Philosophy 405: Knowledge, Truth and Mathematics — Hamilton College
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Class #7: Modern Empiricism: Locke, Berkeley and Hume

I. Restriction and Reclamation

As a rule, the empiricist has difficulty explaining our knowledge of mathematics.
The empiricist claims that all knowledge arises from sense experience.
Mathematical objects are not sensible.
It is difficult to see how sense experience can support claims about mathematical objects.
Furthermore, many mathematical claims are universal in nature.
But, our experience is limited, and finite.
Again, it is difficult to see how sense experience can support universal mathematical claims.

The rationalists, like Descartes and Leibniz, appealed to innate ideas to explain how we can have knowledge of universal claims about mathematical objects.
Descartes appealed to our knowledge of infinitely many geometric objects with more characteristics than we could possibly sense.
He used the example of a chiliagon.
We have certain knowledge of an object that we could not have gained from the senses.
Leibniz claimed that we have intuitive knowledge of the basic claims of mathematics, and intuitive knowledge of all of the steps by which we justify our beliefs in more complex statements.
Again, such intuition could not, Leibniz claimed, be merely sensible, since we have universal knowledge of abstract objects.

Any empiricist account of our knowledge of mathematics can not rely on innate ideas.
There are thus two obvious empiricist options.
First, the empiricist can deny that we have mathematical knowledge.
Berkeley chooses this first option, claiming that mathematics rests on a fundamental error of assuming the existence of non-sensible objects.
In contemporary philosophy, fictionalists like Hartry Field also choose this first option, claiming that existential mathematical claims are false and universal mathematical claims are only vacuously true.

Alternately, the empiricist can try to account for mathematical knowledge using only our sense experience.
Following Leibniz, Locke and Hume rely on demonstration and a limited form of intuition.
Starting with ideas of sensation, they argue, we can use reason to discover relations among them.

> I do not doubt but it will be easily granted that the *knowledge* we have of *mathematical truths* is not only certain, but *real knowledge*, and not the bare empty vision of vain insignificant *chimeras* of the brain. And yet, if we will consider, we shall find that it is only of our own *ideas* (Locke, *Essay*, IV.4.6, p 8).

This latter empiricist strategy, the reclamation project, has two fronts.
In one direction, it gives up some of the general principles supposedly known innately.
Let's call this their restriction strategy.
In the other direction, it attempts to reclaim some of the knowledge that was formerly thought to rely on innate ideas.

The empiricists have two sets of tools for their reclamation project.
First, they have sensation, and any ideas which can be attributed to our sense experience.
Second, they have the psychological capacities of our minds, including memory and the ability to reflect on our ideas.
While they reject innate principles, they do not deny our natural capacity to reason and intuit.

II. Relations of Ideas

I take Hume and Locke to be essentially in agreement in their development of the empiricist's reclamation project.
Hume, as usual, has a nicer statement of the position.

> All of the objects of human reason or inquiry may naturally be divided into two kinds, namely, *relations of ideas* and *matters of fact*. Of the first kind are the sciences of geometry, algebra, and arithmetic, and, in short, every affirmation which is either intuitively or demonstratively certain. *That the square of the hypotenuse is equal to the square of the two sides* is a proposition which expresses a relation between these figures. *That three times five is equal to the half of thirty* expresses a relation between these numbers. Propositions of this kind are discoverable by the mere operation of thought, without dependence on what is anywhere existent in the universe. Though there never were a circle or triangle in nature, the truths demonstrated by Euclid would forever retain their certainty and evidence (Hume, *Enquiry* IV.1, p 3).

Hume claims that the basic tool for discovering whether a given statement is a relation of ideas is the principle of contradiction.
The principle of contradiction says that if a statement entails a contradiction, then it is necessarily false.
We use the principle of contradiction in proofs by *reductio ad absurdum*, or indirect proof.
We know the mathematical claims that Hume cites because their negations are self-contradictory.
Further, Hume believes that a statements can be known to be necessarily true only if its negation entails a contradiction.
Hume proceeds to argue, in the *Inquiry*, and elsewhere, that many claims that have been accepted as certainly true, like statements of the laws of nature or of the existence and goodness of God, can not be so, since their negations are not contradictory.

> The only objects of the abstract sciences or of demonstration are quantity and number...All other inquiries of men regard only matter of fact and existence and these are evidently incapable of demonstration. Whatever *is* may *not be*. No negation of a fact can involve a contradiction (Hume, *Enquiry* XII.3, p 4).

In other words, the principle of contradiction is both sufficient and necessary for justifying our knowledge of all necessary truths, including those of mathematics.

> We are possessed of a precise standard by which we can judge of the equality and proportion of numbers and, according as they correspond or not to that standard, we determine their relations without any possibility of error (Hume, *Treatise* I.3.1, p 8).

