Knowledge, Truth, and Mathematics

Philosophy 405
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Class #5: Rationalists I

Terminology

- *Platonism*: the view that we ascribe to Plato and his followers, including the theory of forms
- *platonism*: the beliefs that mathematical objects exist outside of space and time and that many mathematical statements are true
- So, Plato is both a Platonist and a platonist.
- Descartes and Leibniz are platonists, but not Platonists.

Descartes's Inversion of the Epistemic Order

- Standard view
  - Sense experience is the source of most or all of our knowledge.
  - Mathematical knowledge should somehow derive from the senses.
- Descartes's inversion
  - Some of our beliefs are impervious to serious doubt.
  - Beliefs which are based on our senses are less secure.
- Galileo (1564-1642) on the secondary properties:
  - ...that external bodies, to excite in us these tastes, these odours, and these sounds, demand other than size, figure, number, and slow or rapid motion, I do not believe, and I judge that, if the ears, the tongue, and the nostrils were taken away, the figure, the numbers, and the motions would indeed remain, but not the odours, nor the tastes, nor the sounds, which, without the living animal, I do not believe are anything else than names (*Opere* IV, 336).
- Descartes's goals:
  - unify the new science of Galileo with central Church doctrine
  - establish the individual's ability to grasp eternal truths

The Church and Aristotle (and Descartes)

- The Catholic Church's world view was in many ways essentially Aristotelian.
  - Final Causes
  - Geocentrism
  - Logic
- But: the soul
  - For the Church, the soul was separate from the body.
  - Aristotle saw the soul as the form of the body, linking thought and sensation.
- By separating thought from sensation, Descartes supports the Church's view of the soul against Aristotelian accounts.
- Our knowledge of mathematics, for Descartes, is the product of this pure, non-sensory, rational thought.

Rationalism

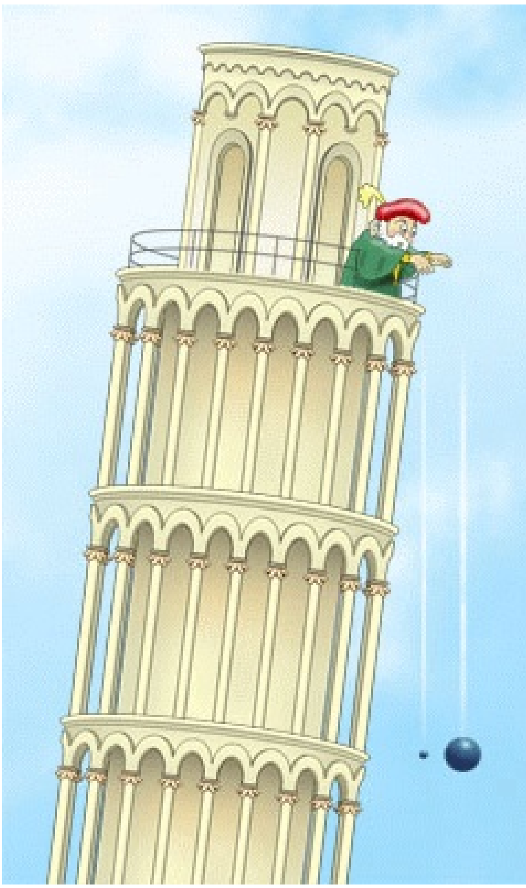
epistemic correlate of platonism

- Leibniz, following Descartes, refines the rationalist view of mathematics.
- Beliefs which depend on the senses are tenuous.
- Mathematical beliefs, unsullied by sensation, are elevated to pure truths.
- We can know them certainly, because they are the result of pure reason.
- Both Descartes and Leibniz were important mathematicians in addition to being seminal philosophers; see secondary readings in Kline.
- Both saw their mathematical developments as arising from their (rationalist) methods.
- Translating their mathematical success to philosophy more generally

Rationalist Methods

- Mathematical developments were spurred by empirical science.
- You might think that this fact supports the Aristotelian/traditional view.
- But the methods of empirical science were as purely rational as they were experimental.
- Scientists presupposed simplicity for both mathematical and empirical theories, rather than finding it.

