonstrative knowledge is ultimately less reliable because some of the surety that applies to intuitive knowledge leaks out of the proof, leaving an ultimately less secure claim. Or, he may be saying that the subjects of our proofs are not as immediately apparent, even though they have the same ultimate surety. In the first case, Locke would be reducing the extent to which our knowledge of mathematics is secure. In the second case, Locke would be merely explaining why some propositions, while proven, remain uncertain to some of us. Later, Frege and Whitehead and Russell attempt to provide a formal language to secure knowledge attained by derivations of any length. For the moderns, questions remain about the security of long proofs.

We have been noticing some affinities between the accounts of mathematics in Locke and Hume, who reject innate ideas, and Leibniz's nativist account. But, we have not discussed Descartes's master argument for innate ideas, the one which uses the chiliagon example. Descartes argues that we must accept innate ideas because without them we lack any account of our knowledge of mathematical objects which can not be acquired through the senses. Locke's mirror analogy may be taken as evidence of the empiricist's restriction strategy.

X. Hume and Berkeley on Restriction

Appeal to the restriction strategy, as a response to Descartes's master argument, is clearer in Hume's Treatise, though in a separate context.

> When we mention any great number, such as a thousand, the mind has generally no adequate idea of it, but only a power of producing such an idea by its adequate idea of the decimals under which the number is comprehended (Hume, Treatise I.1.7, p 6).

We have, Hume claims, no idea of large numbers, or of the chiliagon.
We can only represent it by means of other symbols or ideas.
Hume is particularly worried about the status of geometry, which has a closer connection to sense experience.
If our knowledge of mathematics depends only on a few, very basic psychological abilities, like the ability to recognize sameness and difference, and uses of proof guided by the principle of contradiction, then geometry, the subjects of which are objects extended in space, is suspect.
Spatial relations are generally known by sense experience.

> Geometry, or the art by which we fix the proportions of figures, though it much excels the loose judgments of the senses and imagination both in universality and exactness, yet never attains a perfect precision and exactness. Its first principles are still drawn from the general appearance of the objects, and that appearance can never afford us any security when we examine the prodigious minuteness of which nature is susceptible. Our ideas seem to give a perfect assurance that no two right lines can have a common segment, but if we consider these ideas, we shall find that they always suppose a sensible inclination of the two lines and that where the angle they form is extremely small, we have no standard of a right line so precise as to assure us of the truth of this proposition (Hume, Treatise I.3.1, p 7).

Thus, geometry, for Hume, which appeals to sensory, spatial intuitions fails to retain the certainty that applies to arithmetic.
We have a clear notion of identity in arithmetic.
We never (or only rarely) mistake one sheep for two.
But, we can easily mistake a curved line for straight.

On similar considerations, Berkeley denies that we have any mathematical knowledge.
He has no problem with the practice of mathematicians.

> That the principles laid down by mathematicians are true and their way of deduction from those principles clear and incontestable, we do not deny. But we hold there may be certain erroneous maxims of greater extend than the object of mathematics and for that reason not expressly mentioned, though tacitly supposed throughout the whole progress of that science... (Berkeley, Principles §118).

Berkeley does not complain about Locke's appeals to proof and demonstration.
His concern is more fundamental.
Locke, Berkeley, and Hume, and even Descartes and Leibniz, agree that if we are going to claim knowledge of mathematical statements, we have to have ideas to serve as the subjects of those statements.
All our beliefs are, and are composed of, mental representations.

For the empiricist, those mental representations must be acquired, or traceable back to, initial sense experiences.
We can call their view the picture theory of mind.
According to the picture theory, words, or other inscriptions stand for ideas which are individual pictures of objects.

Concepts are the contents of our ideas, and can be shared.

> Whatsoever doth or can exist, or be considered as one thing is positive: and so not only simple ideas and substances, but modes also, are positive beings: though the parts of which they consist are very often relative one to another: but the whole together considered as one thing, and producing in us the complex idea of one thing, which idea is in our minds, as one picture, though an aggregate of divers parts, and under one name, it is a positive or absolute thing, or idea. Thus a triangle, though the parts thereof compared one to another be relative, yet the idea of the whole is a positive absolute idea (Locke, Essay, II.25.6).

