Philosophy 405: Knowledge, Truth and Mathematics | Hamilton College
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Class #13: Formalism and Incompleteness

I. The Big Three and Hilbert

The nineteenth century in mathematics was full of technical advances (especially in logic); revolutionary new work (especially in set theory, including Cantor's work on transfinites, and in geometry, with the development of non-Euclidean spaces); and foundational treatments (especially the Weierstrass project to arithmetize analysis).
Some new developments led to counter-intuitive results, like Cantor's so-called paradox.
Though Russell's paradox had yet to be discovered, Frege's work on the foundations which was intended to mitigate worries about set theory was not well-known.
In the early twentieth century, three schools of philosophy of mathematics were developed as frameworks for understanding mathematics and for dealing with the paradoxes.
Logicism, developed by Frege and Whitehead and Russell, purported to defend the new developments in mathematics and to easy concerns, especially about the notion of logical consequence, on the basis of a secure, logical foundation.
Intuitionism, which we will examine next week, restricted the range of mathematics, rejecting some of the new developments.
Most intuitionists were finitists, claiming that infinitary results are to be restricted or rejected because of the paradoxes.
Formalism, though widely misunderstood, focused on meta-theoretic proofs, examining the language of mathematics itself to insure the security of the new results.

David Hilbert was among the most important and prominent mathematicians of the late-nineteenth and early-twentieth centuries.
His achievements in mathematics include work on geometry, number theory, algebra and analysis.
He is the progenitor of metatheory in logic and mathematics.
He also contributed to the theory of relativity.
Hilbert shaped the direction of mathematical thought in the twentieth century, most famously by framing the Paris Problems: twenty-three open questions presented at the 1900 International Congress of Mathematicians.

Hilbert is often characterized as a formalist, though that broad term does not effectively capture the subtlety of his thought.
His views contain elements of three different varieties of formalism, as well as of other, non-formalist views.
Shapiro distinguishes term formalism, game formalism, and deductivism.
Term formalism is the claim that mathematical terms refer to inscriptions.
Game formalism is the claim that ideal terms, especially infinitary ones, are meaningless.
Deductivist formalism is the claim that mathematics consists merely of deductions within consistent systems.
All three of these formalist views are present in Hilbert's work.
Hilbert's work also contains elements of finitism, though he is not a strict finitist, either.
In contrast, Hilbert believed both that some mathematical statements were true of real objects, in contrast to some formalists, and that transfinite mathematics was legitimate, in contrast to finitists.

What has become known as Hilbert's project was an attempt to establish finitistic foundations for infinitary mathematics and to prove that mathematical systems were deductively secure by immanently demonstrating their consistency.
Hilbert's project was shown untenable by Gödel's incompleteness theorems, published in 1931.
Gödel's Theorems apply specifically to the deductivist Hilbert but could not have arisen without the emphasis on terms which led to meta-mathematics.
We'll start by looking at the finitist aspects of Hilbert's work.

III. Hilbert's Finitism

The 19th-century project of arithmetizing analysis was motivated mainly by the obscurity of the notion of an infinitesimal.
While calculus was put on firm footing by the epsilon-delta definition of a limit, worries about infinitesimals and infinite numbers were not assuaged by the discoveries, by Russell, of the paradoxes of the naive set theories of Frege and Cantor.
Cantor proposed to work with "inconsistent multitudes," which today we call proper classes.
Such work is mathematically dangerous, at least in classical mathematics.
From an inconsistent theory, using classical logic, any conclusion can be derived, any statement of the language and its negation too.
Frege's definitions of numbers, sets of all n-membered sets, are among these inconsistent multitudes.
By 1925, Zermelo had axiomatized set theory (1908) in a way to avoid the paradoxes.
But worries remained about whether mathematics is consistent.
Perhaps further paradoxes are lurking.

Adding to the worries about consistency, transfinite mathematics seems to dislodge mathematics from its applications in the physical world.
The world does not seem infinite, either in the large or in the small.
In the large, we might think that the universe is finite or no more than denumerably infinite or no larger than the continuum.
In the small, physical matter does not appear to be infinitely divisible.
The classical claim that nature does not make jumps (natura non facit saltus) seems false when considering atomism, or quantum physics.
So, any approach to justifying knowledge of infinitary mathematics which depends mainly on scientific theories constructed to explain our sense experience seems doomed.

