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Class #12: Logicism

I. Ontological Reductions in Mathematics

Frege's grand claim is that much of mathematics is logic in complicated disguise.

> Arithmetic...becomes simply a development of logic, and every proposition of arithmetic a law of logic, albeit a derivative one (Frege, *Grundlagen* §87).

This claim is called logicism.
It is the intellectual heir of Leibniz's proposal to reduce all propositions to elementary identities.
Frege's project of reducing mathematics to logic really requires two steps.
The first step is the reduction of the theory of natural numbers to logic.
This step of his argument is the focus of our reading.
Mathematics is much more than the theory of natural numbers.
The second step is the reduction of the rest of number theory to the theory of natural numbers.

Frege's logicist project failed because of problems with the first step.
Number theory does not reduce to logic in the sense that Frege hoped, as Russell's paradox shows.
In order to define numbers, one needs not just simple logic but some more-controversial claims.
Typically, the more-controversial claims are expressed in set theory.
Mathematicians now typically accept the claim that all of mathematics can be reduced to set theory.

In order to get a feel for the reductionist projects that Frege envisioned, let's look a bit at how to define some of the more-complicated mathematics in terms of the theory of natural numbers.
One principle which guides reductions is called Leibniz's law.
Leibniz's law is a claim about the extensionality of mathematics.

> Things are the same as each other, of which one can be substituted for the other without loss of truth (Frege, *Grundlagen,* §65).

We call this condition substitutivity *salva veritate*.
It is a standard, minimal requirement of any definition.
If 'bachelor' is identical in meaning to 'unmarried man', we have to be able to substitute, *salva veritate*, the one term for the other, wherever they occur.

The standard theory of the natural numbers is based on Dedekind's axioms for arithmetic, which we call the Peano axioms.
There are different ways to present the Peano axioms, but we've seen these:

P1: 0 is a number
P2: The successor (x') of every number (x) is a number
P3: 0 is not the successor of any number
P4: If x'=y' then x=y
P5: If P is a property that may (or may not) hold for any number, and if i. 0 has P; and ii. for any x, if x has P then x' has P; then all numbers have P.
P5 is mathematical induction, actually a schema of an infinite number of axioms.

The Peano axioms refer to a sequence of numbers, which we can call ℕ = 0, 1, 2, 3...
From the Peano axioms, using logic and a bit of set theory, like the notion of an ordered pair, we can define standard arithmetic operations, like addition and multiplication.

We can define the integers, **Z**, in terms of the natural numbers by using subtraction.
Since -3 is 5-8, we can define -3 as the ordered pair <5,8>.
But -3 could also be defined as <17,20>.
We thus take the negative numbers to be equivalence classes of such ordered pairs.
The equivalence class for subtraction is defined using addition: <a,b> ~ <c,d> iff a + d = b + c.
So, we can define **Z** = ...-3, -2, -1, 0, 1, 2, 3... in terms of ℕ, addition, and the notion of an ordered pair.

The rationals, **Q**, can be defined in terms of the integers, **Z**, by using ordered pairs of integers.
a/b::<a,b>, where '<a,b> ~ <c,d> iff ad = bc' is the identity clause.

By adopting the definitions of **Z** and **Q**, guided by Leibniz's law, we have reduced the theory of rationals to the theory of natural numbers (given set theory and logic).
Anything we want to say about the rationals, we can say in a slightly more complicated fashion about the natural numbers.

We can also reduce the theory of the real numbers, ℝ, to the theory of natural numbers (and set theory), though that reduction is a bit more complicated.
The real numbers are all of the numbers found on the number line.
Rational numbers are part of the number line, but there are, as the Pythagoreans discovered, incommensurable numbers: real numbers that are not rational.
The relation between the real numbers and the rational numbers was unclear in the 19^{th} century.
Both the rationals and the reals are dense: between any two there is a third.
The reals are also continuous.
One important question for 19^{th} century mathematicians was how to characterize this continuity.

