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Class #5-6: Modern Rationalism
Sample Introductory Material from
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I. Philosophy of Mathematics and the Early Modern Rationalists

We are often tempted to think that sense experience is the source of most or all of our knowledge and thus that mathematical knowledge should somehow derive from sensation. On such a view, sense experience plays a foundational role, as the source of our knowledge, perhaps, or as the evidence for our theories. What we receive immediately in sensation, on this tempting view, has a primacy, a special status. Other beliefs may be derivative.

For two early modern rationalists, Descartes and Leibniz, this empiricist view gets the hierarchy of knowledge backward. Special status should be conferred not to beliefs directly or indirectly derived or justified by sense experience, but to beliefs based on pure reason, thoughts which are supposed to be independent from, or prior to, sense experience.

At the beginning of the *Meditations*, Descartes worries that he has, or might have, some false beliefs. He wonders about misleading sense perceptions, like illusions; how we know whether we are dreaming; and whether or not a powerful deceiver is putting false thoughts in our minds. The *Meditations* is a broad attempt to account for all of our knowledge and to defeat the skepticism to which such doubts may lead.

Descartes is attempting to understand the consequences of the principles of the scientific revolution. Galilean mechanics and the new Copernican, heliocentric model of the solar system were seen, especially by the Catholic Church, as antagonistic to religion. Galileo's censorship by the Inquisition motivated Descartes to suppress publication of *Le Monde*, his work on the philosophical implications of the new science. Descartes's goal, in the *Meditations*, is a system which will both defeat skepticism and accommodate the new science while not undermining what he thought of as central Church doctrine.

Descartes thus presents the *Meditations* as a proof of both the existence of God and the immortality of the soul as well as a new approach to scientific method. In doing so, Descartes separates thought from sensation. Thinking is the function, indeed the essential aspect, of the soul. Sensation, while it has some conscious correlates, is essentially physical. Descartes can thus support the Church's view of the soul, apart from the body, against Aristotelian accounts on which the soul is the form of the body. Our knowledge of mathematics, for Descartes, is the product of this pure, non-sensory, rational thought. Leibniz follows Descartes, refining the rationalist view of mathematical knowledge. For the rationalists, beliefs which depend on the senses are tenuous, while mathematical beliefs, unsullied by sensation, are elevated to pure truths. We can know them certainly, because they are the result of pure reason.

Both Descartes and Leibniz were important mathematicians in addition to being seminal philosophers. Descartes developed analytic geometry, applying algebraic methods to geometry, and facilitated the development of general theories of curves and functions. The Cartesian plane, defined by the intersection of perpendicular axes, is named for Descartes. Leibniz developed calculus, concurrently with, but independently from, Newton. Both Descartes and Leibniz saw their mathematical developments as arising from their rationalist methods, as did others like Galileo, Huygens, and Newton.

Mathematical developments were spurred by empirical science, but even the methods of empirical science were sometimes as purely rational as they were experimental. Concerning his claim that a stone falling from a ship's mast will drop in the same place whether a ship is stationary or moving at a constant rate, Galileo writes, "So, you have not made a hundred tests, or even one? And yet you so freely declare it to be certain?... Without experiment, I am sure that the effect will happen as I tell you, because it must happen that way" (Galileo, *Dialogue Concerning the Two Chief World Systems*, p 145).

Descartes went further than most of his contemporaries in his view of the role of mathematics, and mathematical methods, in science. Whereas most scientists of the seventeenth and eighteenth centuries were impressed by the advances that mathematics made for science, Descartes thought that the world was essentially mathematical. For Descartes, the objectivity of science and the objectivity of mathematics were inseparable.

II. Descartes's Views On Mathematics

Descartes's work in the philosophy of mathematics can be distilled to two metaphysical claims and one epistemological one. Descartes's metaphysical claims are, first, that mathematical objects have a determinate, objective nature, independent of us; and, second, that mathematical truths are necessary. His argument for the first metaphysical claim depends on our inability to affect mathematical truth.

