SELECTIONS from
KANT'S CRITIQUE
on MATHEMATICS

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trans. Norman Kemp Smith
CRITIQUE of PURE REASON
N.Y. ST. MARTIN'S PRESS
1984

PREFACE TO SECOND EDITION B vii

WHETHER the treatment of such knowledge as lies within the province of reason does or does not follow the secure path of a science, is easily to be determined from the outcome. For if after elaborate preparations, frequently renewed, it is brought to a stop immediately it nears its goal; if often it is compelled to retrace its steps and strike into some new line of approach; or again, if the various participants are unable to agree in any common plan of procedure, then we may rest assured that it is very far from having entered upon the secure path of a science, and is indeed a merely random groping. In these circumstances, we shall be rendering a service to reason should we succeed in discovering the path upon which it can securely travel, even if, as a result of so doing, much that is comprised in our original aims, adopted without reflection, may have to be abandoned as fruitless.

That logic has already, from the earliest times, proceeded B viii. upon this sure path is evidenced by the fact that since Aristotle it has not required to retrace a single step, unless, indeed, we care to count as improvements the removal of certain needless subtleties or the clearer exposition of its recognised teaching, features which concern the elegance rather than the certainty of the science. It is remarkable also that to the present day this logic has not been able to advance a single step, and is thus to all appearance a closed and completed body of doctrine. If some of the moderns have thought to enlarge it by introducing *psychological* chapters on the different faculties of knowledge (imagination, wit, etc.), *metaphysical* chapters on the origin of knowledge or on the different kinds of certainty according to difference in the objects (idealism, scepticism, etc.), or *anthropological* chapters on prejudices, their causes and remedies, this could only arise from their ignorance of the

peculiar nature of logical science. We do not enlarge but disfigure sciences, if we allow them to trespass upon one
B ix. another's territory. The sphere of logic is quite precisely delimited; its sole concern is to give an exhaustive exposition and a strict proof of the formal rules of all thought, whether it be *a priori* or empirical, whatever be its origin or its object, and whatever hindrances, accidental or natural, it may encounter in our minds.

That logic should have been thus successful is an advantage which it owes entirely to its limitations, whereby it is justified in abstracting—indeed, it is under obligation to do so—from all objects of knowledge and their differences, leaving the understanding nothing to deal with save itself and its form. But for reason to enter on the sure path of science is, of course, much more difficult, since it has to deal not with itself alone but also with objects. Logic, therefore, as a propaedeutic, forms, as it were, only the vestibule of the sciences; and when we are concerned with specific modes of knowledge, while logic is indeed presupposed in any critical estimate of them, yet for the actual acquiring of them we have to look to the sciences properly and objectively so called.

Now if reason is to be a factor in these sciences, something in them must be known *a priori*, and this knowledge may be related to its object in one or other of two ways, either as merely *determining* it and its concept (which must be supplied
B x. from elsewhere) or as also *making it actual*. The former is *theoretical*, the latter *practical* knowledge of reason. In both, that part in which reason determines its object completely *a priori*, namely, the *pure* part—however much or little this part may contain—must be first and separately dealt with, in case it be confounded with what comes from other sources. For it is bad management if we blindly pay out what comes in, and are not able, when the income falls into arrears, to distinguish which part of it can justify expenditure, and in which[1] line we must make reductions.

Mathematics and physics, the two sciences in which reason yields theoretical knowledge, have to determine their objects *a priori*, the former doing so quite purely, the latter having to reckon, at least partially, with sources of knowledge other than reason.

In the earliest times to which the history of human reason extends, *mathematics*, among that wonderful people, the Greeks, had already entered upon the sure path of science. But it must not be supposed that it was as easy for mathematics as it was for logic—in which reason has to deal with itself alone—to light upon, or rather to construct for itself, that royal road. B xi.
On the contrary, I believe that it long remained, especially among the Egyptians, in the groping stage, and that the transformation must have been due to a *revolution* brought about by the happy thought of a single man, the experiment which he devised marking out the path upon which the science must enter, and by following which, secure progress throughout all time and in endless expansion is infallibly secured. The history of this intellectual revolution—far more important than the discovery of the passage round the celebrated Cape of Good Hope—and of its fortunate author, has not been preserved. But the fact that Diogenes Laertius, in handing down an account of these matters, names the reputed author of even the least important among the geometrical demonstrations, even of those which, for ordinary consciousness, stand in need of no such proof, does at least show that the memory of the revolution, brought about by the first glimpse of this new path, must have seemed to mathematicians of such outstanding importance as to cause it to survive the tide of oblivion. A new light flashed upon the mind of the first man (be he Thales or some other) who demonstrated the properties of the isosceles triangle. The true method, so he found, was not to inspect what he dis- B xii.
cerned either in the figure, or in the bare concept of it, and from this, as it were, to read off its properties; but to bring out what[1] was necessarily implied in the concepts that he had himself formed *a priori*, and had put into the figure in the construction by which he presented it to himself. If he is to know anything with *a priori* certainty he must not ascribe to the figure anything save what necessarily follows from what he has himself set into it in accordance with his concept.

