Philosophy 405: Knowledge, Truth and Mathematics — Hamilton College
Spring 2014 — Russell Marcus

Class #5: Axioms
Sample Introductory Material from
Marcus and McEvoy, An Historical Introduction to the Philosophy of Mathematics

Mathematics is a broad and varied discipline. In addition to the traditional fields of arithmetic and geometry, mathematicians study real analysis, topology, probability, statistics, linear algebra, set theory, category theory, knots, and graphs and other topics. Any philosophy of mathematics must account for all of the diverse professional activities of mathematicians. In order to focus their questions, though, philosophers often focus on axiomatic formulations of theories. Further, philosophers sometimes focus primarily on the axiomatic formulations of theories they consider most fundamental or foundational. In this section, we review, briefly, some axiomatic approaches to mathematics.

## I. Axiomatic Theories

Euclid's *Elements* is precedental for all future developments in axiomatics. Indeed, for over two millennia, it was the only important axiomatic theory. As late as the seventeenth century, all of mathematics was presumed, by most philosophers and mathematicians, to be geometric. (Our continuing talk of square and cubic numbers is a vestige of the old view.) On that view, all new developments in mathematics could be, theoretically, derived from Euclid's work. The revolutionary development of analytic geometry, by Descartes, Fermat, and others in the early seventeenth century, helped to invert the mathematical order, showing the broader theories of algebra and analysis to be more proper foundations within mathematics. Later, in the nineteenth and twentieth centuries, set theory supplanted analysis as the broadest, most unifying mathematical theory. Today, some philosophers and mathematicians explore a more abstract approach, called category theory, as a potentially better foundation.

Whatever mathematical theory comes to be taken as properly foundational, if any is, a clear apprehension of any theory is provided by a simple, elegant axiomatization. In the late nineteenth century, spurred mainly by Frege's revolutionary work in logic, the method of axiomatization become central to mathematics. We will look briefly at a variety of axiomatic theories, starting with a simple system, from Douglas Hofstadter, called the MIU system.[1]

In the MIU system, we call a string any concatenation of 'M's, 'I's or 'U's. So, 'MIU', 'UMI', and 'MMMUMUUUMUMMU' are all strings.

Only some strings of the MIU system are theorems. We can think of theorems as special or successful strings. If we think of the 'M's, 'I's, or 'U's as words, the theorems would correspond to grammatical sentences of English. So, 'cat mat the on is' is not a theorem while 'the cat is on the mat' is a theorem. We could also think of strings as all possible statements of geometry, while theorems are only the true statements of geometry.

To determine which geometrical statements are true, we have to see which statements are provable from the axioms we adopt. That is not an easy task. But determining which strings of the MIU system are theorems is simple. Hofstadter provides a single starting axiom, 'MI', and some rules from which we can derive new theorems. The string 'MI' is to be taken as a foundational truth. Others are derived.

Axioms and Theorems for Hofstadter's MIU System

[1] Gödel, Escher, Bach: An Eternal Golden Braid, p 33 et seq.

The MIU system takes only one axiom: MI.

A theorem is any string which is either an axiom, or follows from the axioms by using some combination of the rules of inference.

Rules of Inference for Hofstadter's MIU System

R1. If a string ends in 'I' you may append 'U' to the string.
R2. You may append whatever follows an M in a string.
R3. If 'III' appears in that order, then you may replace the three 'I's with a single 'U'.
R4. 'UU' may be dropped from any theorem.

Some strings of the language are theorems. Others are not. Here are some theorems followed by justifications, or demonstrations of how to derive those theorems.

Sample Theorems of the MIU System

| | |
|---|---|
| 1. MI | Axiom |
| 2. MIU | From Step 1, above, and Rule R1 |
| 3. MII | 1, R2 |
| 4. MIIII | 3, R2 |
| 5. MIU | 4, R3 |
| 6. MUI | 4, R3 |
| 7. MIIIIIIII | 4, R2 |
| 8. MIUUI | 7, R3 |

Here is a derivation of the theorem MIIIII.

| | |
|---|---|
| 1. MI | Axiom |
| 2. MII | 1, R2 |
| 3. MIIII | 2, R2 |
| 4. MIIIIIIII | 3, R2 |
| 5. MIIIIIIIIU | 4, R1 |
| 6. MIIIIIUU | 5, R3 |
| 7. MIIIII | 6, R4 |

If you would like to play with the MIU system a bit, you can try to derive the following theorems:

1. MIUUI
2. MIIUIUIUIII
3. MIIUIIIUIIUIIIU
4. MU

Actually, that last string is underivable. For a proof that 'MU' is underivable, see Hofstadter 259-261.

## II. Modern Axiomatics and Logic

Let's turn to some more typical mathematical theories. The standard approach to presenting a formal theory is first to specify a language, including its syntax and a definition of a well-formed-formula (or wff). Then, one presents axioms, or basic assumptions, and rules of inference which allow one to derive theorems from the axioms. Technically, the rules of inference are properties of the logical theory in which the particular theory is embedded. We'll start with a simple logical theory, propositional logic, in an elegant form.

**Propositional Logic (PL)**, following Mendelson, *Introduction to Mathematical Logic*

1. The symbols are ~, ⊃, (, ), and the statement letters $A_i$, for all positive integers I.
2. All statement letters are wffs.
3. If α and β are wffs, so are ~α and (α ⊃ β)
4. If α, β, and γ are wffs, then statements of the form A1, A2, and A3 are axioms.
   A1: $(\alpha \supset (\beta \supset \alpha))$
   A2: $((\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma)))$
   A3: $((\sim\beta \supset \sim\alpha) \supset ((\sim \beta \supset \alpha) \supset \beta))$
5. β is a direct consequence of α and (α ⊃ β)

The language of PL is given at Step 1, with rules for making wffs at steps 2 and 3. These rules are recursive and complete: all and only wffs can be generated starting with a statement letter and using repeated applications, in any order, of the rules at step 3. Step 4 provides three axiom schema; each can yield infinitely many axioms by the substitution of wffs for the variables. Step 5 is the lone rule of inference for PL.