Galileo’s Balls



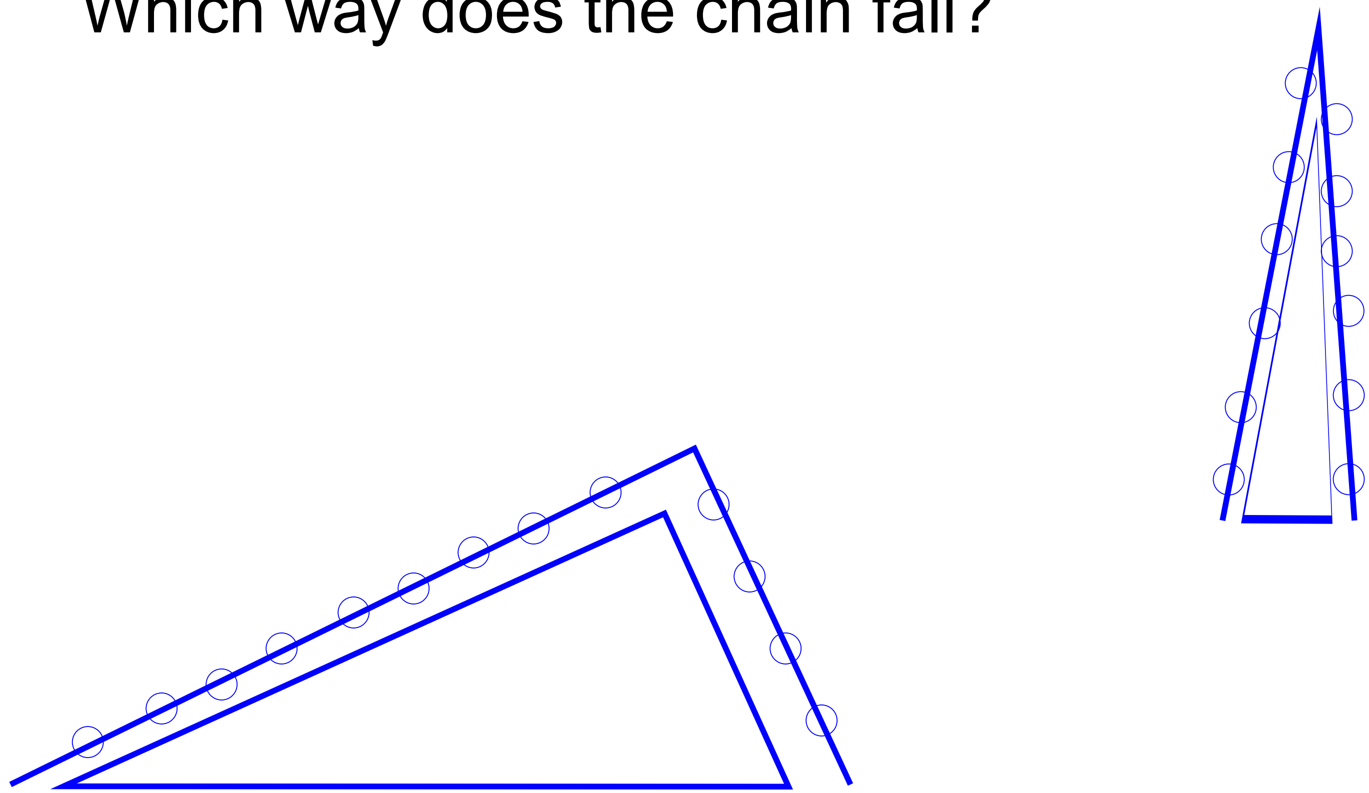
- Aristotle had claimed that heavier bodies fall faster than lighter ones. But...
- Consider a system consisting of the two bodies attached by a string.
- The rate it falls is S.
- Since, the light body falls more slowly than the heavier one, it should act as a drag on the system.
  - So, S < H.
- But, since the system is heavier than the single heavy body, it should fall more quickly.
  - So S > H.
- That’s a contradiction.
- Notice the *a priori* method.