> ...thus it is with our ideas, which are as it were the pictures of things. No one of these mental draughts, however the parts are put together, can be called confused (for they are plainly discernible as they are) till it be ranked under some ordinary name to which it cannot be discerned to belong... (Locke, Essay, II.29.8).

As Descartes's master argument for nativism in mathematics points out, we have no ability to picture many, or all, mathematical objects. For Descartes, this inability motivated an account of ideas which separated thought from sensation. We can think without having a literal picture of an object in our minds. Descartes calls such thoughts pure. Since Locke, Berkeley, and Hume revert to an insistence that thoughts must be sensible, they have no recourse to pure, non-sensory ideas. Kant will pursue, as Descartes did, an alternative account of abstract ideas, ones on which thought is independent of sensation. But, for Locke and Hume, the problem persists.

Locke believes that he has a solution to the problem of accounting for the origins of ideas of mathematical objects, which has become known as the doctrine of abstract ideas.

XI. Abstract Ideas

Locke's doctrine of abstract ideas is essential to his account of our knowledge of mathematics. To understand the doctrine of abstract ideas, we have to think about the nature of language. Language was created to facilitate communication of our ideas, and improvement of our knowledge. In order to communicate, we need some object that can be shared among people. Words were thus devised to stand for ideas in our minds, to be sensible representations of inner states.

Locke's claims about language are controversial because we ordinarily take many words to stand for objects outside of our minds. We normally take ‘this table' to refer to the table, not to my idea of the table. In contrast, Locke says that the main function of words is as names for ideas in our minds. Here's an argument for Locke's claim, derived from §I of our reading, Book III of the Essay.

1. Society depends on our ability to communicate our ideas, so words have to be able to stand for ideas.
2. If ‘book' referred both to my idea of a book and something else (e.g. your idea, or the book itself), then it would be ambiguous in a way in which it is not.

3. Also, since my ideas precede my communication, words must refer to my ideas before they could refer to anything else.

4. So, it is impossible for words also to stand for something other than my ideas.

So, words stand for my ideas.

Locke claims that while names refer to our own ideas, we just suppose them to refer to other people's ideas, or for external objects.

> [It is] perverting the use of words, and bring[ing] unavoidable obscurity and confusion into their signification, whenever we make them stand for anything but those ideas we have in our own minds (Locke, Essay III.2.5).

While particular terms correspond to simple ideas, there are too many particular things for them all to have particular names. First, our capacity to learn and remember names is limited. Second, you don't have names for my ideas and I don't have names for yours. Third, science depends on generality. Thus, we use both particular names, for particular ideas when it is useful. And we devise general names for communication and for science.

General names are the foundation not only of empirical science, but of formal sciences like mathematics. We get knowledge of mathematical objects, which we do not experience, by a process of abstraction. We see doughnuts and frisbees, for examples, and focus only on their common shape to arrive at the idea of a circle. We leave out other properties, form an abstract idea, and coin a general term to stand for it.

> Words become general by being made the signs of general ideas: and ideas become general, by separating from them the circumstances of time and place, and any other ideas that may determine them to this or that particular existence. By this way of abstraction they are made capable of representing more individuals than one; each of which having in it a conformity to that abstract idea, is (as we call it) of that sort (Locke, Essay III.3.6, p 3).

Abstraction is required in other areas, as well. We experience extended things, but not extension itself. Any ideas of extension, size, or shape must arise from abstraction.

Let us consider this process of abstraction in a bit more detail. We start with our sense experiences, of several chairs, for example. We notice that they have common properties: backs, seats, legs. We give a name to whatever has these common properties. This name, ‘chair', is abstract, in the sense that it doesn't refer to a particular chair. Instead, it is a general term, which applies to any chair. The same process yields ‘table'. Now, we can consider the commonalities among tables and chairs, and sofas and desks. This yields an even more general term, ‘furniture'. We have abstracted again.