Unfortunately, the principle of contradiction, by itself, can not do all the work.
We need auxiliary tools to frame an hypothesis, and to determine whether a statement is in fact a

contradiction.
In the nineteenth and twentieth centuries, logicians following Frege developed a syntactic test for contradiction, by developing a formal language in which contradictions could be represented.
A contradiction is any statement of the form $\alpha \bullet \sim\alpha$.
While Hume and the other moderns did not have this criterion, they of course understood that to assert any sentence and its negation was a contradiction.
But, the account of how to know whether one sentence was a negation of another had yet to be developed.
Both Locke and Hume thus appeal to our psychological ability to recognize contradictions.
They also appeal to our ability to recognize identities, statements whose negations are contradictions.

Thus, there are actually two tools for determining whether a statement is a relation of ideas.

> RI1. The principle of contradiction.
> RI2. The imagination's ability to recognize similarity and difference.

III. Intuition and Proof

Leibniz also appealed to these abilities in order to explain our knowledge of mathematics.
Leibniz called an ability to recognize identities intuitive knowledge.
As I mentioned earlier, the differences between Leibniz and Locke can seem small.
Leibniz's account of our knowledge of mathematics appeals to either intuitive or symbolic knowledge of the axioms, along with a weaker class, adequate knowledge, of how theorems are derived from axioms.
Locke appeals to what he calls intuitive and demonstrative knowledge.
Intuitive knowledge is RI2.

> If we will reflect on our own ways of thinking, we shall find that sometimes the mind perceives the agreement or disagreement of two *ideas* immediately by themselves, without the intervention of any other. And this, I think, we may call *intuitive knowledge* (Locke, *Essay* IV.2. 1, p 4).

Hume makes similar claims.

> Only four [philosophical relations], depending solely upon ideas, can be the objects of knowledge and certainty. These four are *resemblance*, *contrariety*, *degrees in quality*, and *proportions in quantity or number*. Three of these relations are discoverable at first sight and fall more properly under the province of intuition than demonstration (Hume, *Treatise* I.III.1, p 7).

Demonstrative knowledge uses RI1, and, more broadly, proofs.

> When the mind cannot so bring its *ideas* together, as by their immediate comparison and as it were juxtaposition or application one to another, to perceive their agreement or disagreement, it is inclined, by the intervention of other *ideas* (one or more, as it happens) to discover the agreement or disagreement which it searches; and this is that which we call *reasoning* (Locke, *Essay* IV.2.2, p 4).

In other words, for both Leibniz and the empiricists engaged in the reclamation project, we have both intuitive knowledge or immediate apprehension of some basic principles, and derivative knowledge of

more complex statements.
Leibniz claimed that intuitive knowledge could not be explained by sense experience.
Locke and Hume, believing it to be just the result of a natural psychological ability to recognize similarities, differences, and contradictions, argue that this ability is acceptable to empiricists, and includes no appeal to innate ideas.

Moreover, both Locke and Leibniz believe that our beliefs based on demonstration are weaker than those which are immediately, or intuitively, apprehended.
Leibniz classifies them as adequate, but neither symbolic nor intuitive, which are both more secure categories.
Locke thinks that the certainty of our claims diminishes the longer our demonstrations extend.

> It is true the perception produced by *demonstration* is also very clear, yet it is often with a great abatement of that evident luster and full assurance that always accompany that which I call *intuitive*; like a face reflected by several mirrors one to another, where as long as it retains the similitude and agreement with the object, it produces a knowledge, but it is still, in every successive reflection, with a lessening of that perfect clearness and distinctiveness which is in the first; until at last, after many removes, it has a great mixture of dimness, and is not at first sight so knowable, especially to weak eyes. Thus it is with knowledge made out by a long train of proof (Locke, *Essay* IV.2.6, p 5).

Locke's mirror metaphor may be interpreted in two ways.
First, he may be saying that demonstrative knowledge is ultimately less reliable because some of the surety that applies to intuitive knowledge leaks out of the proof, leaving an ultimately less secure claim.
Or, he may be saying that the subjects of our proofs are not as immediately apparent, even though they have the same ultimate surety.
In the first case, Locke would be reducing the extent to which our knowledge of mathematics is secure.
In the second case, Locke would be merely explaining why some propositions, while proven, remain uncertain to some of us.
Later, Frege and Whitehead and Russell attempt to provide a formal language to secure knowledge attained by derivations of any length.
For the moderns, questions remain about the security of long proofs.