Evidence

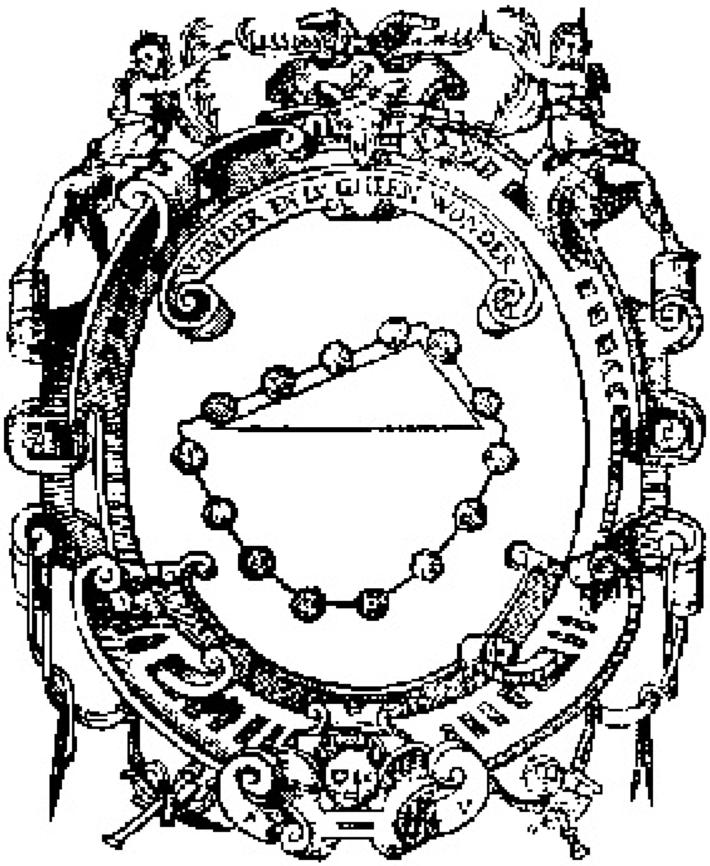
"So, you have not made a hundred tests, or even one? And yet you so freely declare it to be certain?... Without experiment, I am sure that the effect will happen as I tell you, because it must happen that way" (Galileo, *Dialogue Concerning the Two Chief World Systems*, p 145.)

Stevin's Chain

Which way does the chain fall?



Stevin's Solution (1605)



"Unquestionably in the assumption from which Stevin starts, that the endless chain does not move, there is contained primarily only a *purely instinctive* cognition" (Mach).

Descartes's Radical Rationalism

- Most scientists of the seventeenth and eighteenth centuries were impressed by the advances that mathematics made for science.
- Descartes thought that the world was essentially mathematical.
  - Geometry enmaterialed
- The objectivity of science and the objectivity of mathematics were inseparable.

Two Cartesian Claims About Mathematics

a metaphysical claim:
Mathematical objects have a determinate, objective nature, independent of us.

- Mathematical truths are objective.
  1. A thing's nature depends on me if I can make it any way I like.
  2. A thing's nature is objective if I can not make it any way I like.
  3. I can not make mathematical objects any way I would like.

  So, mathematical objects are objective.
- To show that mathematical truths are necessary truths, Descartes relies on the larger argument that error arises mainly from over-reliance on the senses.

  Wax argument

  Mathematical objects are not objects of sensation.

  So, there is no reason to think that our beliefs about them are in error.

Two Cartesian Claims About Mathematics

an epistemological claim:
Our knowledge of mathematics is innate.

1. All ideas must be invented, acquired, or innate.
2. Mathematical truths can not be invented, by the metaphysical claim.
3. Mathematical truths can not be acquired, by the chiliagon claim.

So, they must be innate.