The same process which yields ‘chair' gives us other terms like ‘house' and ‘apartment building'. We can abstract again to get ‘domicile'. Similarly, we arrive at names like ‘animal', and ‘person'. All of the objects we have considered are extended. We can abstract again, and arrive at a term, ‘extension'. Similarly, we get the terms ‘motion' and ‘substance'. These ideas of bodies and motion are the foundations of physical science. A scientist uses ‘motion', for example, when he asserts ‘v = s/ t', that velocity is equal to the change in displacement over time. We can also abstract to the term, ‘physical object'. A progression of abstraction leads us from terms for particular sensations to terms for bodies. Our term ‘bodies' stands for an abstract idea, ‘bodies', which is a representation of an external object. So, the term ‘bodies', which we have constructed to stand for an abstract idea, refers to bodies, which are physical objects.

To account for mathematics, we abstract as well. We abstract the triangularity of triangular-shaped drawings from their specific properties: the chalk, the slight curve in one side, the location on the board. We ignore some properties and focus on others, like the triangularity. General terms, and the abstract ideas to which they refer, apply to particular objects, but only to certain aspects of those objects.

By the same way that they come by the general name and idea of man, they easily advance to more general names and notions. For, observing that several things that differ from their idea of man, and cannot therefore be comprehended under that name, have yet certain qualities wherein they agree with man, by retaining only those qualities, and uniting them into one idea, they have again another and more general idea; to which having given a name they make a term of a more comprehensive extension: which new idea is made, not by any new addition, but only as before, by leaving out the shape, and some other properties signified by the name man, and retaining only a body, with life, sense, and spontaneous motion, comprehended under the name animal (Locke, Essay III.3.8, p 3).

When we leave out the particular elements of our ideas and focus only on the mathematical elements, we can attain perfect generality. This generality yields the certainty of mathematics, since mathematical claims are only about our abstract ideas, and not about the external world. Furthermore, ethical ideas are, like mathematical ones, based on abstractions and also liable to certainty.

For certainty being but the perception of the agreement or disagreement of our ideas; and demonstration nothing but the perception of such agreement, by the intervention of other ideas or mediums, our moral ideas, as well as mathematical, being archetypes themselves, and so adequate and complete ideas; all the agreement or disagreement which we shall find in them will produce real knowledge, as well as in mathematical figures (Locke, Essay IV.4.7, p 8).

XII. Nominalism

Despite his claims that we have knowledge of mathematics, Locke is a nominalist about the referents of abstract ideas.

Universality does not belong to things themselves, which are all of them particular in their existence, even those words and ideas which in their signification are general. When therefore we quit particulars, the generals that rest are only creatures of our own making, their general nature being nothing but the capacity they are put into by the understanding of signifying or representing many particulars. For the signification they have is nothing but a relation that, by the mind of man, is added to them (Locke, III.3.11).

Similarly, despite claiming that mathematical truths are legitimate relations of ideas, Hume denies that there are any mathematical objects. In contemporary language, we can say that Locke and Hume are sentence (or propositional) realists, while remaining object (or metaphysical) nominalists about mathematics. They believe that mathematical sentences are true without believing that they denote any real objects. The sum of the angles of a triangle is 180 , though there are no real triangles.

Still, for both Locke and Hume, nominalism about objects is not intended to denigrate mathematical knowledge.

All the discourses of the mathematicians about the squaring of a circle, conic sections, or any other part of mathematics, do not concern the existence of any of those figures, but their demonstrations, which depend on their ideas, are the same, whether there is any square or circle existing in the world or not. In the same manner the truth and certainty of moral discourses abstract from the lives of men and the existence of those virtues in the world of which they treat (Locke, Essay IV.4.8, p 8).

XIII. Berkeley and Hume Against Abstract Ideas

Locke's faith in the doctrine of abstract ideas leads him to eschew the restriction strategy. Indeed, Locke believes that he has demonstrated that mathematics is securely known, and about our abstract ideas.

The knowledge we have of mathematical truths is not only certain, but real knowledge...The mathematician considers the truth and properties belonging to a rectangle or circle only as they are in ideas in his own mind (Locke, Essay IV.4.6, p 8).