DM1
1. A thing's nature depends on me if I can make it any way I like.
2. A thing's nature is objective if I can not make it any way I like.
3. I can not make mathematical objects any way I would like.
So, mathematical objects are objective.

When Descartes claims that mathematical truths are necessary truths, he intends mainly to characterize them as innate and eternal. Still, his commitment to the omnipotence of God debars him from asserting that their necessity transcends or constrains God in any way. In fact, Descartes does not think that any truths are necessary in the sense that they are independent of God's will. But even for Descartes, mathematical truths are necessary as far as we humans can understand necessity.

Descartes's epistemological claim is that our knowledge of the truths of mathematics can not come from the senses. So, it must be innate. His argument for the epistemological claim is explicit in the Third Meditation.

DE
1. All ideas must be invented, acquired, or innate.
2. Mathematical truths can not be invented.
3. Mathematical truths can not be acquired.
So, they must be innate.

Premise 1 describes three options. Descartes's support for premise 2 comes directly from the first metaphysical claim, DM1. Descartes defends premise 3 with an example of a one-thousand-sided figure, called a chiliagon. We can not imagine such a figure, certainly not distinctly enough to derive certain truths about it. But we can derive, using pure thought, a variety of theorems which apply to the chiliagon. For example, the sum of the angles of any n-sided polygon is $(n-2)180°$. So the sum of the angles of the chiliagon is $998 \times 180°$ or $179{,}640°$, precisely, a figure impossible to determine using sensation and measurement. Since that conclusion is both exactly right and not the result of sense experience or imagination, our mathematical knowledge must be innate.

Still, one might object to Descartes that we do not learn mathematics before we have sense experience. It seems odd to claim that mathematics is innate since our mathematical learning seems to depend, in some way, on our having the experiences involved in, say, learning the names of figures and acquiring the ability to calculate. But, Descartes believes, mathematical knowledge can be innate even if we require sense experiences to put us in a position to grasp it with pure reason. Descartes is distinguishing between the order of knowledge as it comes to us and the order as it is justified. Leibniz makes this point explicitly.

> Although the senses are necessary for all our actual knowledge, they are not sufficient to provide it all, since they never give us anything but instances, that is particular or singular truths. But however many instances confirm a general truth, they do not suffice to establish its universal necessity; for it does not follow that what has happened will always happen in the same way (Leibniz, *New Essays on Human Understanding*, 49).

This may be the most important argument that we can take from Descartes and Leibniz: there is a genetic fallacy in assuming that because evidence from the senses temporally precedes evidence for mathematics, the beliefs which are more closely connected to our sense experience are more secure than our mathematical beliefs. Rather, they argue, we should look not at which beliefs come first, but which are most secure.

Because of the profundity of the doubts which motivate his project and the way in which his response depends on secure foundations, Descartes's arguments for the objectivity and innateness of mathematical beliefs are only as good as his broader system. Descartes's more general goal is to secure all of his beliefs from doubt.

> I realized that it was necessary, once in the course of my life, to demolish everything completely and start again right from the foundations if I wanted to establish anything at all in the sciences that was stable and likely to last (Meditation One, AT 17).

Descartes uses Euclid's *Elements* as a model for his broader project, in places explicitly, though he believes that the method of the *Meditations*, which he calls analytic, surpasses Euclid's method, which he categorizes as synthetic, in its security. In his replies to the second set of objections to the *Meditations*, Descartes makes his Euclidean influence clear, or at least capitulates to pressure to make his axioms and postulates explicit. There, Descartes's underlying goal is clear. We call Descartes a foundationalist because he wants to ground our knowledge on basic, indubitable truths. In the *Meditations*, it looks like his foundational truth is the cogito. In the geometric presentation, it looks like his foundation is the existence and goodness of God. In either case, Descartes's plan is to present an account of all of our knowledge that makes mathematical knowledge, if not primary, then certainly ahead of the beliefs based on our senses.