The Order of Acquisition and the Order Of Justification

- Granted: we do not learn mathematics before we have sense experience.
- But we can distinguish between the order of knowledge as it comes to us and the order as it is justified.
- Leibniz:

  "Although the senses are necessary for all our actual knowledge, they are not sufficient to provide it all, since they never give us anything but instances, that is particular or singular truths. But however many instances confirm a general truth, they do not suffice to establish its universal necessity; for it does not follow that what has happened will always happen in the same way" (Leibniz, *New Essays on Human Understanding*, 49).
- Genetic fallacy: assuming that because evidence from the senses temporally precedes evidence for mathematics, the beliefs which are more closely connected to our sense experience are more secure than our mathematical beliefs.

An Aside on Chomsky’s Poverty-of-the-Stimulus Argument

Linguistic Nativism

the most fundamental rules of language are innate in our minds

- Chomsky argues that language is grown, like an appendage, rather than learned.
- In order to generate the indefinite set of sentences of a natural language, we combine lexical particles according to certain rules of formation, called a generative grammar.
- A generative grammar is a formal system that produces the infinite set, like a set of logical axioms from which we can derive all logical truths.
- Chomsky argues that universal grammar (UG) is innate.
- Children learn too much grammar too quickly to be explained by exposure to (experience with) language.

Chomsky On Grammatical Transformations

- Children learn too much grammar too quickly for us to account for their grammatical abilities on the basis of behavioral stimulus (i.e. induction).
  1. I wonder who the men expected to see them.
  2. The men expected to see them.
- If children were learning grammar behaviorally, they would make the reasonable inductive conclusion that ‘them’ has the same reference in each case.
- But, children just do not make that kind of mistake.
- More transformations
  3. John is easy to please.
  4. John is eager to please.
  5. It is easy to please John.
  6. It is eager to please John.
- Children will make the transformation from 3 to 5, but not from 4 to 6.
- If they were learning grammar inductively, we would expect that they would form sentences like 6, sometimes, requiring instruction to eliminate that formation.
- Such instruction is never necessary, leading us to believe that the grammatical rules are built into the brain, in some way, rather than learned.

Chomsky on Vocabulary

- The POTS argument also relies on the claim that children learn the lexicon (vocabulary) of their first language too quickly to be explained purely behaviorally.
- While they learn the specific words behaviorally, these words must hook onto pre-existing concepts.
- “It is a very difficult matter to describe the meaning of a word, and such meanings have great intricacy and involve the most remarkable assumptions, even in the case of very simple concepts, such as what counts as a possible “thing.” At peak periods of language acquisition, children are “learning” many words a day, meaning that they are in effect learning words on a single exposure. This can only mean that the concepts are already available, with all or much of their intricacy and structure predetermined, and the child’s task is to assign labels to concepts, as might be done with very simple evidence” (Chomsky, “Language and Problems of Knowledge,” 689).
- Our abilities to use language must be built into our brains.

Chomsky and Leibniz

- Poverty of the evidence arguments undermine arguments, like those we will see from Locke, which arise from the temporal order of learning.
  - “A child knows not that three and four are equal to seven, till he comes to be able to count seven, and has got the name and idea of equality; and then, upon explaining those words, he presently assents to, or rather perceives the truth of that proposition” (Locke, *Essay*, §I.2.16).
- We don’t know that, say, a whale is a mammal until we have knowledge of what those terms mean.
- Once we learn the meanings of the terms, then we can see that ‘whales are mammals’ is true.
- Similarly, according to Locke, we learn that 3+4=7 when we learn the meanings of 3, 4, 7, +, and =.
- Locke says that empiricism accounts for the temporal difference in learning 3+4=7 and 18+19=37.
- We learn the terms of the latter sentence later.
- Poverty of evidence arguments show that the temporal order of our belief acquisition is irrelevant.