We have been noticing some affinities between the accounts of mathematics in Locke and Hume, who reject innate ideas, and Leibniz's nativist account.
But, we have not discussed Descartes's master argument for innate ideas, the one which uses the chiliagon example.
Descartes argues that we must accept innate ideas because without them we lack any account of our knowledge of mathematical objects which can not be acquired through the senses.
Locke's mirror analogy may be taken as evidence of the empiricist's restriction strategy.

IV. Hume and Berkeley on Restriction

Appeal to the restriction strategy, as a response to Descartes's master argument, is clearer in Hume's *Treatise*, though in a separate context.

> When we mention any great number, such as a thousand, the mind has generally no adequate idea

> of it, but only a power of producing such an idea by its adequate idea of the decimals under which the number is comprehended (Hume, *Treatise* I.1.7, p 6).

We have, Hume claims, no idea of large numbers, or of the chiliagon.
We can only represent it by means of other symbols or ideas.
Hume is particularly worried about the status of geometry, which has a closer connection to sense experience.
If our knowledge of mathematics depends only on a few, very basic psychological abilities, like the ability to recognize sameness and difference, and uses of proof guided by the principle of contradiction, then geometry, the subjects of which are objects extended in space, is suspect.
Spatial relations are generally known by sense experience.

> Geometry, or the *art* by which we fix the proportions of figures, though it much excels the loose judgments of the senses and imagination both in universality and exactness, yet never attains a perfect precision and exactness. Its first principles are still drawn from the general appearance of the objects, and that appearance can never afford us any security when we examine the prodigious minuteness of which nature is susceptible. Our ideas seem to give a perfect assurance that no two right lines can have a common segment, but if we consider these ideas, we shall find that they always suppose a sensible inclination of the two lines and that where the angle they form is extremely small, we have no standard of a right line so precise as to assure us of the truth of this proposition (Hume, *Treatise* I.3.1, p 7).

Thus, geometry, for Hume, which appeals to sensory, spatial intuitions fails to retain the certainty that applies to arithmetic.
We have a clear notion of identity in arithmetic.
We never (or only rarely) mistake one sheep for two.
But, we can easily mistake a curved line for straight.

On similar considerations, Berkeley denies that we have any mathematical knowledge.
He has no problem with the practice of mathematicians.

> That the principles laid down by mathematicians are true and their way of deduction from those principles clear and incontestable, we do not deny. But we hold there may be certain erroneous maxims of greater extend than the object of mathematics and for that reason not expressly mentioned, though tacitly supposed throughout the whole progress of that science... (Berkeley, *Principles* §118).

Berkeley does not complain about Locke's appeals to proof and demonstration.
His concern is more fundamental.
Locke, Berkeley, and Hume, and even Descartes and Leibniz, agree that if we are going to claim knowledge of mathematical statements, we have to have ideas to serve as the subjects of those statements.
All our beliefs are, and are composed of, mental representations.
For the empiricist, those mental representations must be acquired, or traceable back to, initial sense experiences.
We can call their view the picture theory of mind.
According to the picture theory, words, or other inscriptions stand for ideas which are individual pictures of objects.

Concepts are the contents of our ideas, and can be shared.

> Whatsoever doth or can exist, or be considered as one thing is positive: and so not only simple ideas and substances, but modes also, are positive beings: though the parts of which they consist are very often relative one to another: but the whole together considered as one thing, and producing in us the complex idea of one thing, which idea is in our minds, as one picture, though an aggregate of divers parts, and under one name, it is a positive or absolute thing, or idea. Thus a triangle, though the parts thereof compared one to another be relative, yet the idea of the whole is a positive absolute idea (Locke, *Essay*, II.25.6).
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> ...thus it is with our ideas, which are as it were the pictures of things. No one of these mental draughts, however the parts are put together, can be called confused (for they are plainly discernible as they are) till it be ranked under some ordinary name to which it cannot be discerned to belong... (Locke, *Essay*, II.29.8).

As Descartes's master argument for nativism in mathematics points out, we have no ability to picture many, or all, mathematical objects.
For Descartes, this inability motivated an account of ideas which separated thought from sensation.
We can think without having a literal picture of an object in our minds.
Descartes calls such thoughts pure.
Since Locke, Berkeley, and Hume revert to an insistence that thoughts must be sensible, they have no recourse to pure, non-sensory ideas.
Kant will pursue, as Descartes did, an alternative account of abstract ideas, ones on which thought is independent of sensation.
But, for Locke and Hume, the problem persists.

Locke believes that he has a solution to the problem of accounting for the origins of ideas of mathematical objects, which has become known as the doctrine of abstract ideas.

V. Abstract Ideas

Locke's doctrine of abstract ideas is essential to his account of our knowledge of mathematics.
To understand the doctrine of abstract ideas, we have to think about the nature of language.
Language was created to facilitate communication of our ideas, and improvement of our knowledge.
In order to communicate, we need some object that can be shared among people.
Words were thus devised to stand for ideas in our minds, to be sensible representations of inner states.