In the early 19^{th} century, worries about the infinite had put pressure on mathematicians.
Cantor had not yet produced his set theory, which founded his theory of transfinite numbers.
There was a growing pressure to provide a solid underpinning of calculus and its infinitesimals.
Niels Abel, for example, complained in the very early nineteenth century about,

> the tremendous obscurity which one unquestionably finds in analysis. It lacks so completely all plan and system that it is peculiar that so many men could have studied it. The worst of it is, it has never been treated stringently. There are very few theorems in advanced analysis which have been demonstrated in a logically tenable manner. Everywhere one finds this miserable way of concluding from the special to the general and it is extremely peculiar that such a procedure has led to so few of the so-called paradoxes (Abel, quoted in Kline, *Mathematical Thought from Ancient to Modern Times* 947).

Analysis is the broader field which contains calculus, differential equations, theories of real and complex numbers, analytic functions, and measurement theory.
Karl Weierstrass, approaching the question of the relation between the reals and the rationals in the mid-late nineteenth century, was concerned, in part, with the geometric foundation of the calculus.
Weierstrass, like Abel before him, wanted to put analysis on firm footing.
His project is now known as the arithmetization of analysis.

Bolzano and Cauchy also made contributions, as did Dedekind.
While Weierstrass's influence on Frege is a matter of historical debate, the pressures which led to the project of the arithmetization of analysis were surely the same as those which led to Frege's logicism.
The problem facing analysis is that the geometric foundation of the reals and their continuity is insufficient, as Dedekind noticed.

> In discussing the notion of the approach of a variable magnitude to a fixed limiting value, and especially in proving the theory that every magnitude which grows continually, but not beyond all limits, must certainly approach a limiting value, I had recourse to geometric evidences. Even now, such resort to geometric intuition in a first presentation of the differential calculus, I regard as exceedingly useful, from the didactic standpoint, and indeed indispensable, if one does not wish to lose too much time. But that this form of introduction into the differential calculus can make no claim to being scientific, no one will deny. For myself this feeling of dissatisfaction was so overpowering that I made the fixed resolve to keep meditating on the question till I should find a purely arithmetic and perfectly rigorous foundation for the principles of infinitesimal analysis (Dedekind, cited in Gillies, *Frege, Dedekind, and Peano on the Foundations of Arithmetic*, p 4).

There is a mapping from the rationals to the number line, but not backwards.
That is, for every rational, we can construct a distance from an arbitrary origin to a point on the line.
But it is not the case that every distance on the line can be expressed as a rational.
That is the problem of incommensurables.

At the core of the problem were the definitions of 'continuity' and 'function'.
Part of the success of the project was the epsilon-delta definition of continuous functions, due to Weierstrass in the 1860s, but based on ideas from Cauchy, 1821

> a function $f(x)$ is continuous at a if for any $\varepsilon > 0$ (no matter how small) there is a $\delta > 0$ such that
> for all x such that $|x - a| < \delta$, $|f(x) - f(a)| < \varepsilon$.

From this definition, we can find a definition of limits.

> A function $f(x)$ has a limit L at a if for any $\varepsilon > 0$ there is a $\delta > 0$ such that
> for all x such that $|x - a| < \delta$, $|f(x) - f(a)| < \varepsilon$.

The Bolzano-Cauchy-Weierstrass group proceeded to provide formal proofs in terms of these rigorous definitions, to replace the earlier, loose presentations.
The ε-δ definition of limits is arithmetic rather than relying on the geometric notion of an infinitesimal.

In addition to the arithmetic definitions of continuity and limits, Weierstrass, Dedekind, and Cantor pursued more rigorous definition of the reals, in terms of the rationals.
I will mention only Dedekind's derivation, from 1872, though there are others.
Cauchy's definition, for example, is easier to work with, for some purposes.
The key concept of Dedekind's definition of the real numbers is that of a cut, which has become known as a Dedekind cut.
The real numbers are identified with separations of the rationals, **Q**, into two sets, Q_1 and Q_2, such that every member of Q_1 is less than or equal to the real number and every member of Q_2 is greater.

So, even though $\sqrt{2}$ is not a rational, it divides the rationals into two such sets.
Not all cuts are produced by rational numbers.
So, we can distinguish the continuity of the reals from the discontinuity of the rationals on the basis of these cuts.
Real numbers are thus defined in terms of sets of rationals, the set of rationals below the cut.
These sets have no largest member, since for any rational less than $\sqrt{2}$, for example, we can find another one larger.
But, they do have an upper bound in the reals.

