Hamilton College
Russell Marcus

Class #10: Radical Empiricism

I. Mill's Revolutionary Anti-Platonism

Mill's work generally may be characterized as radically empiricistic.
Earlier empiricists adopted some claims that were hardly distinguishable, on close analysis, from contentious claims of some rationalists.
Mill presents what we can recognize today as a more thoroughgoing empiricism.
To begin, I will characterize Mill's views on mathematics in relation to views we have already studied.
In large part, Mill is responding to his contemporary, Whewell.
Whewell's work in the philosophy of mathematics is, as far as I know, rarely original.
Instead, Whewell mainly appeals to arguments present in the work of earlier authors.
Thus, I will locate Mill's work in response to those views, rather than Whewell's.

Mill, following Aristotle and Berkeley, denies that there are mathematical objects existing independently of, or separate from, the objects available to our senses.

> There are no such things as numbers in the abstract (*System of Logic* §II.VI.2, p 189).

Like Aristotle, Mill believes that the grounds for our mathematical beliefs are observational.
Like Berkeley, Mill rests his rejection of platonism, in part, on our conceptual limitations.

> We can not *conceive* a line without breadth; we can form no mental picture of such a line: all the lines which we have in our minds are lines possessing breadth (*System of Logic* §II.V.1, p 169).

When we discussed Aristotle, I mentioned that among anti-platonists there are revolutionaries and reinterpreters.
Revolutionaries believe that mathematical statements are false and that mathematical objects do not exist.
Reinterpreters believe that mathematical statements are true when reconstrued so as not to refer to mathematical objects.
While platonist mathematical objects do not exist, for the reinterpreter, we can understand mathematical terms as shorthand for other kinds of objects.
Aristotle is a reinterpreter, believing that mathematical theorems are true of physical objects under a mode of thinking that Lear called the qua operation and which we can see as a kind of abstraction.
Mill, like Berkeley, is a revolutionary.
He alleges that the mathematician's claim to have ideas of mathematical objects is false.
The mathematician posits ideas of mathematical objects in order to serve as the subjects, or as mental correlates of the subjects, of mathematical theorems.

> I much question if any one who fancies that he can conceive what is called a mathematical line, thinks so from the evidence of his consciousness: I suspect it is rather because he supposes that unless such a conception were possible, mathematics could not exist as a science" (*System of Logic* §II.V.1, p 169).

If, in opposition to the standard, platonist view, we give up the claim that mathematical theorems are true, then we do not need to posit ideas of mathematical objects at all.
We just need ordinary ideas, of ordinary, sensible objets.

It is a generally-accepted principle of theory choice that one choose the least revolutionary theory compatible with one's evidence.
Theory choice is conservative: we give up as little as possible of our ordinary view when we find an inconsistency in it.
The more conservative choice will be the least upsetting to the rest of one's views, and thus the most plausible.
Berkeley didn't worry too much about the plausibility of his revolutionary mathematical anti-platonism. He had other, materialist opposition with which to deal.
But Mill considers two reinterpretive positions: modalism and conceptualism.

There are a variety of different modalist mathematical theories.
According to one version of modalism, let's call it naive modalism, mathematics is the study of possible concrete objects.
When we say that five plus seven is twelve, this modalist interprets our claim as meaning that if we were to combine five objects and seven objects, we would get twelve objects.
Contemporary versions of this version of modalism have been explored by Geoffrey Hellman and Charles Chihara.
One problem with naive modalism is that possible concrete objects, like actual concrete objects, have (or would have) physical properties rather than mathematical ones.
Possible pizzas, like actual pizzas, are not circles.

The version of modalism which Mill considers, in contrast, avoids this difficulty by taking mathematical theorems to refer to possible mathematical objets.
Against this version of modalism, Mill argues that mathematical objects can not even possibly exist, for the same reasons that they do not exist.

> Their existence, so far as we can form any judgment, would seem to be inconsistent with the physical constitution of our planet, at least, if not of the universe (*System of Logic* §II.V.1, p 168).

Thus, whether we take mathematical theorems to refer to possible mathematical objects or to possible concrete objects, we encounter the same difficulties as we would if we took them to refer to actual objects.
If we could develop a rigorous theory of modalities, and explain our knowledge of possibilities, we could revive modalism.
But, as far as Mill is concerned, the appeals to modality are only as good as the original appeals to either concrete (sensible) or abstract (ideal) objects.