Since we have immediate access to our own ideas, Locke need not worry about our making mistakes about our abstract ideas. But, the doctrine of abstract ideas is not itself secure.

To be plain, we suspect the mathematicians are, as well as other men, concerned in the errors arising from the doctrine of abstract ideas without the mind (Berkeley, Principles §118).

According to Locke, our ideas of primary qualities, like extension, correspond to real properties of real, material objects. But those ideas do not correspond to particular sensations. We experience an extended chair, say, but not the extension itself. In order to form the idea of extension in general, or even the extension of a particular chair, we have to strip away the other qualities in our minds to form a new and abstract idea. Berkeley and Hume claim that we can not form an abstract idea of body.

Berkeley further claims that since we can have no abstract ideas, there is no reason to claim that there are any bodies. Locke should extend his nominalism to the objects which purportedly correspond to all general terms, including terms for physical objects. Berkeley argues that the belief in the existence of the material world is based on mistaken reliance on the doctrine of abstract ideas.

If we thoroughly examine this tenet [materialism] it will, perhaps, be found at bottom to depend on the doctrine of abstract ideas. For can there be a nicer strain of abstraction than to distinguish the existence of sensible objects from their being perceived, so as to conceive them existing unperceived? Light and colors, heat and cold, extension and figures - in a word, the things we see and feel - what are they but so many sensations, notions, ideas, or impressions on the sense? And is it possible to separate, even in thought, any of these from perception? For my part, I might as easily divide a thing from itself. I may, indeed, divide in my thoughts, or conceive apart from each other, those things which, perhaps I never perceived by sense so divided. Thus, I imagine the trunk of a human body without the limbs, or conceive the smell of a rose without thinking on the rose itself. So far, I will not deny, I can abstract, if that may properly be called abstraction which extends only to the conceiving separately such objects as it is possible may really exist or be actually perceived asunder. But my conceiving or imagining power does not extend beyond the possibility of real existence or perception. Hence, as it is impossible for me to see or feel anything without an actual sensation of that thing, so is it impossible for me to conceive in my thoughts any sensible thing or object distinct from the sensation or perception of it. In truth, the object and the sensation are the same thing and cannot therefore be abstracted from each other (Principles §5, AW 447b-445a).

We will not pursue Berkeley's idealism here. Our concern is with how the problems with the doctrine of abstract ideas affect the empiricist's account of our knowledge of mathematics. Consider an abstract idea that corresponds to the general term 'triangle'. Locke claimed that such an idea stands for all triangles, whether scalene, isosceles, or equilateral. Berkeley denies that any such idea is possible.

If any man has the faculty of framing in his mind such an idea of a triangle as is here described, it is in vain to pretend to dispute him out of it, nor would I go about it. All I desire is that the reader would fully and certainly inform himself whether he has such an idea or not. And this, methinks, can be no hard task for anyone to perform. What is more easy than for anyone to look a little into his own thoughts, and there try whether he has, or can attain to have, an idea that shall correspond with the description that is... given [by Locke] of the general idea of a triangle, which is neither oblique nor rectangle, equilateral, equicrural nor scalenon, but all and none of these at once? (Berkeley, Principles Introduction §13, p 466).

This claim is the core of Berkeley's argument against abstract ideas. No idea, no picture in our minds, could have all of these properties at once. We can not have an idea of chair, because it would have to apply to all chairs. Some chairs are black, others are blue, or green. An idea which corresponds to all of these is impossible. No image will do as the idea of man, for it would have to be an image of a short man and a tall man, of a hairy man, and of a bald man.

Berkeley's claim against the doctrine of abstract ideas rests on his denial that we can have such ideas in our minds. He believes it is obvious that we have no abstract ideas.

Philonous: It is a universally received maxim that everything which exists is particular. How then can motion in general, or extension in general, exist in any corporeal substance?

Hylas: I will take time to solve your difficulty.