Revenge of the Genetic Fallacy

confusing the origin of one's ideas with their justification

- I may learn that 2+2=4 by counting apples, but the truth of that claim is independent of how I learned it.
- There is an empirical element in the learning of terms, whether mathematical terms or empirical ones, and of associating terms with ideas.
- But, justification is independent of the temporal order.
- We are aware of particular claims before we know general ones.
- But, the general truths are more fundamental, in the order of justification, than the particular ones.
- We seek to reduce our knowledge to knowledge of axioms, which are simple and most perspicuous.

Epistemology

Intellectual ideas, from which necessary truths arise, do not come from the senses...It is true that explicit knowledge of truths is subsequent (in temporal or natural order) to the explicit knowledge of ideas; as the nature of truths depends upon the nature of ideas, before either are explicitly formed, and truths involving ideas which come from the senses are themselves at least partly dependent on the senses. But the ideas that come from the senses are confused; and so too, at least in part, are the truths which depend on them; whereas intellectual ideas, and the truths depending on them, are distinct, and neither [the ideas nor the truths] originate in the senses; though it is true that without the senses we would never think of them (Leibniz, *New Essays*, 81).

Burden of Proof

- The question whether mathematical truths are innate can not be decided by observing how we learn them.
- We have to look at the way in which we justify them, and their character.
- We have to see whether sense experience is sufficient for their justification, or whether we have to posit an innate capacity, or disposition, to learn them.
- The rationalist accepts, in general, all the psychological capacities that the empiricist accepts.
- Thus, from a purely methodological perspective, the burden of proof is on the rationalists to establish the existence of innate ideas.
- Still, if the empiricist wants us to believe that there are no innate ideas, she must present a plausible positive account of our knowledge of mathematics.
- That's Locke's task, and ours, for next week.

Foundationalism

- Descartes's arguments for the objectivity and innateness of mathematical beliefs are only as good as his broader system.
- Descartes's more general goal is to secure all of his beliefs from doubt.
  - "I realized that it was necessary, once in the course of my life, to demolish everything completely and start again right from the foundations if I wanted to establish anything at all in the sciences that was stable and likely to last" (Meditation One, AT 17).
- Euclid's *Elements* as a model
  - though Descartes believes that his method surpasses Euclid's work in its security
  - The geometric presentation emphasizes Descartes's underlying foundationalism.
- Mathematical knowledge, if not primary, is certainly ahead of the beliefs based on our senses.
- For Leibniz, the foundations, or primary truths, are identities, known by pure (non-sensory) intuition.
  - All other truths reduce to primary truths by definitions.

Axiomatic Theories

- Euclid's *Elements* was, in the seventeenth century, the only important axiomatic theory.
- All of mathematics was presumed to be geometric.
- Thus, new developments could be, theoretically, derived from Euclid's work.
- In the late nineteenth century, spurred mainly by Frege's revolutionary work in logic, the method of axiomatization become central to mathematics.

The MIU System

from Hofstadter's *Gödel, Escher, Bach*

- Any string of Ms Is and Us is a string of the MIU system.
- MIU, UMI, and MMMUMUUUMUMMU are all strings.
- Only some strings will be theorems.
  - Only some strings of letters are English words.
  - Only some strings of words are grammatical sentences.
- The MIU system takes only one axiom: MI
- Theorems
  - any string which is either an axiom
  - or which follows from the axioms by using some combination of the rules of inference.

Four Rules of Inference

- R1. If a string ends in I you may append U.
- R2. From Mx, you can infer Mxx.
- R3. If III appears in that order, then you can replace the three Is with a U
- R4. UU can be dropped from any theorem.

So, each of these are theorems:

1. MI	Axiom
2. MIU	From Step 1 and R1
3. MII	1, R2
4. MIIII	3, R2
5. MIU	4, R3
6. MUI	4, R3
7. MIIIIIIII	4, R2
8. MIUUI	7, R3
9. MII	8, R4

etc.

Derive MIIIII

(That’s five ‘I’s.)

For Later: Derive ‘MU’

- For help, see Hofstadter’s book, pp 259-261.
- Do not spend too much time on this puzzle without consulting Hofstadter, who provides a solution.