Like Descartes, Leibniz presents a foundational system to account for all of our knowledge. For Leibniz, the foundations, or primary truths, are identities, known by pure (non-sensory) intuition. All other truths reduce to primary truths by definitions. The foundational theories of both Descartes and Leibniz rely on the precedents set by Euclid and play an influential role in the development of axiomatic theories in the nineteenth and early twentieth centuries.

III. Axioms, Security and God

One advantage to reducing and encapsulating a vast system of knowledge, like a mathematical theory, in a simple axiomatic form is that it can focus our most basic questions about the larger theory. In particular, questions about formal systems of mathematics may be focused on their two central components: axioms and rules of inference.

In the *Meditations*, as a response to his First-Meditation doubts, Descartes presents the cogito as the foundation of our knowledge. The Cogito is secure, but it is not much of an axiom. While doubt about one's own existence is self-defeating, it is just about the only claim to have that characteristic. Descartes claims that any clear and distinct proposition is true and can be known to be true. But the truth of clear and distinct propositions is secured by the goodness of a creator God. Without knowing that there is a benevolent and omnipotent God, we can wonder whether a deceiving demon is making us believe even that which we hold most certain, like mathematical propositions, when such claims are really false. At what might be taken as the dramatic climax of the *Meditations*, Descartes despairs of justifying our mathematical beliefs without first proving the existence of God.

> But what about when I considered something very simple and easy in the areas of arithmetic or geometry, for example that two plus three make five, and the like? Did I not intuit them at least clearly enough so as to affirm them as true? To be sure, I did decide later on that I must doubt these things, but that was only because it occurred to me that some God could perhaps have given me a nature such that I might be deceived even about matters that seemed most evident. But whenever this preconceived opinion about the supreme power of God occurs to me, I cannot help admitting that, were he to wish it, it would be easy for him to cause me to err even in those matters that I think I intuit as clearly as possible with the eyes of the mind. On the other hand, whenever I turn my attention to those very things that I think I perceive with such great clarity, I am so completely persuaded by them that I spontaneously blurt out these words: "let him who can deceive me; so long as I think that I am something, he will never bring it about that I am nothing. Nor will he one day make it true that I never existed, for it is true now that I do exist. Nor will he even bring it about that perhaps two plus three might equal more or less than five, or similar items in which I recognize an obvious contradiction." And certainly, because I have no reason for thinking that there is a God who is a deceiver (and of course I do not yet sufficiently know whether there even is a God), the basis for doubting, depending as it does merely on the above hypothesis, is very tenuous and, so to speak, metaphysical. But in order to remove even this basis for doubt, I should at the first opportunity inquire whether there is a God, and, if there is, whether or not he can be a deceiver. For if I am ignorant of this, it appears I am never capable of being completely certain about anything else (AW 47b-48a).

Further progress in the *Meditations* thus requires an argument for the existence and goodness of God to secure the criterion of clear and distinct perception. In the geometric presentation, the centrality of this argument is such that Descartes's first proposition is the existence of God. The Cogito, which seems central in the *Meditations*, is only mentioned in passing there.

The *Meditations* and the geometric presentation in the second set of replies both contain two arguments for the existence of God. The first one, the causal argument, is in the Third Meditation but is not included in our selection. The ontological argument is in the Fifth Meditation. Both of these arguments appear up front in the geometric presentation.

Descartes's ontological argument, which derives from Anselm's eleventh-century ontological argument, is simple. Anselm argued that an object which corresponds to the concept 'something greater than which can not be thought' must exist. For, if we thought that the object which corresponded to that concept did not exist, then it would not be the object which corresponded to that concept. There would be something greater, i.e. the object which does exist. So, we give the name 'God' to that best possible object.