Locke's claims about language are controversial because we ordinarily take many words to stand for objects outside of our minds.
We normally take 'this table' to refer to the table, not to my idea of the table.
In contrast, Locke says that the main function of words is as names for ideas in our minds.
Here's an argument for Locke's claim, derived from §I of our reading, Book III of the *Essay*.

1. Society depends on our ability to communicate our ideas, so words have to be able to stand for ideas.
2. If 'book' referred both to my idea of a book and something else (e.g. your idea, or the book itself), then it would be ambiguous in a way in which it is not.

3. Also, since my ideas precede my communication, words must refer to my ideas before they could refer to anything else.
4. So, it is impossible for words also to stand for something other than my ideas.
So, words stand for my ideas.

Locke claims that while names refer to our own ideas, we just suppose them to refer to other people's ideas, or for external objects.

> [It is] perverting the use of words, and bring[ing] unavoidable obscurity and confusion into their signification, whenever we make them stand for anything but those ideas we have in our own minds (Locke, *Essay* III.2.5).

While particular terms correspond to simple ideas, there are too many particular things for them all to have particular names.
First, our capacity to learn and remember names is limited.
Second, you don't have names for my ideas and I don't have names for yours.
Third, science depends on generality.
Thus, we use both particular names, for particular ideas when it is useful.
And we devise general names for communication and for science.

General names are the foundation not only of empirical science, but of formal sciences like mathematics.
We get knowledge of mathematical objects, which we do not experience, by a process of abstraction.
We see doughnuts and frisbees, for examples, and focus only on their common shape to arrive at the idea of a circle.
We leave out other properties, form an abstract idea, and coin a general term to stand for it.

> Words become general by being made the signs of general ideas: and ideas become general, by separating from them the circumstances of time and place, and any other ideas that may determine them to this or that particular existence. By this way of abstraction they are made capable of representing more individuals than one; each of which having in it a conformity to that abstract idea, is (as we call it) of that sort (Locke, *Essay* III.3.6, p 3).

Abstraction is required in other areas, as well.
We experience extended things, but not extension itself.
Any ideas of extension, size, or shape must arise from abstraction.

Let us consider this process of abstraction in a bit more detail.
We start with our sense experiences, of several chairs, for example.
We notice that they have common properties: backs, seats, legs.
We give a name to whatever has these common properties.
This name, 'chair', is abstract, in the sense that it doesn't refer to a particular chair.
Instead, it is a general term, which applies to any chair.
The same process yields 'table'.
Now, we can consider the commonalities among tables and chairs, and sofas and desks.
This yields an even more general term, 'furniture'.
We have abstracted again.

The same process which yields 'chair' gives us other terms like 'house' and 'apartment building'.

We can abstract again to get 'domicile'.
Similarly, we arrive at names like 'animal', and 'person'.
All of the objects we have considered are extended.
We can abstract again, and arrive at a term, 'extension'.
Similarly, we get the terms 'motion' and 'substance'.
These ideas of bodies and motion are the foundations of physical science.
A scientist uses 'motion', for example, when he asserts '$v = \Delta s/\Delta t$', that velocity is equal to the change in displacement over time.
We can also abstract to the term, 'physical object'.
A progression of abstraction leads us from terms for particular sensations to terms for bodies.
Our term 'bodies' stands for an abstract idea, 'bodies', which is a representation of an external object.
So, the term 'bodies', which we have constructed to stand for an abstract idea, refers to bodies, which are physical objects.

To account for mathematics, we abstract as well.
We abstract the triangularity of triangular-shaped drawings from their specific properties: the chalk, the slight curve in one side, the location on the board.
We ignore some properties and focus on others, like the triangularity.
General terms, and the abstract ideas to which they refer, apply to particular objects, but only to certain aspects of those objects.

> By the same way that they come by the general name and idea of man, they easily advance to more general names and notions. For, observing that several things that differ from their idea of man, and cannot therefore be comprehended under that name, have yet certain qualities wherein they agree with man, by retaining only those qualities, and uniting them into one idea, they have again another and more general idea; to which having given a name they make a term of a more comprehensive extension: which new idea is made, not by any new addition, but only as before, by leaving out the shape, and some other properties signified by the name man, and retaining only a body, with life, sense, and spontaneous motion, comprehended under the name animal (Locke, *Essay* III.3.8, p 3).

When we leave out the particular elements of our ideas and focus only on the mathematical elements, we can attain perfect generality.
This generality yields the certainty of mathematics, since mathematical claims are only about our abstract ideas, and not about the external world.
Furthermore, ethical ideas are, like mathematical ones, based on abstractions and also liable to certainty.