[1] [Reading, with Erdmann, *von welchem* for *von welcher*.]

[3]IV. The Distinction between Analytic and Synthetic Judgments

In all judgments in which the relation of a subject to the predicate is thought (I take into consideration affirmative judgments only, the subsequent[4] application to negative judgments being easily made), this relation is possible in two different ways. Either the predicate B belongs to the subject A, as something which is (covertly) contained in this concept A; or B lies outside the concept A, although it does indeed stand in connection with it. In the one case I entitle the judgment analytic, in the other synthetic. Analytic judgments (affirmative) are therefore those in which the connection of the predicate with the subject is thought through identity; those in which this connection is thought without identity should be entitled synthetic. The former, as adding nothing through the predicate to the concept of the subject, but merely breaking it up into those constituent concepts that have all along been thought in it, although confusedly, can also be entitled explicative. The latter, on the other hand, add to the concept of the subject a predicate which has not been in any wise thought in it, and which no analysis could possibly extract from it; and they may therefore be entitled ampliative. If I say, for instance, 'All bodies are extended', this is an analytic judgment. For I do not require to go beyond the concept which I connect with 'body'[5] in order to find extension as bound up with it. To

A 6 / A 7 / B 11

[1] [In A: attaches *a priori* to given concepts others completely foreign to them.]
[3] [In A: This question.] [3] ["IV" added in B.] [4] [*nachher* added in B.]
[5] [In A: outside the concept which I connect with the word body.]

meet with this predicate, I have merely to analyse the concept, that is, to become conscious to myself[1] of the manifold which I always think in that concept. The judgment is therefore analytic. But when I say, 'All bodies are heavy', the predicate is something quite different from anything that I think in the mere concept of body in general; and the addition of such a predicate therefore yields a synthetic judgment.

* Judgments of experience, as such, are one and all synthetic. For it would be absurd to found an analytic judgment on experience. Since, in framing the judgment, I must not go outside my concept, there is no need to appeal to the testimony of experience in its support. That a body is extended is a proposition that holds *a priori* and is not empirical. For, before appealing to experience, I have already in the concept of body all the conditions required for my judgment. I have only to extract from it, in accordance with the principle of contradiction, the required predicate, and in so doing can at the same time become conscious of the necessity of the judgment—and that is what experience could never have taught me. On the other hand, though I do not include in the concept of a body in general the predicate 'weight', none the less this concept indicates an object of experience through one of its parts, and I can add to that part other parts of this same experience, as in this way belonging together with the concept. From the start

B 12

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* ["Judgments of experience" to end of paragraph substituted in B in place of the following:]

Thus it is evident: 1. that through analytic judgments our knowledge is not in any way extended, and that the concept which I already have is merely set forth and made intelligible to me; 2. that in synthetic judgments I must have besides the concept of the subject something else (X), upon which the understanding may rely, if it is to know that a predicate, not contained in this concept, nevertheless belongs to it.

A 8

In the case of empirical judgments, judgments of experience, there is no difficulty whatsoever in meeting this demand. This X is the complete experience of the object which I think through the concept A—a concept which forms only one part of this experience. For though I do not include in the concept

[1] [*mir* added in B.]

I can apprehend the concept of body analytically through the characters of extension, impenetrability, figure, etc., all of which are thought in the concept. Now, however, looking back on the experience from which I have derived this concept of body, and finding weight to be invariably connected with the above characters, I attach it as a predicate to the concept; and in doing so I attach it synthetically, and am therefore extending my knowledge. The possibility of the synthesis of the predicate 'weight' with the concept of 'body' thus rests upon experience. While the one concept is not contained in the other, they yet belong to one another, though only contingently, as parts of a whole, namely, of an experience which is itself a synthetic combination of intuitions.