By adding our definition of the real numbers in terms of the rational numbers to our definitions of the rationals in terms of the natural numbers, we have defined the reals in terms of the natural numbers.
Such definitions do two things.
First, they make it clear that infinitesimals and real numbers are accessible to finite (or at least denumerable) methods.
Second, they make it plausible that we can reduce the problem of justifying our knowledge of mathematics to the problem of justifying our knowledge of the natural numbers.
We have an ontological reduction of the objects of analysis to the objects of number theory: we need not assume the existence of any objects beyond the natural numbers in order to model analysis.

Moreover, while I have been talking about the reduction of the theory of real numbers to the theory of natural numbers, another reduction is easily achieved.
We can reduce the theory of natural numbers to set theory.
That is, we can write the Peano postulates as statements governing sets.
We can define certain sets to serve as the natural numbers.
So we can reduce analysis to set theory.

Despite the ontological reduction that Dedekind cuts and the definitions of real numbers in terms of rationals achieves, there remain questions about whether mathematics is reducible to the theory of natural numbers or to set theory.

> One can only speculate on the extent to which Fregean logicism might accommodate some of the contemporary branches of mathematics, such as complex analysis, topology, and set theory (Shapiro, *Thinking About Mathematics* 113).

Contemporary reductionists (or foundationalists) in mathematics tend to focus their efforts on reducing mathematics to set theory or even to the more-abstract category theory.
It seems fairly obvious that set theory can serve as a reductionist base for mathematics, ontologically.
But, there remain questions about what we might call a methodological reduction.
We could not prove sophisticated theorems in specialized branches of mathematics in set-theoretic or number-theoretic terms.
Even fairly simple number-theoretic inferences become impossibly complex in set theory.
So, even if we think that there is no need to posit objects other than the natural numbers to serve as models for our mathematical theorems, we do need to posit methods and techniques beyond those of number theory.
This methodological claim seems to undermine Frege's allegation that we need no special mathematical laws.

> The present work will make clear that even an inference like that form n to $n + 1$, which on the face of it is peculiar to mathematics, is based on the general laws of logic, and that there is no need of special laws for aggregative thought (Frege, *Grundlagen* iv).

Still, our concern here is more ontological than methodological (this is, in fact, a seminar in metaphysics) and I will put this concern about methodological reduction to the side.

I said that Frege's logicist claim depended on two arguments: that we can reduce all of mathematics to the theory of natural numbers and that we can reduce the theory of natural numbers to pure logic.
I'll take the first argument to be settled, at least on the ontological level.
Frege's argument, then, depends on whether we can, as he alleges, define the natural numbers in purely logical terms.
Before we get to those arguments, let's place Frege in opposition to two prior philosophers: Kant and Mill.

II. Frege Against Mill and Kant

In previous notes, I discussed Frege's arguments against Mill.
While Frege approved of Mill's attempts to avoid psychologistic explanations of mathematics, he deplored Mill's claim that mathematical theorems were, strictly speaking, false.
Frege attacked Mill's definition of number and his claim that mathematical propositions are mere inductions from sense experience.
Mill said that all numbers were numbers of something, that numbers were properties of physical objects: five apples, ten giraffes.

> Ten must mean ten bodies, or ten sounds, or ten beatings of the pulse (Mill, *A System of Logic* 189).

Against Mill's definition of number, Frege complains that we see only apples, not numbers of apples.
Numbers apply to concepts, not to objects.

> While looking at one and the same external phenomenon, I can say with equal truth both "It is a copse" and "It is five trees," or both "Here are four companies" and "Here are 500 men." Now what changes here from one judgement to the other is neither any individual object, nor the whole, the agglomeration of them, but rather my terminology. But that is itself only a sign that one concept has been substituted for another (Frege, *Grundlagen* §46).

Further against taking numbers to be properties, Frege considers whether 'Pa • Pb' entails 'P(a • b)'.
If we take 'Px' as 'x is a student', and 'a' and 'b' as names of two students, the entailment holds.
But, if we take 'Px' as 'x is one', and take 'a' and 'b' again as names of two students, the entailment fails.
If number were a property, it would attach to objects.
But, numbers attach to concepts.

We can rephrase sentences like 'There are ten bodies', which are paradigmatic for Mill, as 'The number of bodies is ten', in which the number is clearly an object, rather than a predicate: $(\exists x)(Nx \bullet x=10)$

Additionally, even to apply the number zero to anything is to undermine Mill's account.
Zero is a number, but it is not the number of any thing.
Mill's accounts of infinite numbers and even very large finite numbers are also lame.