An Axiom System for Propositional Logic

following Mendelson, *Introduction to Mathematical Logic*

- The symbols are $\sim$, $\supset$, (,), and the statement letters A_i, for all positive integers i.
- All statement letters are wffs.
- If α and β are wffs, so are $\sim\alpha$ and $(\alpha \supset \beta)$
- If α, β, and γ are wffs, then the following are axioms:
  A1: $(\alpha \supset (\beta \supset \alpha))$
  A2: $((\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma)))$
  A3: $((\sim\beta \supset \sim\alpha) \supset ((\sim \beta \supset \alpha) \supset \beta))$
- β is a direct consequence of α and $(\alpha \supset \beta)$

Zermelo-Fraenkel Set Theory (ZF)

again following Mendelson
but with adjustments

- ZF may be written in the language of first-order logic, with one special predicate letter: $\in$
- Substitutivity: $(\forall x)(\forall y)(\forall z)[y=z \supset (y \in x \equiv z \in x)]$
- Pairing: $(\forall x)(\forall y)(\exists z)(\forall u)[u \in z \equiv (u = x \vee u = y)]$
- Null Set: $(\exists x)(\forall y){\sim}x \in y$
- Sum Set: $(\forall x)(\exists y)(\forall z)[z \in y \equiv (\exists v)(z \in v \bullet v \in x)]$
- Power Set: $(\forall x)(\exists y)(\forall z)[z \in y \equiv (\forall u)(u \in z \supset u \in x)]$
- Selection: $(\forall x)(\exists y)(\forall z)[z \in y \equiv (z \in x \bullet \mathscr{F}u)]$, for any formula $\mathscr{F}$ not containing y free.
- Infinity: $(\exists x)(\varnothing \in x \bullet (\forall y)(y \in x \supset Sy \in x)$, where 'Sy' stands for $y \cup \{y\}$
- Most mathematicians would accept a further axiom, called Choice, yielding a theory commonly known as **ZFC**.

The Dedekind/Peano Axioms

Also following Mendelson

P1: 0 is a number

P2: The successor (x') of every number (x) is a number

P3: 0 is not the successor of any number

P4: If x'=y' then x=y

P5: If P is a property that may (or may not) hold for any number, and if
- 0 has P, and
- for any x, if x has P then x' has P,

then all numbers have P.

P5 is mathematical induction, and is actually a schema of an infinite number of axioms.

Birkhoff's Postulates for Geometry

following Smart

- *Postulate I: Postulate of Line Measure*. The points A, B,... of any line can be put into a 1:1 correspondence with the real numbers x so that $|x_B - x_A| = d(A,B)$ for all points A and B.
- *Postulate II: Point-Line Postulate*. One and only one straight line l contains two given distinct points P and Q.
- *Postulate III: Postulate of Angle Measure*. The half-lines l, m... through any point O can be put into 1:1 correspondence with the real numbers a(mod 2π) so that if $A \neq 0$ and $B \neq 0$ are points on l and m, respectively, the difference $a_m - a_l$ (mod 2π) is angle∡AOB. Further, if the point B on m varies continuously in a line r not containing the vertex O, the number a_m varies continuously also.
- *Postulate IV: Postulate of Similarity*. If in two triangles △ABC and △A'B'C', and for some constant k>0, d(A', B') = kd(A, B), d(A', C')=kd(A, C) and ∡B'A'C'=±∡BAC, then d(B', C')=kd(B,C), ∡C'B'A'=±∡CBA, and ∡A'C'B'=±∡ACB.

Descartes's Axioms

- For formal systems, epistemological questions may be focused on their two central components
  - axioms
  - rules of inference.
- What are Descartes's axioms?
  - The Cogito?
  - The existence and goodness of God?
- Two arguments for the existence of God
  - The causal argument in the Third Meditation
  - The ontological argument in the Fifth Meditation
  - Both appear up front in the geometric presentation.