A 9 But in *a priori* synthetic judgments this help is entirely
B 13 lacking. [I do not here have the advantage of looking around in the field of experience.] Upon what, then, am I to rely, when I seek to go beyond[1] the concept A, and to know that another concept B is connected with it? Through what is the synthesis made possible? Let us take the proposition, 'Everything which happens has its cause' In the concept of 'something which happens', I do indeed think an existence which is preceded by a time, etc., and from this concept analytic judgments may be obtained. But the concept of a 'cause' lies entirely outside the other concept, and[2] signifies something different

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of a body in general the predicate 'weight', the concept none the less indicates the complete experience through one of its parts; and to this part, as belonging to it, I can therefore add other parts of the same experience. By prior analysis I can apprehend the concept of body through the characters of extension, impenetrability, figure, etc., all of which are thought in this concept. To extend my knowledge, I then look back to the experience from which I have derived this concept of body, and find that weight is always connected with the above characters. Experience is thus the X which lies outside the concept A, and on which rests the possibility of the synthesis of the predicate 'weight' (B) with the concept (A).

[1] [In A: outside.]
[2] [*liegt ganz ausser jenem Begriffe, und* added in B.]

from 'that which happens', and is not therefore[1] in any way contained in this latter representation. How come I then to predicate of that which happens something quite different, and to apprehend that the concept of cause, though not contained in it, yet belongs, and indeed necessarily belongs,[2] to it? What is here the unknown[3] = X which gives support to the understanding when it believes that it can discover outside the concept A a predicate B foreign to this concept, which it yet at the same time considers to be connected with it?[4] It cannot be experience, because the suggested principle has connected the second representation[5] with the first, not only with greater universality,[6] but also with the character of necessity, and therefore completely *a priori* and on the basis of mere concepts. Upon such synthetic, that is, ampliative principles, all our *a priori* speculative knowledge must ultimately rest; analytic judgments[7] are very important, and indeed necessary, but only for obtaining that clearness in the concepts which is requisite for such a sure and wide synthesis as will lead to a genuinely new addition[8] to all previous knowledge.*

How

A 10

B 14

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* [In A there follows the passage, omitted in B:]

A certain mystery lies here concealed;[a] and only upon its solution can the advance into the limitless field of the knowledge yielded by pure understanding be made sure and trustworthy. What we must do is to discover, in all its proper universality, the ground of the possibility of *a priori* synthetic judgments, to obtain insight into the conditions which make

[a] If it had occurred to any of the ancients even to raise this question, this by itself would, up to our own time, have been a powerful influence against all systems of pure reason, and would have saved us so many of those vain attempts, which have been blindly undertaken without knowledge of what it is that requires to be done.

[1] [*ist also* substituted in B for *und ist*.]
[2] [*und sogar notwendig* added in B.]
[3] [*das Unbekannte = X* substituted in B for *das X*.]
[4] [In A: and yet at the same time connected with it.]
[5] [Reading, with Grillo, *Vorstellung* for *Vorstellungen*.]
[6] [In A: with greater universality than experience can yield, but . . .]
[7] [Adding, with Erdmann, *Urteile*.]
[8] [In B *Erwerb* substituted for *Anbau*.]

[1]V. In all Theoretical Sciences of Reason Synthetic *a priori* Judgments are contained as Principles

1. ***All mathematical judgments, without exception, are synthetic.*** This fact, though incontestably certain and in its consequences very important, has hitherto escaped the notice of those who are engaged in the analysis of human reason, and is, indeed, directly opposed to all their conjectures. For as it was found that all mathematical inferences proceed in accordance with the principle of contradiction (which the nature of all apodeictic certainty requires), it was supposed that the fundamental propositions of the science can themselves be known to be true[2] through that principle. This is an erroneous view. For though a synthetic proposition can indeed be discerned in accordance with the principle of contradiction, this can only be if another synthetic proposition is presupposed, and if it can then be apprehended as following from this other proposition; it can never be so discerned in and by itself.

First of all, it has to be noted that mathematical propositions, strictly so called, are always judgments *a priori*, not empirical; because they carry with them necessity, which
B 15 cannot be derived from experience. If this be demurred to, I am willing to limit my statement to *pure* mathematics, the very concept of which implies that it does not contain empirical, but only pure *a priori* knowledge.