> For certainty being but the perception of the agreement or disagreement of our *ideas*; and demonstration nothing but the perception of such agreement, by the intervention of other *ideas* or mediums, our moral *ideas*, as well as mathematical, being archetypes themselves, and so adequate and complete *ideas*; all the agreement or disagreement which we shall find in them will produce real knowledge, as well as in mathematical figures (Locke, *Essay* IV.4.7, p 8).

VI. Nominalism

Despite his claims that we have knowledge of mathematics, Locke is a nominalist about the referents of abstract ideas.

> Universality does not belong to things themselves, which are all of them particular in their existence, even those words and *ideas* which in their signification are general. When therefore we quit particulars, the generals that rest are only creatures of our own making, their general nature being nothing but the capacity they are put into by the understanding of signifying or representing many particulars. For the signification they have is nothing but a relation that, by the mind of man, is added to them (Locke, III.3.11).

Similarly, despite claiming that mathematical truths are legitimate relations of ideas, Hume denies that there are any mathematical objects.
In contemporary language, we can say that Locke and Hume are sentence (or propositional) realists, while remaining object (or metaphysical) nominalists about mathematics.
They believe that mathematical sentences are true without believing that they denote any real objects.
The sum of the angles of a triangle is $180°$, though there are no real triangles.

Still, for both Locke and Hume, nominalism about objects is not intended to denigrate mathematical knowledge.

> All the discourses of the mathematicians about the squaring of a circle, conic sections, or any other part of mathematics, *do not concern* the *existence* of any of those figures, but their demonstrations, which depend on their *ideas*, are the same, whether there is any square or circle existing in the world or not. In the same manner the truth and certainty of *moral* discourses abstract from the lives of men and the existence of those virtues in the world of which they treat (Locke, *Essay* IV.4.8, p 8).

VII. Berkeley and Hume Against Abstract Ideas

Locke's faith in the doctrine of abstract ideas leads him to eschew the restriction strategy.
Indeed, Locke believes that he has demonstrated that mathematics is securely known, and about our abstract ideas.

> The knowledge we have of mathematical truths is not only certain, but real knowledge...The mathematician considers the truth and properties belonging to a rectangle or circle only as they are in ideas in his own mind (Locke, *Essay* IV.4.6, p 8).

Since we have immediate access to our own ideas, Locke need not worry about our making mistakes about our abstract ideas.
But, the doctrine of abstract ideas is not itself secure.

> To be plain, we suspect the mathematicians are, as well as other men, concerned in the errors arising from the doctrine of abstract ideas without the mind (Berkeley, *Principles* §118).

According to Locke, our ideas of primary qualities, like extension, correspond to real properties of real, material objects.
But those ideas do not correspond to particular sensations.
We experience an extended chair, say, but not the extension itself.
In order to form the idea of extension in general, or even the extension of a particular chair, we have to strip away the other qualities in our minds to form a new and abstract idea.

Berkeley and Hume claim that we can not form an abstract idea of body.

Berkeley further claims that since we can have no abstract ideas, there is no reason to claim that there are any bodies.
Locke should extend his nominalism to the objects which purportedly correspond to all general terms, including terms for physical objects.
Berkeley argues that the belief in the existence of the material world is based on mistaken reliance on the doctrine of abstract ideas.

> If we thoroughly examine this tenet [materialism] it will, perhaps, be found at bottom to depend on the doctrine of *abstract ideas*. For can there be a nicer strain of abstraction than to distinguish the existence of sensible objects from their being perceived, so as to conceive them existing unperceived? Light and colors, heat and cold, extension and figures - in a word, the things we see and feel - what are they but so many sensations, notions, ideas, or impressions on the sense? And is it possible to separate, even in thought, any of these from perception? For my part, I might as easily divide a thing from itself. I may, indeed, divide in my thoughts, or conceive apart from each other, those things which, perhaps I never perceived by sense so divided. Thus, I imagine the trunk of a human body without the limbs, or conceive the smell of a rose without thinking on the rose itself. So far, I will not deny, I can abstract, if that may properly be called *abstraction* which extends only to the conceiving separately such objects as it is possible may really exist or be actually perceived asunder. But my conceiving or imagining power does not extend beyond the possibility of real existence or perception. Hence, as it is impossible for me to see or feel anything without an actual sensation of that thing, so is it impossible for me to conceive in my thoughts any sensible thing or object distinct from the sensation or perception of it. In truth, the object and the sensation are the same thing and cannot therefore be abstracted from each other (*Principles* §5, AW 447b-445a).