Philonous: But I think the point may be speedily decided. Without doubt you can tell whether you are able to frame this or that idea. Now I am content to put our dispute on this issue. If you can frame in your thoughts a distinct abstract idea of motion or extension, divested of all those sensible modes, as swift and slow, great and small, round and square, and the like, which are acknowledged to exist only in the mind, I will then yield the point you contend for. But if you cannot, it will be unreasonable on your side to insist any longer upon what you have no notion of.

Hylas: To confess ingenuously, I cannot (Berkeley, Three Dialogues Between Hylas and Philonous, First Dialogue).

In mathematics, and in science, we need terms, like 'triangle', which stand as universals, so that they refer to various different objects. Berkeley claims that we can use particular terms generally, without pretending to form abstract ideas.

A word becomes general by being made the sign, not of an abstract general idea, but of several particular ideas, any one of which it indifferently suggests to the mind. For example, when it is said the change of motion is proportional to the impressed force, or that whatever has extension is divisible, these propositions are to be understood of motion and extension in general, and nevertheless it will not follow that they suggest to my thoughts an idea of motion without a body moved, or any determinate direction and velocity, or that I must conceive an abstract general idea of extension, which is neither line, surface, nor solid, neither great nor small, black, white, nor red, nor of any other determinate color. It is only implied that whatever particular motion I consider, whether it is swift or slow, perpendicular, horizontal, or oblique, or in whatever object, the axiom concerning it holds equally true (Principles Introduction §11, AW 442a).

Our use of general terms, Berkeley says, should not mislead us into thinking that they correspond to some thing. Only particulars, single discrete sensations, exist.

Berkeley's criticism of Locke relies on the contradiction inherent in supposing that we can have a picture in our minds of a triangle which is scalene, isosceles, and equilateral. Hume points out that another option is to posit an idea of triangle which represents all of those properties by having none of them. But, Hume dismisses both of those possibilities. There can be no such abstract objects.

> It is a principle generally received in philosophy that everything in nature is individual and that it is utterly absurd to suppose a triangle really existent which has no precise proportion of sides and angles. If this, therefore, be absurd in fact and reality, it must also be absurd in idea, since nothing of which we can form a clear and distinct idea is absurd and impossible (Hume, Treatise I.1.7, p 5).

Given the picture theory of ideas, we do have some psychological capacities to alter the ideas of sensation, and to create new ones. We can combine parts of our ideas, as when we think of a centaur. We can consider some portions of an idea apart from others, as when we think about the door of a building, and not the walls or roof or windows. But, we can not form an abstract general idea, like the idea of a triangle, without thinking of a particular triangle, or like the idea of 250,737 without thinking of a particular symbol to stand for that number.

XIII. Empiricism Without Abstract Ideas

Given their rejection of Locke's doctrine of abstract ideas, Berkeley and Hume are faced with a new problem to account for our use of general ideas without admitting a psychological capacity for abstraction. That is, Locke designed the doctrine of abstract ideas in order to account for a serious phenomenon, our ability to speak generally, to use one term to stand for many. We obviously use terms like ‘chicken' to stand for many chickens, even if we only ever encounter individual chickens.

Berkeley and Hume both avoid Locke's problem of abstract ideas by claiming that a particular idea may stand for a variety of objects, for other ideas of the same sort.

> The image in the mind is only that of a particular object, though the application of it in our reasoning be the same as if it were universal (Hume, Treatise I.1.7, p 5).

An ability to speak generally is fundamental to mathematics and empirical science, where universal claims are ubiquitous. But while taking particulars to stand for other particulars avoids a commitment to abstract ideas, it may not succeed in supporting knowledge of those universal claims. The empiricist engaged in the reclamation project needs some account of our knowledge of mathematical objects which does not appeal to innate ideas, in the light of Descartes's master argument for nativism. Berkeley argues that no such account is possible. Since we can have no ideas of mathematical objects, we have no real mathematical knowledge, despite the security of our inferences.

> The theories, therefore, in arithmetic...can be supposed to have nothing at all for their object. Hence we may see how entirely the science of numbers is subordinate to practice and how jejune and trifling it becomes when considered as a matter of mere speculation (Berkeley, Principles §120).