Mill next dismisses conceptualism, on which mathematics is the study of our ideas.
The conceptualist, e.g. Locke, believes that mathematical objects are really just ideas in our minds.
Against conceptualism, Mill offers a Berkeleyan argument: we can not form abstract ideas.

> The points, lines, circles, and squares which any one has in his mind, are (I apprehend) simply copies of the points, lines, circles, and squares which he has known in his experience. Our idea of a point, I apprehend to be simply our idea of the *minimum visible*, the smallest portion of surface which we can see. A line, as defined by geometers, is wholly inconceivable (*System of Logic* §II.V.1, p 169).

Mill's objection to Lockean conceptualism recall Berkeley's concerns about mathematics and his constraints on conception from human imagination.
But his positive account of our knowledge of mathematics, insofar as we have any such knowledge, differs sharply from that of Berkeley.
Indeed, Mill does not endorse Berkeley's idealism, generally.
Berkeley denied the existence of mathematical objects as a corollary of his rejection of the existence of the material world.
Both the materialist and the mathematician depend, Berkeley argues, on a belief in abstract ideas.
Mill, in contrast, embraces common-sense materialism and rejects mathematical objects because of their apparent inconsistency with materialist principles.

Mill's views might even be seen as closer to those of Leibniz than to Berkeley's.
Unlike Mill, Leibniz believes that we have innate ideas.
But, both Mill and Leibniz do not think that mathematical objects are special objects.
Instead, Mill believes that mathematical truths are logical truths, and thus need no special objects.
Leibniz argues that all knowledge can be analyzed into immediately accessible truths, logical identities.
From these, we can derive, through uses of symbols, further mathematical claims.
Mill agrees with Leibniz that definition is central to the development of mathematics, especially in deriving more complex theorems on the basis of simpler statements, deriving theorems from axioms.
Mill, like Leibniz and Hume, argues that mathematics is mainly deductive.
But, unlike Leibniz and Hume, Mill believes that taking mathematics to be deductive is not to laud it, to treat it as uniquely secure.
Mill calls mathematics hypothetical, essentially denigrating the its status.
For Mill, mathematical theorems are not the eternal truths that Descartes and Leibniz think they are.

> When, therefore, it is affirmed that the conclusions of geometry are necessary truths, the necessity consists in reality only in this, that they correctly follow from the suppositions from which they are deduced (*System of Logic* §II.V.1, p 170).

As we saw in our discussion of axiomatic systems, epistemological questions about mathematics may (on a common view) be distilled to questions about our knowledge of the axioms and questions about our knowledge of the rules of inference.
Taking mathematics to be mainly deductive or hypothetical, though, even if we are happy with our logic, still leaves open the question of our knowledge of the axioms.
There are five plausible positions regarding the axioms.

First, the axioms could be arbitrary definitions.
That is, we could just accept them without justification.
This position, on which mathematics is just the manipulation of meaningless symbols according to syntactically-constructed rules of inference, becomes known as formalism.
We might interpret Hume, who takes mathematical theorems to follow from the law of non-contradiction without having any objects of its own, as a proto-formalist, though it is a stretch.
The mathematician David Hilbert, whose work we will study, is often erroneously called a formalist.
Formalism is more often attacked than defended.

Second, mathematical axioms could be merely logical truths.
Both Leibniz and Hume took the foundational theorems to be logical truths.
They both believed that all of mathematics derived from definitions, using the principle of contradiction,

and so could avoid questions about our knowledge of the axioms.
Logical truths apply to everything, and have no special domain of their own.
If mathematics were just logic in disguise, we might avoid positing any special objects.
Neither Hume nor Leibniz thinks that there are mathematical objects, even though they believe that mathematical theorems are true.
This second view, that mathematical theorems are true but that mathematical terms have no special referents, is awkward.
Consider the empty set axiom: $\exists x \forall y\, y \notin x$.
It asserts the existence of a set.
To deny that there are mathematical objects means that we have to find a way to understand that claim without taking it at face value.
Both Hume and Leibniz are thus faced with the challenge of re-interpretation, just like the reinterpretive anti-platonists.

Third, the axioms could be justified by our reflections on our own mental states.
Locke and Kant hold this position, taking mathematical objects to be mental objects.
Kant disagrees with Hume and Leibniz that mathematical inferences are purely deductive.
Mill has already dismissed this possibility in dismissing conceptualism.