We might, indeed, at first suppose that the proposition $7+5=12$ is a merely analytic proposition, and follows by the principle of contradiction from the concept of a sum of 7 and 5. But if we look more closely we find that the concept of the sum of 7 and 5 contains nothing save the union of the two numbers into one, and in this no thought is being taken as to what that single number may be which combines both. The concept of 12 is by no means already thought in merely thinking this union of 7 and 5; and I may analyse my concept of such a possible sum as long as I please, still I shall never find the 12 in it. We have to go outside these concepts, and call in the aid of the intuition which corresponds to one of them, our five fingers, for instance, or, as Segner[1] does in his *Arithmetic*, five points, adding to the concept of 7, unit by unit, the five given in intuition. For starting with the number 7, and for the concept of 5 calling in the aid of the fingers of my hand as intuition, I now add one by one to the number 7 the units which I previously took together to form the number
B 16 5, and with the aid of that figure[2] [the hand] see the number 12 come into being. That 5 should be added to 7,[3] I have indeed already thought in the concept of a sum $=7+5$, but not that this sum is equivalent to the number 12. Arithmetical propositions are therefore always synthetic. This is still more evident if we take larger numbers. For it is then obvious that, however we might turn and twist our concepts, we could never, by the mere analysis of them, and without the aid of intuition, discover what [the number is that] is the sum.

Just as little is any fundamental proposition of pure geometry analytic. That the straight line between two points is the shortest, is a synthetic proposition. For my concept of *straight* contains nothing of quantity, but only of quality. The concept of the shortest is wholly an addition, and cannot be derived, through any process of analysis, from the concept of the straight line. Intuition, therefore, must here be called in; only by its aid is the synthesis possible. What here[4] causes
B 17 us commonly to believe that the predicate of such apodeictic judgments is already contained in our concept, and that the judgment is therefore analytic, is merely the ambiguous character of the terms used. We are required to join in thought a certain predicate to a given concept, and this neces-

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each kind of such judgments possible, and to mark out all this knowledge, which forms a genus by itself, not in any cursory outline, but in a system, with completeness and in a manner sufficient for any use, according to its original sources, divisions, extent, and limits. So much, meantime, as regards what is peculiar in synthetic judgments.

[1] [Sections V. and VI. added in B.]
[2] [In 4th edition *erkannt* changed to *anerkannt*.]

[1] [*Anfangsgründe der Arithmetik*, translated from the Latin, second edition, Halle, 1773, pp. 27, 79.]
[2] [*an jenem meinem Bilde*.]
[3] [Reading, with Erdmann, 5 *zu* 7.]
[4] [As Vaihinger has pointed out (*Commentar*, i. pp. 303-4), this passage, which in both A and B is made to follow "Some few fundamental propositions ... exhibited in intuition", is quite obviously displaced. In the above translation the necessary rearrangement has been made.]

sity is inherent in the concepts themselves. But the question is not what we *ought* to join in thought to the given concept, but what we *actually* think in it, even if only obscurely; and it is then manifest that, while the predicate is indeed attached necessarily to the concept,[1] it is so in virtue of an intuition which must be added to the concept, not as thought in the concept itself.

B 16 Some few fundamental propositions, presupposed by the geometrician, are, indeed, really analytic, and rest on the principle of contradiction. But, as identical propositions, they
B 17 serve only as links in the chain of method and not as principles; for instance, $a=a$; the whole is equal to itself; or $(a+b)>a$, that is, the whole is greater than its part. And even these propositions, though they are valid according to pure concepts, are only admitted in mathematics because they can be exhibited in intuition.

2. *Natural science (physics) contains* a priori *synthetic judgments as principles*. I need cite only two such judgments: that in all changes of the material world the quantity of matter remains unchanged; and that in all communication of motion, action and reaction must always be equal. Both propositions, it is evident, are not only necessary, and therefore in their origin
B 18 *a priori*, but also synthetic. For in the concept of matter I do not think its permanence, but only its presence in the space which it occupies. I go outside and beyond the concept of matter, joining to it *a priori* in thought something which I have not thought *in* it. The proposition is not, therefore, analytic, but synthetic, and yet is thought *a priori*; and so likewise are the other propositions of the pure part of natural science.