Hume, in contrast to Berkeley, pursues the reclamation project. He thus has to explain how our particular ideas can support universal claims, by function as general ideas while remaining particular. In order to make our particular idea function as a general one, Hume claims, we re-purpose the idea, which is a psychological capacity different from abstraction.

> A particular idea becomes general by being annexed to a general term, that is, to a term which, from a customary conjunction, has a relation to many other particular ideas and readily recalls them in the imagination (Hume, Treatise I.1.7, p 6).

Hume believes that unlike Locke's doctrine of abstract ideas, this capacity to annex a particular idea to a general term is psychologically defensible. We can take objects to be of the same sort if they have any properties in common. All (Euclidean) triangles have their angle sums in common, so they are the same sort of triangles. But they do not have their side lengths in common, so they are not all scalene, etc.

Hume defends our ability to re-purpose individual ideas by providing examples.

> The most proper method, in my opinion, of giving a satisfactory explication of this act of the mind is by producing other instances which are analogous to it and other principles which facilitate its operation (Hume, Treatise I.1.7, p 6).

We use symbols, like numerical inscriptions. One particular idea or word can lead us to think of many different ones, as when the first notes of a song give us the whole tune. We can recall different component aspects of a general term, depending on the appropriate context. These psychological capacities may be unexplained or inexplicable, but they are also undeniable.

> Nothing is more admirable than the readiness with which the imagination suggests its ideas and presents them at the very instant in which they become necessary or useful (Hume, Treatise I.1.7, pp 6-7).

Just as Hume re-interpreted 'cause' to be a mental phenomenon, and explains inductions to be psychological habits, he explains general terms as arising from habits of use.

> If ideas be particular in their nature and at the same time finite in their number, it is only by custom they can become general in their representation and contain an infinite number of other ideas under them (Hume, Treatise I.1.7, p 7).

Thus, Berkeley and Hume differ on the lesson to be learned from the failure of Locke's doctrine. Berkeley pursues the nihilist project for mathematics, denying the existence of mathematical objects. Hume pursues the reclamation project, basing our knowledge of mathematics on the principle of contradiction and our bare psychological capacities.

XIV. Infinity and Infinitesimals

In addition to disagreeing with Hume over the infinite, or universal, representative ability of our ideas, Berkeley denies that we can have ideas of infinitely small things. And, this denial entails even more destructive claims about mathematics. Most famously, Berkeley was an early critic of the Newtonian/Leibnizian calculus, specifically for its reliance on infinitesimal quantities. In short, the central achievement of the calculus is to measure the area under a curve by adding up an infinite number of infinitely small areas (by integrating). But, these infinitely small areas, infinitesimals, made some people uncomfortable. Infinity was taken to be a property of God, or a property of God's properties. Infinitesimals were imagined to be nonsensical. Newton tried to explain away his reliance on infinitesimals (or 'indivisibles').

> Proofs are rendered more concise by the method of indivisibles. But since the hypothesis of indivisibles is problematical and this method is therefore accounted less geometrical, I have preferred to make the proofs of what follows depend on the ultimate sums and ratios of vanishing quantities and the first sums and ratios of nascent quantities, that is, on the limits of such sums and rations, and therefore to present proofs of those limits beforehand as briefly as I could. For the same result is obtained by these as by the method of indivisibles, and we shall be on safer ground using principles that have been proved (Newton, Principia Mathematica, Book I,§1, Lemma 11 scholium).

Infinitesimals were replaced (or explicated) in the late nineteenth century, by the work of Weierstrass (and others) who developed the rigorous epsilon-delta definition of a limit. Berkeley's criticisms of the calculus can thus seem amusing to the contemporary mathematician, who understands its foundations. But, at the time, Berkeley's worries were both serious and prescient.

Berkeley's central commitment is to all knowledge deriving from experience. From his empiricism, he derives an idealism: all that exists are either perceptions or perceivers; there are no material objects. Any purported object, like an infinitesimal, which is insensible in principle, must not be real. Belief in infinitesimals is the major fundamental error of mathematicians to which Berkeley alludes.