Still, infinite numbers were mathematically useful, and transfinite set theory seemed mathematically legitimate, if worrisome.
Despite its lack of its utility in physical science, Hilbert did not want to let go of the transfinites.

> Wherever there is any hope of salvage, we will carefully investigate fruitful definitions and deductive methods. We will nurse them, strengthen them, and make them useful. No one shall drive us out of the paradise which Cantor has created for us (Hilbert, "On the Infinite" 191).

To determine whether a mathematical theory is legitimate, Hilbert examines theories themselves.
The first and most important criterion for mathematical legitimacy is consistency, which is a purely mathematical criterion.
An inconsistent theory is unacceptable.

Beyond that, there are still further mathematical criteria which are independent of the application of a theory to physical science.

> If, apart from proving consistency, the question of the justification of a measure is to have any meaning, it can consist only in ascertaining whether the measure is accompanied by commensurate success. Such success is in fact essential, for in mathematics as elsewhere success is the supreme court to whose decisions everyone submits (Hilbert, "On the Infinite" 184).

Call this claim Hilbert's success argument.
It raises and does not answer the question how to measure mathematical success.
We might take Hilbert to mean that we evaluate mathematical theories by inspection of their theorems.
We might, alternatively, take Hilbert to favor evaluating theories by their applications in empirical science?.
As we will see, these distinct measures of success lead to very different views of mathematical epistemology.
For now, we will put them aside to focus on the first criterion: consistency.
If we can prove that a mathematical theory, like Cantor's transfinite set theory, is consistent then, Hilbert believes, we may justify our finitary beliefs in infinitary objects.
Worries about our abilities to grasp infinite quantities, or to reason using infinitary measures, will be as unfounded as worries about complex numbers once were.

> We find writers insisting, as though it were a restrictive condition, that in rigorous mathematics only a *finite* number of deductions are admissible in a proof - as if someone had succeeded in making an infinite number of them (Hilbert, "On the Infinite" 184).

To use our finite methods to justify transfinite mathematics, Hilbert distinguishes between ideal mathematical statements and finitary (or real) mathematical statements.
This distinction has been explored and refined.
It is now fairly common to accept that Hilbert's finite statements are the primitive recursive functions; see Shapiro 161, and the references therein.
To explain the notion of a primitive recursive function, I'll take a small detour.

IV. An Aside on Primitive Recursive Functions and Church's Thesis

Church's thesis is an interesting and somewhat controversial attempt to make an intuitive idea about our finite reasoning processes mathematically precise.
Understanding Church's thesis requires a bit of technical work.
Let's start with the notion of an algorithm.
An algorithm is anything computable by means that Hilbert would have called finitistic.
The notion of computability is an ordinary-language notion.
In 1936, Alonzo Church, Emil Post, and Alan Turing each proposed independent explications of the informal notion of an effectively computable function or algorithm.
The three formal notions were later shown to select the same class of mathematical functions.
Further equivalent formulations have been produced by Gödel and others.
The resulting thesis, that the computable functions are the recursive functions, is known as Church's Thesis, or the Church-Turing thesis, after Alonzo Church, the founding and long-time editor of the *Journal of Symbolic Logic*, and Alan Turing, whose amazing and tragic life is worth some of your time.

Church's Thesis is important because we want to know whether some problems have algorithmic solutions.
For example, Church initially formulated the thesis in an attempt to answer the question of whether first-order logic was decidable.
A theory is decidable if there is a procedure for determining whether any given formula is a theorem.
Since recursion is formally definable, Church's Thesis provides a method for determining whether a particular problem has an effective solution.
It provides a formal characterization of an intuitive concept.

Church's Thesis says that the computable functions are the recursive functions.
The name 'recursive function' comes from Gödel, in his incompleteness paper.
The following presentation is derived from Hunter, *Metalogic*, p 232 et seq.
For an alternative presentation, see Mendelson, *Introduction to Mathematical Logic*, p 174 et seq.

A recursive definition of a function f(x) is one that is given by mathematical induction.
We give the value of the function for $x = 0$.
Then, we give the value of the function for $x = n + 1$ in terms of the value for $x = n$.
We can just churn out any values of the function that we want.
Given such a characterization, clearly the recursive functions are computable.
The big question is which mathematical functions can be characterized recursively.
We want to know which functions are the recursive functions.
This seems like a difficult question to answer.
But it can be answered by specifying a list of simple computable functions and showing how to generate other recursive functions from them.
Such a project is ambitious.
It might remind you of any axiomatic theory or foundationalist philosophical project, Euclid's *Elements* or Descartes's *Meditations*.
We'll specify a list of initial functions and them some operations on them.