There are lots of different logical theories, including variations of any particular theory. For example, PL is often presented without axioms at all: just a language, rules of inference, and a so-called semantic method for determining when a wff is a theorem, ordinarily using truth tables. Depending on how we define theorems, such variations can produce theories equivalent to PL or quite different theories.

The most important logical theory other than PL (and its equivalents) is called first-order (or, sometimes, predicate or quantificational) logic. In first-order logic, quantifiers are prepended to wffs and propositions are analyzed into subjects and predicates, which are properties of, or relations among, subjects. First-order logic often is extended to include an identity particle, '=', and rules governing the logical properties of identity.

More specific mathematical theories are ordinarily presumed to be couched in the languages of propositional and predicate logic. They thus do not, generally, explicitly include rules of inference, relying on the logic to do the appropriate inferential work.

## III. Some Contemporary Mathematical Axiomatizations

The most widely-accepted foundational mathematical theory is set theory. There are a variety of competing set theories, but ZF is standard. ZF may be written in the language of first-order logic, with one special predicate letter, ∈.

**Zermelo-Fraenkel Set Theory (ZF)**, again following Mendelson, but with adjustments

Substitutivity: $(\forall x)(\forall y)(\forall z)[y=z \supset (y\in x \equiv z\in x)]$
Pairing: $(\forall x)(\forall y)(\exists z)(\forall u)[u\in z \equiv (u = x \vee u = y)]$
Null Set: $(\exists x)(\forall y){\sim}x\in y$
Sum Set: $(\forall x)(\exists y)(\forall z)[z\in y \equiv (\exists v)(z\in v \bullet v\in x)]$
Power Set: $(\forall x)(\exists y)(\forall z)[z\in y \equiv (\forall u)(u\in z \supset u\in x)]$
Selection: $(\forall x)(\exists y)(\forall z)[z\in y \equiv (z\in x \bullet \mathscr{F}u)]$, for any formula $\mathscr{F}$ not containing y free.
Infinity: $(\exists x)(\varnothing\in x \bullet (\forall y)(y\in x \supset Sy\in x)$, where 'Sy' stands for $y\cup\{y\}$, the definitions for the components of which are standard.

Most mathematicians adopt a further axiom, called Choice, yielding a theory commonly known as **ZFC**. Choice says that given any set of sets, there is a set which contains precisely one member of each of the subsets of the original set. The axiom of choice has many equivalents, some of which are much less intuitively pleasing than some simple formulations. Further, Choice, when added to ZF, leads to some strange results, like the well-ordering theorem.

It is widely accepted that all mathematical theories can be reduced to set theory. A theory can be reduced to set theory if with the proper definitions, all the theorems of the higher-level theory can be written with just the language of set theory, and can be proved, in principle, with just the axioms of set theory. Set theory is thus seen as a unifying framework for mathematics: all mathematical results can be brought together as complex set-theoretic statements. Some philosophers and mathematicians now have given up on set theory as the most basic unifying mathematical theory, preferring a more abstract theory called category theory.

Regardless of our choice of foundation, if our interests are more local, more mathematical, we can formulate axioms for particular mathematical theories. For example, we can construct axioms for number theory, or geometry, or topology, embedding those axioms in a logical theory. Here is a classic formulation of number theory, called Peano arithmetic, which was developed by Richard Dedekind, but gets Peano's name. (Peano himself credited Dedekind.)

**Peano Arithmetic**, again, following Mendelson with adjustments

P1: 0 is a number
P2: The successor (x') of every number (x) is a number
P3: 0 is not the successor of any number
P4: If x'=y' then x=y
P5: If P is a property that may (or may not) hold for any number, and if I. 0 has P; and ii. for any x, if x has P then x' has P; then all numbers have P.

P5 is mathematical induction, a schema of an infinite number of axioms.

The first modern axiomatization of geometry is due to Hilbert in the late nineteenth century. Hilbert's axiomatization is notable for its completely pure geometric form, which eschews all number theory. Here is an alternative axiomatization of geometry which uses real numbers.

**Birkhoff's Postulates for Geometry**, following James Smart, *Modern Geometries*

*Postulate I: Postulate of Line Measure*. The points A, B,... of any line can be put into a 1:1 correspondence with the real numbers x so that $|x_B - x_A| = d(A,B)$ for all points A and B.

*Postulate II: Point-Line Postulate*. One and only one straight line l contains two given distinct points P and Q.

*Postulate III: Postulate of Angle Measure*. The half-lines l, m... through any point O can be put into 1:1 correspondence with the real numbers a(mod $2\pi$) so that if $A \neq 0$ and $B \neq 0$ are points on l and m, respectively, the difference $a_m - a_l \pmod{2\pi}$ is angle$\triangle$AOB. Further, if the point B on m varies continuously in a line r not containing the vertex O, the number $a_m$ varies continuously also.

*Postulate IV: Postulate of Similarity*. If in two triangles $\triangle$ABC and $\triangle$A'B'C', and for some constant k>0, d(A', B') = kd(A, B), d(A', C')=kd(A, C) and $\triangle$B'A'C'=±$\triangle$BAC, then d(B', C')=kd(B,C), $\triangle$C'B'A'=±$\triangle$CBA, and $\triangle$A'C'B'=±$\triangle$ACB.