Fourth, the axioms could be justified by our intuitions about abstract mathematical objects.
Descartes holds this position, as do some contemporary platonists.
Whewell defends mathematical intuition.
Mill argues that such intuition is just the result of induction on sense experiences.
The source and validity of our intuitions are deep, perplexing questions, which we can not resolve here.
More importantly, here, both the third and fourth positions commit us to contentious psychological abilities like abstraction or intuition.
They also commit us to contentious objects of those processes, like abstract mathematical objects.

Fifth and last, the axioms could be justified by their reference to concrete objects.
Mill takes fundamental mathematical claims to be empirical generalizations, inductions from sense experience

> [Mathematical axioms] are experimental truths; generalizations from observation. The proposition Two straight lines can not inclose a space...is an induction from the evidence of our senses (*System of Logic* §II.V.4, p 172).

Mill thus improves on the Hume-Leibniz view, by giving referents to mathematical objects, without committing to contentious psychological abilities or contentious mental or abstract objects.
Strictly speaking, Mill takes the axioms to be false.
His claim is that given a mathematical claim, there is, or may be, a related claim regarding concrete objects that is justifiable.

II. Enumerative Induction in Geometry

The claim that our knowledge of mathematical axioms and theorems is the result of enumerative induction seems most plausible for geometry.
Mill, like the empiricists before him, believes that our mathematical beliefs arise from reflection on our

sense experience.
But whereas Locke posited a psychological capacity for abstraction to explain how we construct mental objects which satisfy mathematical theorems, Mill denies that we can construct such ideas.

> We are thinking, all the time, of precisely such objects as we have seen and touched, and with all the properties which naturally belong to them; but, for scientific convenience, we feign them to be divested of all properties, except those which are material to our purpose, and in regard to which we design to consider them (*System of Logic* §II.V.1, p 169).

Mill believes that theorems of geometry, which seem to refer to ideal objects, actually refer to physical objects.
But, since no physical objects actually satisfy the geometric theorems, they are false, or merely approximately true.
We can pretend that they hold, but this is just a pretense.

> Since an hypothesis framed for the purpose of scientific inquiry must relate to something which has real existence (for there can be no science respecting nonentities), it follows that any hypothesis we make respecting an object, to facilitate our study of it, must not involve any thing which is distinctly false, and repugnant to its nature: we must not ascribe to the thing any property which it has not; our liberty extends only to slightly exaggerating some of those which it has (by assuming it to be completely what it really is very nearly), and suppressing others, under the indispensable obligation of restoring them whenever, and in as far as, their presence or absence would make any material difference in the truth of our conclusions (*System of Logic* §II.V.2, p 171).

Let's call Mill's view radical empiricism, for his insistence that the true subjects of mathematical claims are ordinary, concrete objects.
Mill not only claims that we exaggerate the properties of the objects we sense, he also claims that we rely on a method of proof that enables us to make mathematical conclusions on the basis of objects that we sense.
He considers the following imagined objection, in a footnote (pp 173-4).

IO
1. Either mathematical theorems refer to ideal objects or they refer to objects that we sense.
2. If they refer to ideal objects, the radical empiricist can not defend our knowledge of them, since we never sense such objects.
3. If they refer to objects that we sense, they are false.

So, for the radical empiricist, mathematical theorems are either unknowable or false. In either case, the radical empiricist can not justify any proof of a mathematical theorem.

Mill responds to IO by appealing explicitly to the legitimacy of exaggeration.

> Those who employ this argument to show that geometrical axioms can not be proved by induction, show themselves unfamiliar with a common and perfectly valid mode of inductive proof: proof by approximation (*System of Logic* §II.V.4, p 174).

Mill claims that our approximations in mathematics are analogous to our adjustments of experimental results in light of intervening phenomena.
Imagine that an experiment fails to produce a predicted result.
We may not abandon the theory which produced the prediction.
Instead, we can adduce further considerations about the experiment: perhaps there is wind resistance, or friction, which should have altered our prediction.

> The correctness of those generalizations, *as* generalizations, is without a flaw: the equality of all the radii of a circle is true of all circles, so far as it is true of any one: but it is not exactly true of any circle; it is only nearly true; so nearly that no error of any importance in practice will be incurred by feigning it to be exactly true. When we have occasion to extend these inductions, or their consequences, to cases in which the error would be appreciable - to lines of perceptible breadth or thickness, parallels which deviate sensibly from equidistance, and the like - we correct our conclusions, by combining with them a fresh set of propositions relating to the aberration; just as we also take in propositions relating to the physical or chemical properties of the material, if those properties happen to introduce any modification into the result; which they easily may, even with respect to figure and magnitude, as in the case, for instance, of expansion by heat (*System of Logic* §II.V.1, p 169).