Initial functions (functions from ordered n-tuples[1] on $\mathbb{N}$ to $\mathbb{N}$)

Successor:	$f_1(x) = x + 1$	
Sum:	$f_2(x,y) = x + y$	
Product:	$f_3(x,y) = x \bullet y$	
Power:	$f_4(x,y) = x^y$	calling $0^0 = 1$, in order to have all values defined
Arithmetic difference:	$f_5(x,y) = x \dot{-} y$	where $x \dot{-} y = x - y$, if $x > y$, and $x \dot{-} y = 0$ if $y \geq x$

Operations on Computable Functions

Combination: Any combination of computable functions is computable

The μ-operation: Let $f(x_1,...x_n,y)$ be a computable function such that for each n-tuple of natural numbers $<x_1,...x_n>$, there is a natural number y, such that $f(x_1,...x_n,y) = 0$. The μ-operation returns the least such y. That is, the function $g(x_1,...x_n,y) = \mu y[f(x_1,...x_n,y) = 0]$ is given by the μ-operation. Functions obtainable by the μ-operation (given the conditions in the first sentence) are computable.

[1] N-tuples are just ordered pairs, triples, etc.

Recursive Functions

> Any function which is obtainable from the initial functions by a finite number of steps, using combination or the μ-operation is recursive.

The set of recursive functions is provably equivalent to what a Turing machine can calculate.
There are other ways to characterize the recursive functions, due to Post (like Turing's) and Markov (algebraic).
The subclass of primitive recursive functions, those which interested Hilbert, are those obtainable without the use of the μ-operation.
It has become standard to identify the primitive recursive functions as any obtained from the initial functions using composition of functions and primitive recursion (defining the value of a function at an argument in terms of its value at the previous argument).
The superclass of partial recursive functions are those obtainable by a weaker μ-operation, sometimes called least search or minimization.

Church's Thesis is that the recursive functions are exactly the computable ones.

Church's Thesis is widely accepted.
In one direction, Church's Thesis is obvious: all recursive functions are computable.
It also appears to many folks that every effectively computable function is recursive.
Still, there is some debate over whether Church's Thesis is provable.
This debate has focused on whether any identification of an informal concept with a formal notion can be proven.
Some philosophers consider Church's Thesis to be a working hypothesis.
Others take it to be merely another mathematical refinement of a commonsense notion like set, function, limit, or logical consequence.

I mention Church's Thesis both because of its inherent interest and because of its relevance to Hilbert's program.
Hilbert's finitism is just an insistence that we show that our mathematical proofs are primitive recursive.
It is not, clearly, a rejection of infinitary methods.
Hilbert's finitism is only one aspect of his work and it is a broad kind of finitism.

V. Hilbert's Formalism

Term formalism is the claim that the statements of mathematics actually concern the symbols themselves.

> The subject matter of mathematics is, in accordance with this theory, the concrete symbols themselves whose structure is immediately clear and recognizable (Hilbert " On the Infinite" 192).

Term formalism has obvious limits.
One lesson of Cantor's diagonal argument is that the real numbers do not each have names.
A term formalist has difficulty defending knowledge of objects, like the real numbers, which can't be correlated with concrete symbols, like inscriptions.
Thus, Hilbert, motivated by transfinite mathematics and his success argument, distinguished ideal elements of mathematics from real elements and shifted toward a game formalist view.

On game formalism, mathematics consists of manipulations of meaningless symbols.
The game-formalist Hilbert ascribes real meaning to the real elements of mathematics but no meaning at all to the ideal elements of mathematics.
Real mathematical elements are those which can be used in physical science.
For Hilbert, it was an open question whether the objects of Euclidean geometry are real or not.

> Although euclidean geometry is indeed a consistent conceptual system it does not thereby follow that euclidean geometry actually holds in reality. Whether or not real space is euclidean can be determined only though observation and experiment (Hilbert, "On the Infinite" 186).

The reality of Euclidean geometry depends for Hilbert on whether space is actually Euclidean.
Since space turns out to be hyperbolic, the objects of Euclidean geometry turn out to be ideal, not real.
We still posit such objects, Hilbert claims, for the fruitfulness they provide to mathematics and because of the simplicity and elegance they add to our theories.