The infinite divisibility of finite extension...is throughout the same everywhere supposed and thought to have so inseparable and essential a connexion with the principles and demonstrations in geometry, that mathematicians never admit it into doubt, or make the least question of it. And, as this notion is the source from whence do spring all those amusing geometrical paradoxes which have such a direct repugnancy to the plain common sense of mankind, and are admitted with so much reluctance into a mind not yet debauched by learning; so it is the principal occasion of all that nice and extreme subtilty which renders the study of mathematics so difficult and tedious. Hence, if we can make it appear that no finite extension contains innumerable parts, or is infinitely divisible, it follows that we shall at once clear the science of geometry from a great number of difficulties and contradictions which have ever been esteemed a reproach to human reason, and withal make the attainment thereof a business of much less time and pains than it hitherto has been (Berkeley, Principles §123).

Instead of being infinitely divisible, Berkeley believes that objects of sense are composed of small, finite extensions (or, appearances of extension). The smallest part, the atom, is called the minimum sensibilia. It is the smallest perceivable extension. To get a sense of how big the minimum sensibilia is, Berkeley claims that the full moon is about thirty minimum sensibilia wide. If the minimum sensibilia is the smallest thing, not only the calculus, but any continuous geometry is fundamentally in error.

To say a finite quantity or extension consists of parts infinite in number is so manifest a contradiction that everyone at first sight acknowledges it to be so (Berkeley, Principles §124).

Berkeley accounts for this fundamental error by considering a map. Consider a map on which one inch represents a mile. There are certainly five thousand parts to a mile. So, one might conclude that there are five thousand parts to the inch, which on the map stands for a mile. Berkeley claims that this inference is the erroneous source of belief in the infinite divisibility of matter.

There is no such thing as the ten-thousandth part of an inch; but there is of a mile or diameter of the earth, which may be signified by that inch. When therefore I delineate a triangle on paper, and take one side not above an inch, for example, in length to be the radius, this I consider as divided into 10,000 or 100,000 parts or more; for, though the ten-thousandth part of that line considered in itself is nothing at all, and consequently may be neglected without an error or inconveniency, yet these described lines, being only marks standing for greater quantities, whereof it may be the ten-thousandth part is very considerable, it follows that, to prevent notable errors in practice, the radius must be taken of 10,000 parts or more... When we say a line is infinitely divisible, we must mean a line which is infinitely great (Berkeley, Principles §§127-8).

Berkeley's criticisms of the calculus are more sustained than we have seen. In particular, he has an essay, The Analyst, which makes the criticisms outlined here in greater detail.

XV. Conclusions

Descartes and Leibniz gave us certainty about mathematics, which seemed to inform also everything else, including science. That view seemed implausible, and relied on innateness. Locke and Berkeley tried to remove innate ideas, but fell upon the rocks of abstract ideas. Berkeley and Locke fail to separate mathematics as a distinct domain, untouched by the skepticism which Hume shows is the inevitable consequence of empiricism. The relations of ideas/matters of fact distinction helps Hume avoid both the skepticism which infects Locke's account and the nihilism which affects Berkeley's account. Hume is avoiding both abstract ideas and innate ideas, and still trying to find room for mathematics. The problem with Hume is that he still falls on the attempt to derive all ideas from sense impressions. Hume argues that our knowledge of geometry depends on its use in science, and that the objects of geometry are the same as scientific objects. So, he seems to be missing something about the nature of mathematics which is independent of science. Also, geometry becomes impugned along with all ideas, derived from impressions.

Berkeley's concerns are mainly metaphysical. He is so concerned to deny the existence of mathematical objects, that he insists that the claims that mathematicians make must be in error. Locke, in contrast, argues for the truth of mathematical claims on the basis of the fact that mathematical terms do not refer to anything outside our ideas. Hume does not even engage the metaphysical question, about the nature of mathematical objects. His concern is mainly to defend a Lockean view of the truth of mathematics, that it is just a system of relations of ideas and so our knowledge of it is protected.