Mill's appeal to an analogy with scientific methodology itself does not settle anything.
The problem with arguing by analogy is that one always has to assert that the analogy continues to hold in the respect one wishes it to hold.
Unless we have an independently compelling reason to affirm that the methods of science hold in mathematics, instead of more properly mathematical methods, we have no reason to take the approximations in science to be useful in mathematics.

Much of Mill's *System of Logic* is dedicated to characterizing methods of inductive inference.
I will not pursue his claim that proof by approximation is legitimate.
It seems to me that IO is compelling.
But, the radical empiricist encounters an even more profound problem.
The claims of mathematics are often taken not just to be true, but necessarily true.
Even if Mill could argue that enumerative induction leads us to true beliefs about concrete objects, it could not explain our knowledge of the necessity of mathematical theorems.

III. Against Apriority and Necessity

Mill's argument against the apriority of mathematics is brief and unsatisfying.
He assumes that mathematics is inductive.
He concludes that we can not justify our knowledge of mathematics by pure reasoning, or intuition.
The argument seems circular.

More charitably, Mill's argument against the apriority of mathematics may be best seen as a particular instance of an extended, general argument against apriority.
Like Locke's argument against innate ideas, Mill is attempting to account for all of our knowledge without appeal to any *a priori* method.
We must look at Mill's system as a whole to decide if it is satisfying.
For now, let's look at the metaphysical question of the necessity of mathematical claims.

Mill's long argument against the necessity of mathematical theorems involves identifying necessity with inconceivability.
Mainly, as usual, he is responding to Whewell.
But, Hume made the same connection.

> Although Dr. Whewell has naturally and properly employed a variety of phrases to bring his meaning more forcibly home, he would, I presume, allow that they are all equivalent; and that what he mans by a necessary truth, would be sufficiently defined, a proposition the negation of which is not only false but inconceivable (*System of Logic* §II.V.6, p 177).

Mill's argument against taking mathematical theorems to hold necessarily, then, is just that conceivability is an accident of our limited minds.

> Our capacity or incapacity of conceiving a thing has very little to do with the possibility of the thing itself; but is in truth very much an affair of accident, and depends on the past history and habits of our own minds (*System of Logic* §II.V.6, p 178).

Mill provides a series of examples in which something people thought was inconceivable turned out not only to be conceivable, but true.
He also argues that what people think is true may turn out to be false, and later be perceived as inconceivable.
Mill's examples include how people thought about Newtonian gravitation, and how we think about the indestructibility of matter.
Consider Newton's first law of motion.

> That a body, once in motion, would continue forever to move in the same direction with undiminished velocity unless acted upon by some new force, was a proposition which mankind found for a long time the greatest difficulty in crediting. It stood opposed to apparent experience of the most familiar kind, which taught that it was the nature of motion to abate gradually, and at last terminate of itself (*System of Logic* §II.V.6, p 181).

Mill also cites Augustus DeMorgan, on how people take what is most obvious to them to be necessary.

> If all mankind had spoken one language, we can not doubt that there would have been a powerful, perhaps a universal, school of philosophers, who would have believed in the inherent connection between names and things... (DeMorgan, *Formal Logic*, cited in Mill, *System of Logic* §II.V.6, p 178 (fn)).

Mill's caution to beware of ascribing necessity to that which is just merely obvious or usual is prudent.
But, it need not follow that all ascriptions of necessity are of this type.
Notice that Mill's examples mainly regard our beliefs about physical laws.
Such claims may not translate to mathematics.

Mill makes two attempts to show that mathematical theorems do not have the necessity ordinarily ascribed to them.
First, he argues that mathematical statements are so firmly entrenched in our minds, from early on, that it is harder to see that they are just empirical generalizations.

> Philosophers, for generations, have the most extraordinary difficulty in putting certain ideas together; they at last succeed in doing so; and after a sufficient repetition of the process, they first fancy a natural bond between the ideas, then experience a growing difficulty, which at last, by the continuation of the same progress, becomes an impossibility, of severing them from one another. If such be the progress of an experimental conviction of which the date is of yesterday, and which is in opposition to first appearances, how must it fare with those which are conformable to appearances familiar from the first dawn of intelligence, and of the conclusiveness of which, from the earliest records of human thought, no skeptic has suggested even a momentary doubt? (*System of Logic* §II.V.6, pp 181-2).

Second, Mill argues that mathematics does have a weaker kind of necessity.
We might call this weaker form conditional, or hypothetical, necessity.