> The method of ideal elements is used even in elementary plane geometry. The points and straight lines of the plane originally are real, actually existent objects...There is no theorem that two straight lines always intersect at some point...for the two straight lines might well be parallel. Still we know that by introducing ideal elements, viz., infinitely long lines and points at infinity, we can make the theorem that two straight lines always intersect at one and only one point come out universally true. These ideal "infinite" elements have the advantage of making the system of connection laws as simple and perspicuous as possible. Moreover, because of the symmetry between a point and a straight line, there results the very fruitful principle of duality for geometry (Hilbert " On the Infinite" 187).

Ideal elements allow generality in mathematical formulas.
Still, they require acknowledging both ideal Euclidean geometry and infinitary statements.
Even the logical claim that a rule of inference applies requires exceeding any particular finitary claims.
All universal claims are infinitary, even if their negations are not.
Conversely, '(∃x)Px' is a perfectly finitary statement
But '~(∃x)Px' is equivalent to '(∀x)~Px', which is infinitary.
So we are committed to infinitary statements even in the most basic logic and mathematics.
To express, for example, the commutativity of addition, we assert:

CA $(\forall a)(\forall b)(a + b = b + a)$

The quantifiers of CA range over all numbers, and so are not finitary.
We have to wonder about the meanings of terms in ideal statements, for example about the references of the 'a' and the 'b' in CA.

> Such an extension into the infinite is, unless further explanation and precautions are forthcoming, no more permissible than the extension from finite to infinite products in calculus. Such extensions, accordingly, usually lapse into meaninglessness (Hilbert " On the Infinite" 194).

So, CA is not finitistically acceptable and includes meaningless terms.

> We therefore conclude that a, b, =, +, as well as the whole formula $a + b = b + a$ mean nothing in themselves... (Hilbert " On the Infinite" 196).

We could introduce bounded quantifiers, ranging over only some finite set of numbers, to turn CA into a meaningful statement.
If we limit the sentence to hold over finite natural numbers, it can have significance.
But that meaningfulness has a theoretical cost.
The resulting theory is awkwardly gerrymandered.
We have to determine how much awkwardness in a theory is acceptable.

Hilbert's game-formalist claim that ideal terms are meaningless is motivated by finitistic concerns.
Despite his desire to retain Cantor's paradise, Hilbert is worrying about finite minds having access to infinities.
Since, he claims, we can not think of an actually infinite number, a completed infinite sequence, we have to re-interpret infinitary statements.

Even if we adopt game-formalism, though, we need criteria of acceptability for the infinitary elements.
We need precautionary measures to ensure that we do not engender paradox and inconsistency in the extensions beyond those which are finitistically verifiable.
Hilbert, rejecting claims that infinitary statements could be true (at least in the same way that real statements could be true) must show the consistency of mathematical theories.

> There is just one condition, albeit an absolutely necessary one, connected with the method of ideal elements. That condition is a *proof of consistency*, for the extension of a domain by the addition of ideal elements is legitimate only if the extension does not cause contradictions to appear in the old, narrower domain, or, in other words, only if the relations that obtain among the old structures when the ideal structures are deleted are always valid in the old domain (Hilbert "On the Infinite" 199).

Hilbert thus took the consistency of a set of axioms, even if they referred to ideal elements, as sufficient evidence for their acceptability.
Mathematical theories which include ideal elements thus must be tested for their consistency.
The systems themselves have to be studied.

VI. Deductivism, Consistency, and Completeness

Hilbert thus turned mathematicians' interests meta-mathematical, to the study of mathematical systems.
Here we see Hilbert's deductivist formalism.

> The leading idea of Hilbert's theory of proof is that, even if the statements of classical mathematics should turn out to be false as to content, nevertheless, classical mathematics involves an internally closed procedure which operates according to fixed rules known to all mathematicians, and which consists basically in constructing successively certain combinations of primitive symbols which are considered "correct" or "proved." This construction-procedure, moreover, is "finitary" and directly constructive (Von Neumann, "The Formalist Foundations of Mathematics" 61-2).

In addition to consistency, if one can show that a mathematical theory is complete, that every true theorem is provable, then one could hope for a solution to all open mathematical problems.