We will not pursue Berkeley's idealism here.
Our concern is with how the problems with the doctrine of abstract ideas affect the empiricist's account of our knowledge of mathematics.
Consider an abstract idea that corresponds to the general term 'triangle'.
Locke claimed that such an idea stands for all triangles, whether scalene, isosceles, or equilateral.
Berkeley denies that any such idea is possible.

> If any man has the faculty of framing in his mind such an idea of a triangle as is here described, it is in vain to pretend to dispute him out of it, nor would I go about it. All I desire is that the reader would fully and certainly inform himself whether he has such an idea or not. And this, methinks, can be no hard task for anyone to perform. What is more easy than for anyone to look a little into his own thoughts, and there try whether he has, or can attain to have, an idea that shall correspond with the description that is... given [by Locke] of the general idea of a triangle, which is *neither oblique nor rectangle, equilateral, equicrural nor scalenon, but all and none of these at once*? (Berkeley, *Principles* Introduction §13, p 466).

This claim is the core of Berkeley's argument against abstract ideas.
No idea, no picture in our minds, could have all of these properties at once.
We can not have an idea of chair, because it would have to apply to all chairs.
Some chairs are black, others are blue, or green.
An idea which corresponds to all of these is impossible.

No image will do as the idea of man, for it would have to be an image of a short man and a tall man, of a hairy man, and of a bald man.

Berkeley's claim against the doctrine of abstract ideas rests on his denial that we can have such ideas in our minds.
He believes it is obvious that we have no abstract ideas.

> *Philonous*: It is a universally received maxim that *everything which exists is particular*. How then can motion in general, or extension in general, exist in any corporeal substance?
> *Hylas*: I will take time to solve your difficulty.
> *Philonous*: But I think the point may be speedily decided. Without doubt you can tell whether you are able to frame this or that idea. Now I am content to put our dispute on this issue. If you can frame in your thoughts a distinct abstract idea of motion or extension, divested of all those sensible modes, as swift and slow, great and small, round and square, and the like, which are acknowledged to exist only in the mind, I will then yield the point you contend for. But if you cannot, it will be unreasonable on your side to insist any longer upon what you have no notion of.
> *Hylas*: To confess ingenuously, I cannot (Berkeley, *Three Dialogues Between Hylas and Philonous*, First Dialogue).

In mathematics, and in science, we need terms, like 'triangle', which stand as universals, so that they refer to various different objects.
Berkeley claims that we can use particular terms generally, without pretending to form abstract ideas.

> A word becomes general by being made the sign, not of an abstract general idea, but of several particular ideas, any one of which it indifferently suggests to the mind. For example, when it is said *the change of motion is proportional to the impressed force*, or that *whatever has extension is divisible*, these propositions are to be understood of motion and extension in general, and nevertheless it will not follow that they suggest to my thoughts an idea of motion without a body moved, or any determinate direction and velocity, or that I must conceive an abstract general idea of extension, which is neither line, surface, nor solid, neither great nor small, black, white, nor red, nor of any other determinate color. It is only implied that whatever particular motion I consider, whether it is swift or slow, perpendicular, horizontal, or oblique, or in whatever object, the axiom concerning it holds equally true (*Principles* Introduction §11, AW 442a).

Our use of general terms, Berkeley says, should not mislead us into thinking that they correspond to some thing.
Only particulars, single discrete sensations, exist.

Berkeley's criticism of Locke relies on the contradiction inherent in supposing that we can have a picture in our minds of a triangle which is scalene, isosceles, and equilateral.
Hume points out that another option is to posit an idea of triangle which represents all of those properties by having none of them.
But, Hume dismisses both of those possibilities.
There can be no such abstract objects.

> It is a principle generally received in philosophy that everything in nature is individual and that it is utterly absurd to suppose a triangle really existent which has no precise proportion of sides and

> angles. If this, therefore, be absurd in *fact and reality*, it must also be absurd in *idea*, since nothing of which we can form a clear and distinct idea is absurd and impossible (Hume, *Treatise* I.1.7, p 5).

Given the picture theory of ideas, we do have some psychological capacities to alter the ideas of sensation, and to create new ones.
We can combine parts of our ideas, as when we think of a centaur.
We can consider some portions of an idea apart from others, as when we think about the door of a building, and not the walls or roof or windows.
But, we can not form an abstract general idea, like the idea of a triangle, without thinking of a particular triangle, or like the idea of 250,737 without thinking of a particular symbol to stand for that number.

VII. Empiricism Without Abstract Ideas

Given their rejection of Locke's doctrine of abstract ideas, Berkeley and Hume are faced with a new problem to account for our use of general ideas without admitting a psychological capacity for abstraction.
That is, Locke designed the doctrine of abstract ideas in order to account for a serious phenomenon, our ability to speak generally, to use one term to stand for many.
We obviously use terms like 'chicken' to stand for many chickens, even if we only ever encounter individual chickens.