> [The conclusions of mathematics] are only true on certain suppositions, which are, or ought to be, approximations to the truths, but are seldom, if ever, exactly true; and to this hypothetical character is to be ascribed the peculiar certainty, which is supposed to be inherent in demonstration (*System of Logic* §II.VI.1, p 187).

Thus, Mill's argument against the necessity of mathematical claims really relies on the failure of conceivability as a guide to necessity.
He is willing to take mathematical theorems as following from the axioms.
But, he is not willing to take them as having any kind of necessary truth in themselves.

Against Mill's argument, inconceivability seems like a weak interpretation of what the traditional philosopher of mathematics takes as the intent behind our ordinary ascriptions of necessity.
To interpret 'p is necessary' as 'the falsity of p is inconceivable' makes such claims psychological, or, at best, epistemological.
But necessity is supposed to be a metaphysical property, about the claim itself, and not merely about those persons asserting or believing the claim.
To say that p is necessary is to say that it could not be false.
There are problems with understanding what 'could not be false' means.
But, those who claim that mathematical theorems are necessary tend not to think of that necessity as psychological or epistemic.
If Mill were to establish that the only way to understand the notion of necessity is as a psychological term, then his argument would be compelling.
But, that argument is lacking.

IV. Enumerative Induction and Arithmetic

Mill's claim that mathematics consists of empirical inductions is most compelling in geometry.
We do seem to prove theorems in geometry which hold only approximately of physical objects.
It might not seem that we lose too much by taking geometric theorems to be mere exaggerations, only approximately true.
Mill's attempts to extend these claims to arithmetic is less plausible.
When applying arithmetic to the world, we seem not to have approximate, but exact results.
Two chickens and three chickens yield precisely five chickens.

Mill attempts to account for the difficulty extending his view of mathematics as inductive to arithmetic by appealing to our lack of specific ideas corresponding to numerical or algebraic terms.
In geometry, we think of frisbees and pizzas when we think of circles.
But, in number theory, we have no such supporting images.
Thus, we are tempted to think that mathematical reasoning is, as Hume claimed, mere relations of ideas.
Or, as Mill puts it, we think that arithmetical reasoning is secure because it consists of merely verbal transformations.

> We do not carry any ideas along with us when we use the symbols of arithmetic or of algebra. In a geometrical demonstration we have a mental diagram, if not one on paper...Nothing, then, being in the [arithmetical] reasoner's mind but the symbols, what can seem more inadmissable than to contend that the reasoning process has to do with any thing more? (*System of Logic* §II.VI.2, p 188).

To support his view of arithmetic as the result of enumerative induction, Mill alleges that all numbers are numbers of something.
We learn them from experiences with collections of actual objects.

> All numbers must be numbers of something: there are no such things as numbers in the abstract. *Ten* must mean ten bodies or ten sounds, or ten beatings of the pulse (*System of Logic* §II.VI.2, p 189).

Mill continues to define the numbers in terms of things that we experience using our senses.

> And thus we may call "Three is two and one" a definition of three; but the calculations which depend on that proposition do not follow from the definition itself, but from an arithmetical theorem presupposed in it, namely, that collections of objects exist, which while they impress the senses thus, ∵, may be separated into two parts, thus, ... This proposition being granted, we term all such parcels Threes, after which the enunciation of the above-mentioned physical fact will serve also for a definition of the word Three (*System of Logic* §II.VI.2, pp 190-1).

Thus, for Mill, number terms are really convenient references to physical objects.
A three is nothing other than three dots, or three fingers, or three Wankel rotary engines.

V. Confirmation and Disconfirmation

If mathematical theorems are inductive generalizations, then they are confirmed by our experiences.
Similarly, mathematical falsehoods should be disconfirmed by experience.
Mill describes how our experience confirms the geometric theorem that crossed lines diverge as we travel away from their point of intersection.

> Whether the axiom [that two straight lines can not inclose a space] needs confirmation or not, it receives confirmation in almost every instant of our lives; since we can not look at any two straight lines which intersect one another, without seeing that from that point they continue to diverge more and more (*System of Logic* §II.V.4, p 173).

Of course, we can not actually view any lines, even concrete inscriptions, to infinity.
Mill claims that we can still imagine ourselves as viewing lines in different places.

> We can transport ourselves thither in imagination, and can frame a mental image of the appearance which one or both of the lines must present at that point, which we may rely on as being precisely similar to the reality (*System of Logic* §II.V.5, p 175).