> We are all convinced that [every mathematical problem is solvable]. In fact one of the principle attractions of tackling a mathematical problem is that we always hear this cry within us: There is the problem, find the answer; you can find it just by thinking, for there is no *ignorabimus* in mathematics (Hilbert "On the Infinite" 200).

Von Neumann descries four steps in the deductivist's meta-mathematical project.
The first three steps designate the syntax of the object theory.
For those of you who have taken logic, he is just insisting on providing formation rules (steps 1 and 2) and inference rules (step 3).
The fourth step is the key to Hilbert's program.

> 4. To show (in a finitary combinatorial way) that those formulas which correspond to statements of classical mathematics which can be checked by finitary arithmetical methods can be proved (i.e. constructed) by the process described in [step 3] if and only if the check of the corresponding statement shows it to be true (Von Neumann, "The Formalist Foundations of Mathematics" 63).

Hilbert tried to establish that mathematical theories were both consistent and complete by studying mathematical systems themselves.
He thus founded the fields of meta-mathematics and metalogic that characterize much of contemporary logical research.

A meta-theory is one in which you discuss a different, subordinate theory, which is called an object theory.
We construct metalogical theories to explore formal systems to find the strength, extent, and limits of an object theory, say first-order logic.
Similarly, we can explore mathematical theories by doing meta-mathematics.
Metalogic, with its central notions of proof theory and model theory, blossomed in the twentieth century as a new and fruitful way of doing mathematics.
Model theory, in which a formal theory is examined from within a meta-theory, gave mathematicians novel tools for their proofs.

Let's look at some meta-theoretic terms and goals.
A system will be *consistent* if no contradiction is derivable from the axioms of that system.
We can prove that a system is consistent if there is some formula that is not provable in the system.
For, if a contradiction is present in the system, then any formula is derivable in classical logic.
If one can prove that some statement, say '0=1', is not provable in a system, then the system must be consistent.
(This technique only works in classical logic, which contains the law of the excluded middle.
The intuitionists will reject this law, as we will see later.)
Consistency is minimal condition on a theory's utility.
In logic, we want only the logical truths to be provable.
In mathematics, we want only true statements to be derivable.

Consistency proofs for portions of mathematics that depend on other portions are relatively easy to come by.
Hilbert, for example, in his 1899 axiomatization of Euclidean geometry had shown an arithmetic interpretation of his geometric system.

Thus, geometry is consistent if arithmetic is consistent.
We know Euclidean geometry to be relatively consistent, but not absolutely consistent.

> In geometry and physical theory, proof of consistency is effected by reducing their consistency to that of the axioms of arithmetic. But obviously we cannot use this method to prove the consistency of arithmetic itself. Since our theory of proof, based on the method of ideal elements, enables us to take this last important step, it forms the necessary keystone of the doctrinal arch of axiomatics (Hilbert, "On the Infinite" 200).

In other words, Hilbert wants a consistency proof for all of mathematics.

Consistency is related to *soundness*.
In a sound theory, every formula that is provable is true.
An inconsistent system is automatically unsound.
But consistency is a syntactic notion, independent of truth.
Soundness is a semantic notion, involving truth.
Hilbert did not make the distinction between syntactic and semantic properties and so ran the notions of soundness and consistency together.
But, we might want more of our system than consistency, like to know if its theorems are true.
And, we might want to avoid all talk of truth when determining consistency.

The converse of soundness is called *completeness*: Every true formula is provable.
Note that inconsistent systems are trivially complete.
If we can prove every statement, then among those statements will be the true ones.
The ultimate goal of formalist meta-mathematics was to prove that mathematics is both sound and complete: all the provable theorems are true and all the true theorems are provable.
It is natural to read von Neumann's fourth step as indicating a desire for proofs of both completeness and soundness.
Given Hilbert's worries about infinitary statements, his emphasis on consistency, rather than soundness, is understandable.

Like Hilbert, the logicists were interested in establishing formal systems to guide mathematical inference.
Both the formalist and the logicist desired completeness proofs.
But the logicists did not worry about consistency since mathematical statements were all supposed to be clearly true.
They were supposed to be logical truths.
For Frege, the consistency of mathematics follows from the presumed truth of its axioms.
The logicist wants a formal system to show that we can derive all of mathematics from logical truths, to show that the system is complete.
Hilbert gives up the idea that we have proven any (infinitary) truths.
He just wants to show that the system we use for our mathematics is consistent, that all we are doing in meta-theory is proving that our systems are acceptable, not that they yield truths.