Berkeley and Hume both avoid Locke's problem of abstract ideas by claiming that a particular idea may stand for a variety of objects, for other ideas of the same sort.

> The image in the mind is only that of a particular object, though the application of it in our reasoning be the same as if it were universal (Hume, *Treatise* I.1.7, p 5).

An ability to speak generally is fundamental to mathematics and empirical science, where universal claims are ubiquitous.
But while taking particulars to stand for other particulars avoids a commitment to abstract ideas, it may not succeed in supporting knowledge of those universal claims.
The empiricist engaged in the reclamation project needs some account of our knowledge of mathematical objects which does not appeal to innate ideas, in the light of Descartes's master argument for nativism.
Berkeley argues that no such account is possible.
Since we can have no ideas of mathematical objects, we have no real mathematical knowledge, despite the security of our inferences.

> The theories, therefore, in arithmetic...can be supposed to have nothing at all for their object. Hence we may see how entirely the science of numbers is subordinate to practice and how jejune and trifling it becomes when considered as a matter of mere speculation (Berkeley, *Principles* §120).

Hume, in contrast to Berkeley, pursues the reclamation project.
He thus has to explain how our particular ideas can support universal claims, by function as general ideas while remaining particular.
In order to make our particular idea function as a general one, Hume claims, we re-purpose the idea,

which is a psychological capacity different from abstraction.

> A particular idea becomes general by being annexed to a general term, that is, to a term which, from a customary conjunction, has a relation to many other particular ideas and readily recalls them in the imagination (Hume, *Treatise* I.1.7, p 6).

Hume believes that unlike Locke's doctrine of abstract ideas, this capacity to annex a particular idea to a general term is psychologically defensible.
We can take objects to be of the same sort if they have any properties in common.
All (Euclidean) triangles have their angle sums in common, so they are the same sort of triangles.
But they do not have their side lengths in common, so they are not all scalene, etc.

Hume defends our ability to re-purpose individual ideas by providing examples.

> The most proper method, in my opinion, of giving a satisfactory explication of this act of the mind is by producing other instances which are analogous to it and other principles which facilitate its operation (Hume, *Treatise* I.1.7, p 6).

We use symbols, like numerical inscriptions.
One particular idea or word can lead us to think of many different ones, as when the first notes of a song give us the whole tune.
We can recall different component aspects of a general term, depending on the appropriate context.
These psychological capacities may be unexplained or inexplicable, but they are also undeniable.

> Nothing is more admirable than the readiness with which the imagination suggests its ideas and presents them at the very instant in which they become necessary or useful (Hume, *Treatise* I.1.7, pp 6-7).

Just as Hume re-interpreted 'cause' to be a mental phenomenon, and explains inductions to be psychological habits, he explains general terms as arising from habits of use.

> If ideas be particular in their nature and at the same time finite in their number, it is only by custom they can become general in their representation and contain an infinite number of other ideas under them (Hume, *Treatise* I.1.7, p 7).

Thus, Berkeley and Hume differ on the lesson to be learned from the failure of Locke's doctrine.
Berkeley pursues the nihilist project for mathematics, denying the existence of mathematical objects.
Hume pursues the reclamation project, basing our knowledge of mathematics on the principle of contradiction and our bare psychological capacities.

VIII. Infinity and Infinitesimals

In addition to disagreeing with Hume over the infinite, or universal, representative ability of our ideas, Berkeley denies that we can have ideas of infinitely small things.
And, this denial entails even more destructive claims about mathematics.
Most famously, Berkeley was an early critic of the Newtonian/Leibnizian calculus, specifically for its reliance on infinitesimal quantities.

In short, the central achievement of the calculus is to measure the area under a curve by adding up an infinite number of infinitely small areas (by integrating).
But, these infinitely small areas, infinitesimals, made some people uncomfortable.
Infinity was taken to be a property of God, or a property of God's properties.
Infinitesimals were imagined to be nonsensical.
Newton tried to explain away his reliance on infinitesimals (or 'indivisibles').

> Proofs are rendered more concise by the method of indivisibles. But since the hypothesis of indivisibles is problematical and this method is therefore accounted less geometrical, I have preferred to make the proofs of what follows depend on the ultimate sums and ratios of vanishing quantities and the first sums and ratios of nascent quantities, that is, on the limits of such sums and rations, and therefore to present proofs of those limits beforehand as briefly as I could. For the same result is obtained by these as by the method of indivisibles, and we shall be on safer ground using principles that have been proved (Newton, *Principia Mathematica*, Book I,§1, Lemma 11 scholium).