Against Mill's claim that mathematical propositions are the results of enumerative inductions, Frege claims that induction can not support universal and modal claims and can not support knowledge of mathematical objects which do not appear in nature.
Frege adds that all induction presumes probabilistic reasoning but probabilistic reasoning itself relies on arithmetic.

> Induction [then, properly understood,] must base itself on the theory of probability, since it can never render a proposition more than probably. But how probability theory could possibly be developed without presupposing arithmetical laws is beyond comprehension (Frege, *Grundlagen* §10).

Frege agrees with Kant that geometry is synthetic *a priori*.

> I consider Kant did great service in drawing the distinction between synthetic and analytic judgements. In calling the truths of geometry synthetic a priori, he revealed their true nature (Frege, *Grundlagen* §89).

But Frege denies Kant's claims that arithmetic is synthetic.
He argues that arithmetic is analytic, and cites as the cause of Kant's error Kant's wrong concept of analyticity.

> Kant obviously - as a result, no doubt, of defining them too narrowly - underestimated the value of analytic judgements, though it seems that he did have some inkling of the wider sense in which I have used the term (Frege, *Grundlagen* §88).

Kant analyzes judgments as linking subject concepts with object concepts.
To say that apples are sweet is to join the concept of apples with the concept of being sweet.
Since the concept of being an apple does not by itself include the concept of being sweet, the judgment is synthetic.
But, when we say that every mother has a child, the concept of being a mother does include the concept of having a child.

Against Kant's analysis, Frege first worries about when the sentence contains an individual object as the subject.
Consider 'Matt is a cat'.
'Matt' seems not to stand for a concept, but for a thing.
Here, we might be tempted to say that the statement's analyticity conditions depend not on whether the subject concept is contained in the predicate concept, but whether the object to which 'Matt' refers necessarily has the property of being a cat.
Or, we might want to know if the concept of being a cat contains an object, Matt.
But, Matt does not seem to be a concept.
This worry is at the foundation of twentieth-century philosophy of language.
It is ancillary here.

Second, Frege worries about existential statements, e.g. 'There are electrons'.
Such statements do not seem to fit into Kant's analysis of all sentences into subject-predicate form.
One of Frege's advances in logic is to see a predicate not as a closed concept, but on analogy with a function, including the kind of hole, or holes, a function takes.
In 'Matt is a cat', the predicate is not catness, but '...is a cat'.
If we take all sentences to be of subject-predicate form, the B1 and B2 have different predicates.

> B1 Clinton is between Syracuse and Albany
> B2 New York is between Philadelphia and Boston.

For Kant, B1 has the predicate, 'is between Syracuse and Albany' but B2 has the predicate 'is between Philadelphia and Boston'.
These are completely distinct properties.
Frege showed that those sentences are better understood as sharing a three-place relation among objects, betweenness.
Frege was thus able to portray betweenness more generally.

Third, and most important, Frege criticizes Kant's notion of a concept as a list waiting to be unpacked.
Frege claims that the elements of a concept are linked.
They are not merely appended to each other, but tied together.
And, we can unpack them, tracing them back to their justificatory grounds.
Definitions, then, should not be mere lists of properties, but chains of inference.

> The more fruitful type of definition is a matter of drawing boundary lines that were not previously given at all. What we shall be able to infer from it, cannot be inspected in advance; here, we are not simply taking out of the box again what we have just put into it. The conclusions we draw from it extend our knowledge, and ought therefore, on Kant's view, to be regarded as synthetic; and yet they can be proved by purely logical means, and are thus analytic. The truth is that they are contained in the definitions, but as plants are contained in their seeds, not as beams are contained in a house. Often we need several definitions for the proof of some proposition, which consequently is not contained in any one of them alone, yet does follow purely logically from all of them together (Frege, *Grundlagen* §88).

For Kant, mathematical theorems had to be constructed, synthetically, in intuition.
But Frege laments our lack of ability to construct some mathematical objects.

> Nought and one are objects which cannot be given to us in sensation. And even those who hold that the smaller numbers are intuitable, must as least concede that they cannot be given in intuition any of the numbers greater than $1000^{1000^{1000}}$ (*Frege, Grundlagen* §89).