Hilbert's desire for consistency proofs gives the formalist project another affinity to logicism.
The logicist wants to show that all of mathematics is just logic in disguise.
The deductivist, by emphasizing meta-theoretic proofs of the inferential health of a mathematical system, turns mathematics into the study of what follows from what.
Meta-mathematics becomes the study of logical systems and the relations among statements.

At the time of the von Neumann article, Hilbert's program had some hope.
Before Gödel proved his incompleteness theorems, in 1931, he proved the completeness of first-order logic.
Propositional logic and first-order logic are both sound and complete.
Von Neumann alludes to consistency proofs for small, non-classical systems.

> Hilbert's system has passed the first test of strength: the validity of a non-finitary, not purely constructive mathematical system has been established through finitary constructive means (Von Neumann, "The Formalist Foundations of Mathematics" 65).

In contrast, Gödel's incompleteness theorems struck a decisive blow against Hilbert's pursuits of completeness and consistency proofs for mathematics.
Gödel's first theorem showed that for even quite weak mathematical theories a consistent theory could not be complete.
Gödel's second theorem proved that the consistency of a theory could never be proven within the theory itself.
We can only prove that mathematical theories are consistent relative to other theories.

VII. Gödel First Theorem

I will leave details of Gödel's incompleteness proofs for those of a mathematical bent.
Smullyan's account is nice, but technical.
Prof. Cockburn has constructed this excellent handout.
There are lots of other good approaches.
One other that I recommend is in Boolos, Burgess, and Jeffrey, *Computability and Logic*.
They approach the same concepts through the notions of Turing machines and computation.

Gödel provided two theorems.
The first shows that completeness is impossible in a sufficiently strong consistent theory.
Gödel's proof uses a version of the liar paradox.
He constructs a sentence within the formal theory that says of itself that it is not provable.

The key to Gödel's theorems is the arithmetization of proof procedures.
This is called Gödel-numbering.
We assign numerical values to all the expressions of the system.
To the rules of inference, we correlate arithmetic equivalents.
Thus, deriving a statement becomes equivalent to performing some arithmetic operation.

We show that the Gödel-sentence is not provable by using a version of the diagonalization argument.
We construct a sentence which says, in arithmetic language, that it is not provable.
For Gödel's proof to work, to be able to construct the Gödel sentence, a theory must be able to Gödel-number.
The theory must have names for each of the natural numbers, for example.
There are other requirements on a theory.

The following selection on what it takes for a theory to be sufficiently strong to admit of the Gödel results is from Peter Suber, http://www.earlham.edu/~peters/courses/logsys/g-proof.htm

Remember that Gödel's theorem does not apply to all systems of arithmetic, only to those that are "sufficiently powerful." This is what creates the dilemma of incompleteness: either a system is incomplete because it is too weak for Gödel's theorem to apply, or it is incomplete because the theorem does apply... If we imagine a thief who only robs the "sufficiently rich" and who accosts all travelers on a certain road, then we know that travelers on that road will always be poor: either because they are not sufficiently rich, or because they have been robbed. Here in summary are the conditions of eligibility that describe when a system is "sufficiently powerful," or when it is rich enough to be robbed by G.

1. It must be a formal system of arithmetic.
    1. On its intended interpretation, some of its theorems must express truths of arithmetic.
    2. The formal language of the system is capable of naming each of the natural numbers, and does so on its intended interpretation.
    3. The formal language of the system has a finite alphabet, and all wffs are only finitely long.

2. It must be a "respectable" system of arithmetic.
    1. It must be consistent.
    2. It must represent (in the technical sense) every decidable set of natural numbers.
    3. It must be the case that each of its wffs with free variables is a theorem iff some closure of it is a theorem.

3. It must be a "well-made" system of arithmetic.
    1. Its set of axioms must be decidable.
    2. Its set of rules of inference must be decidable.

The first wave of conditions means that the system must be a system of finitary polyadic predicate logic, extended with proper (as opposed to logical) axioms so that it is capable of proving at least some arithmetical wffs as theorems. The second wave of conditions makes it powerful. The third wave ensures that its set of proofs is decidable, which gives it an effective test of proofhood, which allows the predicate for proof-pairhood to be decidable.