Infinitesimals were replaced (or explicated) in the late nineteenth century, by the work of Weierstrass (and others) who developed the rigorous epsilon-delta definition of a limit.
Berkeley's criticisms of the calculus can thus seem amusing to the contemporary mathematician, who understands its foundations.
But, at the time, Berkeley's worries were both serious and prescient.

Berkeley's central commitment is to all knowledge deriving from experience.
From his empiricism, he derives an idealism: all that exists are either perceptions or perceivers; there are no material objects.
Any purported object, like an infinitesimal, which is insensible in principle, must not be real.
Belief in infinitesimals is the major fundamental error of mathematicians to which Berkeley alludes.

> The infinite divisibility of finite extension...is throughout the same everywhere supposed and thought to have so inseparable and essential a connexion with the principles and demonstrations in geometry, that mathematicians never admit it into doubt, or make the least question of it. And, as this notion is the source from whence do spring all those amusing geometrical paradoxes which have such a direct repugnancy to the plain common sense of mankind, and are admitted with so much reluctance into a mind not yet debauched by learning; so it is the principal occasion of all that nice and extreme subtilty which renders the study of mathematics so difficult and tedious. Hence, if we can make it appear that no finite extension contains innumerable parts, or is infinitely divisible, it follows that we shall at once clear the science of geometry from a great number of difficulties and contradictions which have ever been esteemed a reproach to human reason, and withal make the attainment thereof a business of much less time and pains than it hitherto has been (Berkeley, *Principles* §123).

Instead of being infinitely divisible, Berkeley believes that objects of sense are composed of small, finite extensions (or, appearances of extension).
The smallest part, the atom, is called the *minimum sensibilia*.
It is the smallest perceivable extension.
To get a sense of how big the *minimum sensibilia* is, Berkeley claims that the full moon is about thirty *minimum sensibilia* wide.
If the *minimum sensibilia* is the smallest thing, not only the calculus, but any continuous geometry is

fundamentally in error.

> To say a finite quantity or extension consists of parts infinite in number is so manifest a contradiction that everyone at first sight acknowledges it to be so (Berkeley, *Principles* §124).

Berkeley accounts for this fundamental error by considering a map.
Consider a map on which one inch represents a mile.
There are certainly five thousand parts to a mile.
So, one might conclude that there are five thousand parts to the inch, which on the map stands for a mile.
Berkeley claims that this inference is the erroneous source of belief in the infinite divisibility of matter.

> There is no such thing as the ten-thousandth part of an inch; but there is of a mile or diameter of the earth, which may be signified by that inch. When therefore I delineate a triangle on paper, and take one side not above an inch, for example, in length to be the radius, this I consider as divided into 10,000 or 100,000 parts or more; for, though the ten-thousandth part of that line considered in itself is nothing at all, and consequently may be neglected without an error or inconveniency, yet these described lines, being only marks standing for greater quantities, whereof it may be the ten-thousandth part is very considerable, it follows that, to prevent notable errors in practice, the radius must be taken of 10,000 parts or more... When we say a line is infinitely divisible, we must mean a line which is infinitely great (Berkeley, *Principles* §§127-8).

Berkeley's criticisms of the calculus are more sustained than we have seen.
In particular, he has an essay, *The Analyst*, which makes the criticisms outlined here in greater detail.

IX. Conclusions

Descartes and Leibniz gave us certainty about mathematics, which seemed to inform also everything else, including science.
That view seemed implausible, and relied on innateness.
Locke and Berkeley tried to remove innate ideas, but fell upon the rocks of abstract ideas.
Berkeley and Locke fail to separate mathematics as a distinct domain, untouched by the skepticism which Hume shows is the inevitable consequence of empiricism.
The relations of ideas/matters of fact distinction helps Hume avoid both the skepticism which infects Locke's account and the nihilism which affects Berkeley's account.
Hume is avoiding both abstract ideas and innate ideas, and still trying to find room for mathematics.
The problem with Hume is that he still falls on the attempt to derive all ideas from sense impressions.
Hume argues that our knowledge of geometry depends on its use in science, and that the objects of geometry are the same as scientific objects.
So, he seems to be missing something about the nature of mathematics which is independent of science.
Also, geometry becomes impugned along with all ideas, derived from impressions.

Berkeley's concerns are mainly metaphysical.
He is so concerned to deny the existence of mathematical objects, that he insists that the claims that mathematicians make must be in error.
Locke, in contrast, argues for the truth of mathematical claims on the basis of the fact that mathematical terms do not refer to anything outside our ideas.
Hume does not even engage the metaphysical question, about the nature of mathematical objects.

His concern is mainly to defend a Lockean view of the truth of mathematics, that it is just a system of relations of ideas and so our knowledge of it is protected.