Frege's goal, then, is to show that mathematics is analytic by providing a logical system in which one can define and then derive all of mathematics.
Instead of taking mathematical statements to be true on the basis of our intuitive apprehension of them, we can take them to be true because they are derivable from logical truths using a secure logic.

> The fundamental propositions of arithmetic should be proved, if in any way possible, with the utmost rigour; for only if every gap in the chain of deductions is eliminated with the greatest care

> can we say with certainty upon what primitive truths the proof depends; and only when these are known shall we be able to answer our original questions (*Frege, Grundlagen* §4).

The task of constructing a system for defining and deriving all of mathematics unites Frege's three main projects.

III. Three Books, Three Principles, One Project

Almost all of Frege's work traces back to his logicist project.
Frege wrote three books.
The *Begriffsschrift* (1879) formulates his logical language.
While Frege's syntax appears unfamiliar to us, standard contemporary predicate logic is mainly just a notational variant of Frege's logic.
The *Grundlagen* (1884), which we have been reading, presents a philosophical defense of the logicist project, and criticisms of Mill, Kant, and others.
The *Grundgesetze*, in two (of three planned) volumes (1893 and 1903), did some of the technical work promised in the *Grundlagen*.
After the discovery of Russell's paradox, Frege despaired of completing the work; he never wrote the intended third volume of the *Grundgesetze*.

Russell developed a way to work around the paradoxes.
He and Alfred North Whitehead produced the massive *Principia Mathematica* in three volumes (1910, 1912, and 1913), pursuing the logicist project.
Moreover, many philosophers today work on what we call the neo-Fregean (or neo-logicist) project, attempting to show how much of Frege's reduction persists despite the paradoxes.
Frege's work was almost entirely ignored when he initially published it, but became the cornerstone of twentieth-century English-language philosophy, largely due to Russell's influence.

While Mill was continuing the empiricist's project of attempting to trace our knowledge of mathematics to our sense experience, Frege was pursuing Leibniz's distinction between the origins of our beliefs and their justifications.

> It can...be asked, on the one hand, by what path a proposition was gradually reached, and on the other hand, in what way it is now finally to be most firmly established. The former question possibly needs to be answered differently for different people; the latter is more definite, and its answer is connected with the inner nature of the proposition concerned. The firmest proof is obviously the purely logical, which, prescinding from the particularity of things, is based solely on the laws on which all knowledge rests. Accordingly, we divide all truths that require justification into two kinds, those whose proof can be given purely logically and those whose proof must be grounded on empirical facts (Frege, Preface to *Begriffsschrift*, III).

In the introduction to the *Grundlagen*, Frege presents three principles.

> always to separate sharply the psychological from the logical, the subjective from the objective;
> never to ask for the meaning of a word in isolation, but only in the context of a proposition;
> never to lose sight of the distinction between concept and object (Frege, *Grundlagen,* x).

We have already seen how Frege uses the third principle against Mill's definition of number.
Leibniz used the first principle against Locke and I claimed that Locke's violation of the principle was a sort of genetic fallacy.
Frege uses the principle against both Mill and Kant.

> Now these distinctions between a priori and a posteriori, synthetic and analytic, concern, as I see it, not the content of the judgement but the justification for making the judgement (Frege, *Grundlagen* §3).

In Mill's case, Frege argues that classifying mathematics as empirical because it requires sense experience errs by missing the distinction between the origin of our beliefs and their justifications.

> We are making a psychological statement, which concerns solely the content of the proposition; the question of its truth is not touched. In this sense, all of Münchhausen's tales are empirical too; for certainly all sorts of observations must have been made before they could be invented (Frege, *Grundlagen* §8).

In Kant's case, Frege argues that making intuition an essential component of mathematical knowledge violates the first principle.
Frege has independent reasons to hold that mathematical objects are objective.
They are independent of us.

> Number is no whit more an object of psychology than, let us say, the North Sea is...If we say, "The North Sea is 10,000 square miles in extent" then neither by "North Sea" nor by "10,000" do we refer to any state of or process in our minds: on the contrary, we assert something quite objective, which is independent of our ideas and everything of the sort (Frege, *Grundlagen* §26).