Descartes's version of the ontological argument does not depend on our actual conception or on our ability to conceive. He merely notes that existence is part of the essence of whatever we call God. The essence of an object is all the properties that necessarily belong to that object. They are the necessary and sufficient conditions for being that object, or one of that type. Something that has all these properties is one. Something that lacks any of these properties is not one. A chair's essence (approximately) is to be an item of furniture for sitting, with a back, made of durable material. The essence of being a bachelor is being an unmarried man. Descartes claims that a person is essentially a body and a mind and that the essence of God is perfection, which includes omnipotence and existence.

The containment invoked by Descartes is similar to how having angles whose measures add up to 180 degrees is part of the essence of the concept of a triangle. Or, as Descartes also notes, like the concept of a mountain necessarily entails the concept of a valley. He assumes that if the concept of an object includes existence, the object to which it corresponds must exist. You can have the concept of a non-existing object very much like God but which does not exist. But such a concept would not be the concept of God, by definition.

In the two centuries following Descartes's work, the ontological argument was examined in great detail. Caterus, a Dutch philosopher and contemporary of Descartes, noted, in the first set of objections to the *Meditations*, that the concept of a necessarily existing lion has existence as part of its essence, but entails no actual lions. We must distinguish more carefully between concepts and objects. Even if the concept contains existence, it is still just a concept.

Kant, who coined the term 'ontological argument' a century and a half later, claimed that existence is not a property the way that other perfections are properties. Existence can not be part of an essence, since it is not a property. Kant's point had been stated, though less elegantly, by Gassendi in the Fifth Objections.

Leibniz worries that the concept of God might be self-contradictory, when analyzed appropriately. He objects that Descartes's argument must first show the concept of God to be possible.

> One must realize that from this argument we can conclude only that, if God is possible, then it follows that he exists. For we cannot safely use definitions for drawing conclusions unless we know first that they are real definitions, that is, that they include no contradictions, because we can draw contradictory conclusions from notions that include contradictions, which is absurd (Leibniz, *Meditations on Knowledge, Truth, and Ideas*, 25).

Leibniz asks us to imagine the fastest motion. It seems like 'the fastest motion' might be a consistent concept. But, he argues, we can construct a faster motion than any real motion. Consider a wheel spinning at our supposed fastest motion. Now, consider a point extended out beyond the rim of the wheel. The extension will move faster than any point on the wheel.

Relativity theory undermines the details of Leibniz's response to Descartes. According to the theory of relativity, there is indeed a fastest motion: the speed of light. Leibniz's thought experiment, according to special relativity, is itself problematic. But the general point still stands. It just turns out that the impossible notion is the impossibility of a fastest motion! In any case, Leibniz supplants Descartes's argument with a claim that the coexistence of all perfections is indeed possible.

While the arguments for the existence of God are not our central concern, the underlying point of those arguments is essential to understand. Plato denied the reality of the sensible world; it was a world of mere belief, not a world of truth. Descartes only denies that our access to truth can come from the senses, not the existence of the physical world. His arguments for the existence of God are *a priori*, from pure thought, even though we have real bodies.

Still, Descartes's arguments about mathematics depend, in part, on his axioms concerning the existence of God. If these arguments are not self-evidently secure, then his whole system can not function the way in which it is supposed to. Thus, one problem for Descartes's account of mathematics is the insecurity of his axioms, the weakness of the arguments for the existence of God.

Let's put aside, for the moment, the general question of whether any beliefs can be secure in the absence of a successful demonstration of the existence and goodness of God. Descartes provides a separate argument for the security of mathematical knowledge and the rest of the so-called eternal truths, including those of logic. According to this argument, since mathematical beliefs are the product of pure thought, there is no evidence that we can make any kinds of systematic errors in mathematics.

Descartes here relies on a broader argument that error primarily arises in sense experience. While we can make, for example, calculational errors, Descartes believes that the possibility of error when we soberly and carefully examine a mathematical proof is minimal or non-existent. So even if we do not accept Descartes's arguments for the existence of God, or we find some other error in Descartes's reasoning, the mere characterization of mathematical truths and their distinctness from truths which depend on sense experience may provide some measure of surety and security in mathematics. To put the Cartesian point in a slightly more contemporary context: even if our scientific theories are false, if they get discarded in favor of more powerful or more unifying theories, the mathematics employed in the old theories is not impugned.