Weak systems, for example ones which can not arithmetize their vocabulary, are not susceptible to the Gödel theorems.
That is why first-order logic can be complete.
There are even some limited versions of arithmetic, like Presburger arithmetic, which are complete.
Unfortunately, Presburger arithmetic omits multiplication!

VIII. Gödel's Second Theorem

Gödel's second theorem shows that proofs of consistency are impossible, within a single system.
Here is the argument, following Shapiro:

1. For any sufficiently strong theory, T, we can regiment predicates for consistency and derivability, and a Gödel sentence, G, which can not be proven within the theory.
2. So, we can write a sentence within a theory, that says: If T is consistent then G is not derivable in T.
3. So, if T is consistent, then G, and we can write this within the theory.
4. Assume T is consistent.
5. Then, we can derive G.
6. But, we know that ~Der(G)
7. So, no consistent theory can prove its own consistency.

1. $G \equiv \sim Der_T(G)$ By definition
2. $Con\ (T) \supset \sim Der_T(G)$ By the first theorem
3. $Con\ (T) \supset G$ fr 1, 2
    4. Provable: Con (T)
    5. Provable: G
    6. Not provable: G
7. So not provable Con (T)

Given that Hilbert's program depends on developing proofs of consistency for formal theories of mathematics, Gödel's second theorem effectively shows that Hilbert's program can not work.

IX. Ways Out?

We have seen four different Hilberts:

1. the term formalist (mathematical terms refer to inscriptions);
2. the game formalist (ideal terms are meaningless);
3. the deductivist formalist (mathematics consists of deductions within consistent systems); and
4. the finitist (mathematics must proceed on a finitary, but not foolishly so, basis).

The formalist aspects of Hilbert's view have affinities to logicism, in that he focuses on the deductive aspects of mathematics.
But, the finitist side of Hilbert's view is closer to Kant's work than to Frege's.

> Kant taught - and it is an integral part of his doctrine - that mathematics treats a subject matter which is given independently of logic. Mathematics, therefore, can never be grounded solely on logic. Consequently, Frege's and Dedekind's attempts to so ground it were doomed to failure (Hilbert "On the Infinite" 192).

Hilbert refers to preconditions for the legitimacy of our finitary reasoning.
These preconditions include the cognition of mathematical objects themselves.
One of Hilbert's achievements was an axiomatization of geometry.

Here is its epigraph:

> All human knowledge thus begins with intuitions, proceeds thence to concepts and ends with ideas. - Kant, *Critique of Pure Reason*, "Elements of Transcendentalism," *Second Part*, II (Hilbert, *Foundations of Geometry* 10^{th} ed., p 2).

We will return to Kantian views of mathematics in our discussion of intuitionism, the last of the three major philosophical views of mathematics developed in the early twentieth century.
Today, Hilbert's views survive in Hartry Field's fictionalism, which emphasizes the consistency of mathematical theories, and which we will examine in a few weeks; in Mark Balaguer's plenitudinous platonism, which asserts that every consistent mathematical theory truly describes a mathematical realm; and in defenses of limited versions of Hilbert's Program.

One way to try to revive Hilbert's program, in the wake of Gödel's theorems, is to weaken the demand for consistency proofs.
Shapiro even believes that Hilbert made no such demand.

> The dream of finding a single formal system for all of ideal mathematics was not an official (or essential) part of the Hilbert programme (Shapiro 166).

While I believe that Shapiro's claim is revisionist, there are options for the contemporary formalist.

The formalist can seek sound mathematical theories.
Of course, some weak systems, like Presburger arithmetic, are sound, and even complete, but not sufficient for full mathematical purposes.
The contemporary formalist can also redefine consistency for mathematical theories in ways which resist their susceptibility to Gödel results; see Shapiro 167-8.

Hilbert's program was largely a response to his worries about the paradoxes discovered by those investigating infinitary mathematics and its underlying set theory.
As the connection between Hilbert's program and Church's Thesis shows, it also has a broader basis.
Church's Thesis is foremost a question in computer science about the abilities of machines to use formal systems to construct derivations.
It also leads us to fundamental questions about our own abilities to construct proofs.
If we are, as we seem to be, merely sophisticated computers, complex information processors, then identifying and explicating the notion of an algorithm has profound implications for our understanding of human intelligence.
Hilbert's program was precisely an attempt to reconcile the grand reach of infinitary mathematics with our human limitations.