Frege is not claiming that numbers are concrete, just that they are objects.
Part of the objectivity of numbers comes from the truth value of expressions which contain numbers.
So, the reason that 'snow is white' is true is that there is snow, and it has the property of being white.
By analogy, 'five is prime' is true because there is a five, and it has the property of being prime.
Numbers are not spatio-temporal objects, but they are objects nonetheless.

> Not every objective object has a place (Frege, *Grundlagen* §61).

For Frege, the objectivity of mathematics consists in part in its independence from us.
But, it also depends on the logicist project more generally.
If mathematical theorems are really just logical truths, then they can be seen as applicable most generally, most objectively.
Every mathematical theorem will be justified by a purely objective, secure derivation.
Frege constructs his concept-writing in the *Begriffsschrift* precisely order to establish the logicist claim.
There had been work in the nineteenth century, particularly by Boole and Schröder, unifying Aristotle's categorical logic with propositional logic, which traced back (at least) to the Stoics.
Frege made advances in predicate logic, including the use of quantifiers, and in the adoption of a unique, but cumbersome, notation.
The goal, remember, is to trace all of mathematics back, without any gaps, to return to, "The Old Euclidean standards of rigour" (Frege, *Grundlagen* §1).

In the *Begriffsschrift* and the *Grundlagen,* Frege expresses his view that there is just one, most estimable proof per proposition.
Such proofs will be completely gap-free, if long.
All appeals to intuition must be eliminated, and every step must be guided purely syntactically.

> The demand is not to be denied: every jump must be barred from our deductions (Frege, *Grundlagen* §91).

To prove that an arithmetic statement is analytic, its derivation from basic logical principles must be possible.
Only such proofs will determine whether a proposition is analytic or synthetic.
Analytic statements will be provable from mere logic.
Synthetic proofs will rely on assumptions given in intuition.

> The problem [of determining whether a statement is analytic or synthetic] becomes, in fact, that of finding the proof of the proposition, and of following it up right back to the primitive truths (Frege, *Grundlagen* §3).

The success of Frege's work, over all three books, depends on whether he can define the numbers and the axioms governing them in purely logical terms.

IV. Frege's Definitions of Numbers

Frege needs a definition of number which takes them as objects, and which allows for precise identity conditions for those objects.
He establishes definitions of one-one correspondence, the property of having the same number as, number, zero, successor, and natural number.
From these definitions, he can derive definitions of each individual number.
In the *Grundlagen*, he sketches these definitions and derivations.
In the *Grundgesetze*, he develops the derivations fully.

For his definitions, Frege relies on his second principle, that the meaning of a term can only be determined as part of a complete proposition and not in isolation.
This principle has become known as Frege's Context Principle.
It entails that he defines numbers within sentences, rather than as individual objects.

Frege believes that violating the context principle is the source of the Berkeley problem.
If we look for meanings of individual number terms, and any other individual terms, we are tempted to think of them as particular ideas in our minds.
Lockean conceptualism, Berkeleyan nihilism, or Kantian intuitionism follows.

> If the second principle is not observed, one is almost forced to take as the meanings of words mental pictures or acts of the individual mind, and so to offend against the first principle as well (Frege, *Grundlagen* x).

Frege's central point here is the argument against any of the other reductive views about numbers.
Kant reduced numbers to intuition; Locke reduced them to abstract ideas, which are psychological; and

Mill reduced them to empirical objects.
Frege's logicism rejects all of these views: mathematical objects are logical.

Frege's definitions of numbers are not simple.
The short explanation is that numbers are certain kinds of sets, indeed sets of sets.

Frege takes sets to be logical objects, extensions of predicates.
The extension of a concept is the set of things which fall under that concept, or which have that property.
Frege believes that the concept of the extension of a predicate is more precise than the concept of a set.

> The terms "multitude", "set" and "plurality" are unsuitable, owing to their vagueness, for use in defining number (Frege, *Grundlagen* §45).

This view, though, is an artifact of his era, in which set theory was still undeveloped.
Frege's *Begriffsschrift* defined extensions for predicates in the process of developing a rigorous logical system.
For Fregean logic, a predicate is a mathematical function.
The extension of a predicate is the range of that function.
The extension of a concept is just the set of things which have the property assigned by the concept.
Nowadays, we model predicate logic using the tools of set theory, so Frege would accept sets, I presume.

For one-one correspondence, and the property of having the same number as, Frege relies on what has come to be known as Hume's principle and which we saw in Cantor's work.