Definitions of ‘God’

- There are various characterizations of ‘God’, to many of which Descartes alludes.
  - Whatever necessarily exists
  - All perfections, including omniscience, omnipotence, and omnibenevolence
  - Creator and preserver
- Anselm (1033-1109) uses a different characterization: ‘something greater than which can not be thought’.
- These are definitions of a term, or a word, but not an object.
- There is no presupposition in this characterization that such a thing exists.
  - Or, so it seems.

Anselm's Ontological Argument

AO1. I can think of 'God'

AO2. If 'God' were just an idea, or term, then I could conceive of something greater than 'God' (i.e. an existing God).

AO3. But 'God' is that than which nothing greater can be conceived

AO4. So 'God' can not be just an idea

AOC. So, God exists.

- Anselm further argues that one can not even conceive of God not to exist.

Descartes's Ontological Argument

- Descartes's version does not depend on our actual conception, or on our ability to conceive.
- Existence is part of the essence of 'God'.
  - having angles whose measures add up to 180 degrees is part of the essence of a 'triangle'.
  - the concept of a mountain necessarily entails a valley.
- The essence of an object is all the properties that necessarily belong to that object.
  - necessary and sufficient conditions for being one of that type.
  - Something that has all these properties is one.
  - Something that lacks any of these properties is not one.
  - A chair's essence (approximately) is to be an item of furniture for sitting, with a back, made of durable material.
  - The essence of being a bachelor is being an unmarried man.
  - A human person is essentially a body and a mind.
- The essence of 'God' is perfection.
  - the three omnis
  - existence

Objections to the Ontological Argument

- Gaunilo
  - My idea of the most perfect island does not entail that it exists.
  - A non-existing island would be free of imperfections.
- Caterus
  - The concept of a necessarily existing lion has existence as part of its essence, but it entails no actual lions.
  - We must distinguish more carefully between concepts and objects.
  - Even if the concept contains existence, it is still just a concept.
- Kant, following Hume, following Gassendi
  - Existence is not a property, the way that the perfections are properties.
  - Thus, existence can not be part of an essence.
  - Logic should make no existence assertions.

Leibniz on the Ontological Argument

- Descartes's argument must first show the concept of God to be possible.
- "One must realize that from this argument we can conclude only that, if God is possible, then it follows that he exists. For we cannot safely use definitions for drawing conclusions unless we know first that they are real definitions, that is, that they include no contradictions, because we can draw contradictory conclusions from notions that include contradictions, which is absurd" (Leibniz, *Meditations on Knowledge, Truth, and Ideas*, 25).
- The fastest motion
  - Consider a wheel spinning at the fastest motion.
  - Now, consider a point extended out beyond the rim of the wheel.
  - The extension will be moving at a faster speed than any point on the wheel.
- The concept of God might be self-contradictory, when analyzed appropriately.

Relativity Theory and Leibniz's Criticism

- Relativity theory undermines the details of Leibniz's response to Descartes, but not his more general point.
- According to the theory of relativity, there is indeed a fastest motion: the speed of light.
- Leibniz's thought experiment, according to special relativity, is itself problematic.
- But the general point still stands.
- It just turns out that the impossible notion is the impossibility of a fastest motion!

Two Problems for Descartes

- The insecurity of his axioms
  - His arguments about mathematics depend on his axioms concerning the existence of God.
  - If these arguments are not self-evidently secure, then his whole system can not function the way in which it is supposed to.
- To secure the method of inference
  - Clear and Distinct Ideas

Clear and Distinct Ideas

- I ask my readers to ponder on all the examples that I went through in my *Meditations*, both of clear and distinct perception, and of obscure and confused perception, and thereby accustom themselves to distinguishing what is clearly known from what is obscure. This is something that it is easier to learn by examples than by rules, and I think that in the *Meditations* I explained, or at least touched on, all the relevant examples...When they notice that they have never detected any falsity in their clear perceptions, while by contrast they have never, except by accident, found any truth in matters which they grasp only obscurely, I ask them to conclude that it is quite irrational to cast doubt on the clear and distinct perceptions of the pure intellect merely because of preconceived opinions based on the senses, or because of mere hypotheses which contain an element of the unknown. And as a result they will readily accept the following axioms as true and free of doubt (Descartes, Second Replies, ATVII.164).
- Descartes's rule of inference, the method of clear and distinct ideas, has seemed to many subsequent philosophers to lack the security that Descartes imputed to it.
  - "[O]ften what is obscure and confused seems clear and distinct to people careless in judgment" (Leibniz, *Meditations on Knowledge, Truth, and Ideas*, 26).