IV. Clear and Distinct Ideas

In addition to axioms, any formal theory must include a procedure by which one infers theorems from axioms. Specific formal theories are usually presumed to be embedded in more general logical theories which include rules of inference, like modus ponens. Descartes's method of clear and distinct ideas may be thought of as a rule of inference used in a formal system. The sixth and seventh postulates of the geometric presentation correspond to the method of clear and distinct ideas.

> I ask my readers to ponder on all the examples that I went through in my *Meditations*, both of clear and distinct perception, and of obscure and confused perception, and thereby accustom themselves to distinguishing what is clearly known from what is obscure. This is something that it is easier to learn by examples than by rules, and I think that in the *Meditations* I explained, or at least touched on, all the relevant examples...
>
> When they notice that they have never detected any falsity in their clear perceptions, while by contrast they have never, except by accident, found any truth in matters which they grasp only obscurely, I ask them to conclude that it is quite irrational to cast doubt on the clear and distinct perceptions of the pure intellect merely because of preconceived opinions based on the senses, or because of mere hypotheses which contain an element of the unknown. And as a result they will readily accept the following axioms as true and free of doubt (Descartes, Second Replies, ATVII.164).

A logical system, and any foundational theory, is only as secure as its rules of inference. In the mid-nineteenth century, mathematics was shaken by some unsettling results in geometry, with odd results concerning non-Euclidean spaces; in number theory, with odd results concerning infinity; and in analysis, with persistent worries about the methods of the calculus and its reliance on infinitesimals. Frege's work in logic was impelled by a desire to make sure that every inference in a mathematical proof was secure. To do so, he formulated a mathematical theory of logical consequence.

> The course I took was first to seek to reduce the concept of ordering in a series to that of *logical* consequence, in order then to progress to the concept of number. So that nothing intuitive could intrude here unnoticed, everything had to depend on the chain of inference being free of gaps (Frege, *Begriffsschrift*, IV).

For modern logical theories, rules of inference are supposed to be syntactic, defined by the shape of the symbols used. For older theories, rules of inference were also identified by their form, though not, perhaps, as fully syntactically.

> At the very least, the argument must reach its conclusion by virtue of its form (Leibniz, *Meditations on Knowledge, Truth, and Ideas*, 27).

What we might take to be Descartes's rule of inference, the method of clear and distinct ideas, has seemed to many subsequent philosophers to lack the security that Descartes imputed to it.

> [O]ften what is obscure and confused seems clear and distinct to people careless in judgment (Leibniz, *Meditations on Knowledge, Truth, and Ideas*, 26).

So, a second problem the rationalists face is to secure a method of inference   Leibniz's solution to the problem is to refine the notions of clarity and distinctness.

V. Leibniz's Hierarchy of Knowledge

Like Descartes, Leibniz believes that we can trace all complex ideas back to foundational ones. Leibniz, however, attempts to reduce all of our knowledge to simple principles. In addition, Leibniz believes that there are foundational objects: all complex objects are built out of simple components. That is, he provides twin reductions: an epistemological reduction and a correlative metaphysical reduction.

The metaphysical reduction is not really our concern, but a few words are in order. Descartes believes that matter is geometry made concrete. The essential property of matter is its extension, which is mathematically describable. Since the real numbers we use to describe extension are infinitely divisible, Descartes concludes that matter is infinitely divisible as well.

Leibniz realizes that if matter were infinitely divisible, then there could be no utterly simple objects to serve as the basic building blocks. In other words, the infinite divisibility of matter blocks metaphysical reductionism. Leibniz thus, in contrast to Descartes, posits foundational objects which are, like Aristotelian substances, active. Leibniz calls the foundational substances monads. Monads are soul-like, and they reflect the entire state of the universe at each moment. Leibniz's metaphysical foundationalism is ancillary to the mathematical question, but it's always worth quoting Voltaire on Leibniz.