> We have to define the sense of the proposition "the number which belongs to the concept *F* is the same as that which belongs to the concept *G*"... In doing this, we shall be giving a general criterion for the identity of numbers. When we have thus acquired a means of arriving at a determinate number and of recognizing it again as the same, we can assign it a number word as its proper name. Hume long ago mentioned such a means: "When two numbers are so combined as that the one has always an unit answering to every unity of the other, we pronounce them equal" (Frege, *Grundlagen* §§62-3).

To define 'number', Frege relies directly on extensions.

> The number which belongs to the concept *F* is the extension of the concept "equal to the concept *F*" (Frege, *Grundlagen* §68).

Frege's definition tells us when a number belongs to a concept.
But numbers are objects themselves, not merely properties of concepts.
Recall Frege's argument against Mill's definition of number, that he takes them as properties of objects when they are really objects themselves.
Frege must provide a definition of the number terms without appealing merely to when they hold of concepts.
Numbers are second-order extensions, extensions of extensions.
In particular, numbers are extensions of all extensions of a particular size.
In Russell's terms, they are sets of sets; two is the set of all two-membered sets.

For Frege, 0 belongs to a concept if nothing falls under the concept.

Thus, Frege can define zero by appealing to a concept with no extension.

> 0 is the Number which belongs to the concept "not identical with itself" (Frege, *Grundlagen* §74).

Again in Russell's terms, 0 is the set of all sets which are not identical to themselves (i.e. the number of x such that $x \neq x$).

The definitions of the rest of the numbers can be generated inductively, using the successor definition.

> There exists a concept *F*, and an object falling under it *x* such that the Number which belongs to the concept *F* is *n* and the Number which belongs to the concept 'falling under *F* but not identical with *x*' is *m*" is to mean the same as "*n* follows in the series of natural numbers directly after *m* (Frege, *Grundlagen* §76.)

1 applies to a concept if that concept a) applies to at least one thing, and b) if it applies to two things, they are the same thing.
In contemporary notation: $(\exists!x)Fx :: (\exists x)[Fx \bullet (y)(Fy \supset y=x)]$
1 then may be defined as the number which belongs to the concept 'identical to 0', since there is only one concept 0.
More succinctly, 1 is the set of all 1-membered sets; 2 is the set of all 2-membered sets.

More details of Frege's project are beyond the scope of this course.
Here is a link to a good discussion of Frege's definitions of number, and his contributions to the philosophy of mathematics.
For a more sophisticated and thorough discussion of Frege's work, see John Burgess's *Fixing Frege* or Richard Heck's *Reading Frege's* Grundgesetze.

V. Russell's Paradox

Russell's paradox showed that Frege's project, as he originally conceived it, was unsuccessful.
Russell sent word of the paradox to Frege just as the second volume of the *Grundgesetze* was being published.
Frege added an attempt to avoid the paradox, but it was, in the end, unsuccessful.
Russell worked out a more thorough, if not fully intuitive, way to avoid the paradox, and used it in his *Principia Mathematica*.
The paradox also applies to Cantor's earlier set theory, though not in a way that undermines the generation of the transfinite numbers.
The source of the paradox is an unrestricted axiom of comprehension.

Frege's logic and Cantor's set theory can both be called naive set theory, for their use of an axiom of comprehension (or abstraction).
The axiom of comprehension says that any property determines a set.
For Frege, the relevant version is that every predicate has an extension.
Frege adds this comprehension claim to his treatment in the *Grundgesetze*, as Axiom 5.

> Axiom 5: $\{x|Fx\} = \{x|Gx\} \equiv (\forall x)(Fx \equiv Gx)$

Axiom 5 leads to Proposition 91.

Proposition 91: $Fy \equiv y \in \{x|Fx\}$

Axiom 5 says that the extensions of two concepts are equal if and only if the same objects fall under the two concepts.
In other words (Russell's), the set of Fs and the set of Gs are identical iff all Fs are Gs.
Proposition 91 says that a predicate F holds of a term iff the object to which the term refers is an element of the set of Fs.
Both statements assert the existence of a set of objects which corresponds to any predicate, though this claim could be made more explicitly with a higher-order quantification.