Improvements to Inferences

- Leibniz's emphasis on form
  - "At the very least, the argument must reach its conclusion by virtue of its form" (Leibniz, *Meditations on Knowledge, Truth, and Ideas*, 27).
- Frege's theory of logical consequence.
  - "The course I took was first to seek to reduce the concept of ordering in a series to that of *logical* consequence, in order then to progress to the concept of number. So that nothing intuitive could intrude here unnoticed, everything had to depend on the chain of inference being free of gaps" (Frege, *Begriffsschrift*, IV).

Leibniz's Twin Reductions

- Leibniz also reduces knowledge to simple principles.
- In addition, Leibniz thought that there were foundational objects.
- He provides twin reductions: an epistemological reduction and its sister metaphysical reduction.

The Metaphysical Reduction

- Descartes thought that matter was just geometry, made concrete.
  - The essential property of matter is its extension.
  - Matter should be infinitely divisible.
- Leibniz: if matter were infinitely divisible, then there could be no utterly simple objects to serve as the basic building blocks.
- The infinite divisibility of matter blocks metaphysical reductionism.
- Leibniz thus posited foundational objects which were, like Aristotelian substances, active: monads.
- Monads are soul-like, and they reflect the entire state of the universe at each moment.
- “Can you really believe that a drop of urine is an infinity of monads, and that each of these has ideas, however obscure, of the universe as a whole?” (Voltaire, *Oeuvres complètes*, Vol. 22, p. 434).

Leibniz Refining Descartes's Criterion of Clear and Distinct Perception, I

- *Obscure* knowledge does not allow us even to identify a thing.
  - For example, I see that something is a leaf, but I don't know what kind of tree it came from.
  - mere belief
- *Clear* knowledge gives us a "means for recognizing the thing represented" (24).
- *Confused* knowledge is working knowledge, like that of color.
  - Consider chicken sexers, or musicians.
- *Distinct* knowledge is connected with marks to distinguish an object from others.
  - We can communicate it, and start to discuss its component parts.
  - There may be many component parts.
- *Inadequate* knowledge is when we do not know, and can not communicate, all of the component notions of a thing.
  - The assayer may know how to distinguish gold from iron pyrite, and aluminum from molybdenum.
  - The assayer may know how to test for atomic weight, but not know what it is.

Leibniz Refining Descartes's Criterion of Clear and Distinct Perception, II

- If I have *adequate* knowledge of p, then I have adequate knowledge of all components of p, all components of components of p, etc.
  - "I don't know whether humans can provide a perfect example of [adequate knowledge], although the knowledge of numbers certainly approaches it" (Leibniz, *Meditations on Knowledge, Truth, and Ideas* 24).
  - In mathematics, we can trace any claim, via its proof, back to the axioms.
  - But, even adequate knowledge is not the ultimate foundation, since we have to justify knowledge of the axioms.
  - The mathematician uses definitions to make his work perspicuous.
- *Symbolic* knowledge is adequate knowledge which appeals to signs (definitions) to represent our knowledge of components.
  - The use of definitions prevents our knowledge from being fully intuitive.
- *Intuitive* knowledge is of distinct primitive notions.
  - An infinite mind would be able to have intuitive knowledge of all propositions.
- For Leibniz, the foundational truths are identities, laws of logic.
  - These would be known intuitively, or directly.
  - We can consider all the component notions of the most perfect knowledge at the same time.
- The *most perfect knowledge*, intuitive and adequate knowledge, would be *a priori*, traced back to the component parts of its real definition (not just its nominal one, p 26).