> Can you really believe that a drop of urine is an infinity of monads, and that each of these has ideas, however obscure, of the universe as a whole? (Voltaire, *Oeuvres complètes*, Vol. 22, p. 434).

Our interest is properly in Leibniz's attempt to refine Descartes's rule of inference, his criterion of clear and distinct perception. To refine Descartes's criterion, Leibniz presents a variety of distinctions among kinds of knowledge. Obscure knowledge does not allow us even to identify a thing. For example, I see that something is a leaf, but I don't know what kind of tree it came from. Obscure knowledge should not even be called knowledge, in our sense of the term. It is mere belief.

Clear knowledge gives us a "means for recognizing the thing represented" (24). Clear knowledge may be divided into confused or distinct knowledge, which are distinguished by our ability to distinguish things from each other.

Confused knowledge is working knowledge, like that of color. Leibniz says that we know how colors appear to us, but not so well how they work, how they are composed. Perhaps we have a better understanding of the physical bases for colors. But, many people, like chicken sexers or musicians, may possess abilities without being able to describe how they work. Even muindane activities like riding a bike can be difficult to understand fully. We can not communicate confused knowledge well.

Distinct knowledge is connected with marks to distinguish an object from others. We can communicate it, and start to discuss its component parts. Still, distinct knowledge may be adequate or not, depending on how many of the component parts we understand.

Inadequate knowledge is when we do not know, and can not communicate, all of the component notions of a thing. The assayer may know how to distinguish gold from iron pyrite, and aluminum from molybdenum. But, part of that distinction has to do with atomic weight. And, the assayer may know how to test for atomic weight, but not know what it is.

If I have adequate knowledge of p, then I have adequate knowledge of all components of p, all components of components of p, etc. This seems like a tall order, and Leibniz admits that we may not have any adequate knowledge. Still, this is the presumed domain of mathematical knowledge.

> I don't know whether humans can provide a perfect example of [adequate knowledge], although the knowledge of numbers certainly approaches it (Leibniz, *Meditations on Knowledge, Truth, and Ideas* 24).

Leibniz, like Descartes, is thinking of the mathematical method, the axiomatic method. In mathematics, we can trace any claim, via its proof, back to the axioms. But, even adequate knowledge is not the ultimate foundation, since we have to justify knowledge of the axioms. The mathematician uses definitions to make his work perspicuous. When we give a proof in, say, linear algebra, we do not present it in its set-theoretic form. Our finite minds have limited abilities to comprehend all the steps in a long and complex proof or proposition.

Symbolic knowledge, then, is adequate knowledge which appeals to signs (definitions) to represent our knowledge of components. The use of definitions prevents our knowledge from being fully intuitive. Intuitive knowledge is of distinct primitive notions. An infinite mind would be able to have intuitive knowledge of all propositions.

For Leibniz, the foundational truths are identities, laws of logic. These would be known intuitively, or directly. We can consider all the component notions of the most perfect knowledge at the same time. Note that the most perfect knowledge, intuitive and adequate knowledge, would be *a priori*, traced back to the component parts of its real definition (not just its nominal one, p 26).

VI. Summary

Like Plato, Descartes exalts our knowledge of mathematics over any beliefs which involve sense perception. But Descartes's account of our knowledge of mathematics, embedded within the scope of the doubts of the First Meditation as it is, depends on the flawed and distal arguments for the security of clear and distinct perception via the goodness of an omnipotent God.

Leibniz's views on mathematics are similar in many ways to those of Descartes. But Leibniz was working at the end of the seventeenth century, having been born only a few years before Descartes's death. He was a contemporary of Locke, whose work was a significant influence on the younger Leibniz. We will look in more detail at Locke's work, and Leibniz's, in the next section.