This power of our imagination will allow us to confirm or disconfirm a mathematical statement, not directly by sense experience, but by an imagination whose contents derive from experience.

> We learn by the evidence of experience, that a line which, after diverging from another straight line, begins to approach it, produces the impression on our senses which we describe by the expression, "a bent line," not by the expression "a straight line" (*System of Logic* §II.V.5, p 175).

But now, consider traveling on the surface of a sphere.
On spheres, lines (or great circles) which cross do return to each other.
Further, we can see no kinks in such lines.
This consideration seems to provide a counter-example to Mill's claim that we can find empirical confirmation or disconfirmation of geometric theorems.
The empirical fact of finding lines which cross and then converge has no bearing on our mathematical theorems.
When non-Euclidean geometries were developed, claims which seemed false, when applied to the world, turned out to be true.
Parallel lines do meet in hyperbolic space.
Empirical evidence leaves mathematical results alone.
Mathematical results are independent of the physical world.
In scientific experiments, when we encounter evidence which disconfirms a theory we hold, we don't take it to undermine the mathematics we use in the theory.
We hold mathematical principles constant.
We revise them only in response to mathematical worries, like the discovery of a contradiction.

VI. Frege's Criticisms

Like the empiricist arguments against innate ideas, Mill's argument that we can account for mathematics on the basis of sense experience depends, in large part, on a principle of parsimony.
He argues that the burden is on the traditionalists to support *a priori* knowledge of necessary truths.

> The burden of proof lies on the advocates of the contrary opinion: it is for them to point out some fact, inconsistent with the supposition that this part of our knowledge of nature is derived from the same sources as every other part (*System of Logic* §II.V.4, p 173).

The argument here relies on Ockham's razor: do not multiply entities unnecessarily.
If we can account for mathematical truth without positing platonic objects, or innate ideas, we should do so.

Frege discusses several problems with Mill's account, several difficulties doing so.
Frege's attacks on Mill are famously amusing, and especially focused on Mill's definition of number.

Recall that Mill defined three in terms of sensible representations of three items, like dots, which we can move around, separating and combining.

> What a mercy, then, that not everything in the world is nailed down; for if it were, we should not be able to bring off this separation, and 2+1 would not be 3" (Frege, *Grundlagen* §7).

I disagree with Shapiro's claims that Frege's criticisms are more thin mockery than substantial argument. Consider that Mill will have difficulty explaining the uses of 'every' and 'all' and 'must' in our mathematical reasoning, as Leibniz noted against Locke.

> Propositions, therefore, concerning numbers, have the remarkable peculiarity that they are propositions concerning *all* things whatever; *all* objects, *all* existences of *every* kind, known to our experience. *All* things possess quantity; consist of parts which can be numbered; and in that character possess all the properties which are called properties of numbers. That half of four is two, *must* be true whatever the word four represents, whether four hours, four miles, or four pounds weight. We need only conceive a thing divided into four equal parts (and all things may be conceived as so divided), to be able to predicate of it *every* property of the number four, that is *every* arithmetical proposition in which the number four stands on one side of the equation (*System of Logic* §II.VI.2, p 189; emphases added).

Remember, too, the Pythagorean discovery that we could never get to pi from sense experience.
Appeals to enumerative induction just can not support universal and modal claims, and can not support knowledge of mathematical objects which do not appear in nature.
Beyond the irrational numbers, for instance, mathematicians derive infinitely many distinct kinds of infinite numbers.
No amount of enumerative induction will justify knowledge of infinity, which is our topic for next class.

Frege adds that all induction presumes probabilistic reasoning.
But, probabilistic reasoning itself relies on arithmetic.

> Induction [then, properly understood,] must base itself on the theory of probability, since it can never render a proposition more than probably. But how probability theory could possibly be developed without presupposing arithmetical laws is beyond comprehension (Frege, *Grundlagen* §10).

Frege's own work is largely motivated by attempts to define the numbers, as we will see.
We will look at his appeals to logic, and his rejection of Kant's claim that arithmetic is synthetic *a priori*, in a future class.

VII. A Thought Experiment

Lastly, here is a thought experiment about whether mathematics could be inductive, one which was asked of me in my dissertation defense.
Imagine that you raise sheep.
You have your sheep divided into eighteen pens.
In each of the eighteen pens, you have fourteen sheep.
When you count the sheep, though, you find that you have only 251.

You count each pen: fourteen in each.
But, when you count them all together, you again have only 251.
There are no trap doors, nor magical sheep.
What do you think?