To derive Russell's paradox, take F to be 'is not an element of itself'.
So, y is not element of itself is expressed:

$y \notin y$ (which is short for '$\sim y \in y$')

Now, take y to be the set of all sets that are not elements of themselves:

$\{x|x \notin x\}$

And substitute that expression for y in the above expression, to get Fy, the left side of proposition 91:

$\{x|x \notin x\} \notin \{x|x \notin x\}$

On the right of proposition 91, we get

$\{x|x \notin x\} \in \{x|x \notin x\}$

Put them together: $\{x|x \notin x\} \notin \{x|x \notin x\} \equiv \{x|x \notin x\} \in \{x|x \notin x\}$

Which is of the form: $\sim P \equiv P$

For those of you who like their contradictions in the form $\alpha \bullet \sim\alpha$, note that:

1. $\sim P \equiv P$	
2. $(\sim P \supset P) \bullet (P \supset \sim P)$	1, Biconditional Exchange
3. $\sim P \supset P$	2, Simplification
4. $P \supset \sim P$	2, Simplification
5. $\sim P \vee \sim P$	4, Conditional Exchange
6. $P \vee P$	3, Conditional Exchange and Double Negation
7. $\sim P$	5, Tautology
8. P	6, Tautology
9. $P \bullet \sim P$	8, 7, Conjunction

QED

The axiom of comprehension is responsible for another paradox, called the Burali-Forti paradox.
The Burali-Forti paradox arises from considering the set of all ordinals.
By the axiom of comprehension, any collection, any property, determines a set.

So, there should be a set of all ordinals.
By definition (which I won't go into), the set of all ordinals will itself be an ordinal larger than itself.
So, any set of ordinals will be necessarily incomplete.
Some collections are just too large to form sets.
We now take the Burali-Forti paradox as a reductio on the existence of a particular set, a set to which every ordinal number belongs.
Similarly, Russell's paradox shows that there can not be a set of all sets, a universal set.

In current axiomatic set theory, we avoid we avoid the paradoxes of unrestricted comprehension by building sets iteratively.
We start with a few basic axioms, and build up the rest from those, as we have seen with ZF.

Zermelo-Fraenkel Set Theory (ZF):

Substitutivity:	$(\forall x)(\forall y)(\forall z)[y=z \supset (y\in x \equiv z\in x)]$
Pairing:	$(\forall x)(\forall y)(\exists z)(\forall u)[u\in z \equiv (u = x \lor u = y)]$
Null Set:	$(\exists x)(\forall y){\sim}x\in y$
Sum Set:	$(\forall x)(\exists y)(\forall z)[z\in y \equiv (\exists v)(z\in v \bullet v\in x)]$
Power Set:	$(\forall x)(\exists y)(\forall z)[z\in y \equiv (\forall u)(u\in z \supset u\in x)]$
Selection:	$(\forall x)(\exists y)(\forall z)[z\in y \equiv (z\in x \bullet \mathscr{F}u)]$, for any formula $\mathscr{F}$ not containing y free.
Infinity:	$(\exists x)(\varnothing\in x \bullet (y)(y\in x \supset Sy\in x)$, where 'Sy' stands for $y\cup\{y\}$

We thus use what is called an iterative concept of set.
We iterate, or list, the sets from the beginning.

VI. After the Paradox

Russell's solution to the paradoxes is to introduce a theory of types.
According to the theory of types, a set can not be a member of itself.
ZF provides a similar solution in that there is no way to generate the problematic sets.
Frege was able to argue that mathematics reduced to logic, because he claimed just the basic insight that every property determined a set.
But both Russell's solution and ZF substitute a substantive set theory for Frege's original logical insight.
Set theory does not appear to be a logical theory, but a mathematical theory.
Thus, given the paradoxes, Frege was able to show that mathematics is reducible to mathematics, but not to logic.

Still, the question remains whether mathematics is analytic.
Frege's method of showing that a statement is analytic is to trace it back to its fundamental assumptions.
We can trace mathematical theorems back to set theory, working the translations I described at the beginning of these notes in reverse.
So, the question becomes whether the axioms of set theory are analytic.

For a long time, Frege's program was considered a failure.
In recent years, renewed interest in Frege's work has led some philosophers to re-think this conclusion.
It turns out that more technical results can be salvaged from Frege's system than had been thought.
Interest in what came to be known as neo-logicism was high over the last twenty years, but seems to be waning now.