3. *Metaphysics*, even if we look upon it as having hitherto failed in all its endeavours, is yet, owing to the nature of human reason, a quite indispensable science, and *ought to contain* a priori *synthetic knowledge*. For its business is not merely to analyse concepts which we make for ourselves *a priori* of things, and thereby to clarify them analytically, but to extend our *a priori* knowledge. And for this purpose we must employ principles which add to the given concept something that was not contained in it, and through *a priori* synthetic judgments venture out so far that experience is quite unable to follow us, as, for instance, in the proposition, that the world must have a first beginning, and such like. Thus metaphysics consists, at least *in intention*, entirely of *a priori* synthetic propositions.

[1] [Reading, with Erdmann, *jenem Begriffe* for *jenen Begriffen*.]

VI. The General Problem of Pure Reason B 19

Much is already gained if we can bring a number of investigations under the formula of a single problem. For we not only lighten our own task, by defining it accurately, but make it easier for others, who would test our results, to judge whether or not we have succeeded in what we set out to do. Now the proper problem of pure reason is contained in the question: How are *a priori* synthetic judgments possible?

That metaphysics has hitherto remained in so vacillating a state of uncertainty and contradiction, is entirely due to the fact that this problem, and perhaps even the distinction between analytic and synthetic judgments, has never previously been considered. Upon the solution of this problem, or upon a sufficient proof that the possibility which it desires to have explained does in fact not exist at all, depends the success or failure of metaphysics. Among philosophers, David Hume came nearest to envisaging this problem, but still was very far from conceiving it with sufficient definiteness and universality. He occupied himself exclusively with the synthetic proposition regarding the connection of an effect with its cause (*principium causalitatis*), and he believed himself to have B 20 shown that such an *a priori* proposition is entirely impossible. If we accept his conclusions, then all that we call metaphysics is a mere delusion whereby we fancy ourselves to have rational insight into what, in actual fact, is borrowed solely from experience, and under the influence of custom has taken the illusory semblance of necessity. If he had envisaged our problem in all its universality, he would never have been guilty of this statement, so destructive of all pure philosophy. For he would then have recognised that, according to his own argument, pure mathematics, as certainly containing *a priori* synthetic propositions, would also not be possible; and from such an assertion his good sense would have saved him.

In the solution of the above problem, we are at the same

time deciding as to the possibility of the employment of pure reason in establishing and developing all those sciences which contain a theoretical *a priori* knowledge of objects, and have therefore to answer the questions:

How is pure mathematics possible?
How is pure science of nature possible?

Since these sciences actually exist, it is quite proper to ask *how* they are possible; for that they must be possible is proved
B 21 by the fact that they exist.[a] But the poor progress which has hitherto been made in metaphysics, and the fact that no system yet propounded can, in view of the essential purpose of metaphysics, be said really to exist, leaves everyone sufficient ground for doubting as to its possibility.

Yet, in a certain sense, this *kind of knowledge* is to be looked upon as given; that is to say, metaphysics actually exists, if not as a science, yet still as natural disposition (*metaphysica naturalis*). For human reason, without being moved merely by the idle desire for extent and variety of knowledge, proceeds impetuously, driven on by an inward need, to questions such as cannot be answered by any empirical employment of reason, or by principles thence derived. Thus in all men, as soon as their reason has become ripe for speculation, there has always existed and will always continue to exist some kind of metaphysics. And so we have the question:

B 22 *How is metaphysics, as natural disposition, possible?*

that is, how from the nature of universal human reason do those questions arise which pure reason propounds to itself, and which it is impelled by its own need to answer as best it can?

But since all attempts which have hitherto been made to answer these natural questions—for instance, whether the world has a beginning or is from eternity—have always met with unavoidable contradictions, we cannot rest satisfied with the mere natural disposition to metaphysics, that is, with the pure faculty of reason itself, from which, indeed, some sort of metaphysics (be it what it may) always arises. It must be possible for reason to attain to certainty whether we know or do not know the objects of metaphysics, that is, to come to a decision either in regard to the objects of its enquiries or in regard to the capacity or incapacity of reason to pass any judgment upon them, so that we may either with confidence extend our pure reason or set to it sure and determinate limits. This last question, which arises out of the previous general problem, may, rightly stated, take the form:

How is metaphysics, as science, possible?

Thus the critique of reason, in the end, necessarily leads to scientific knowledge; while its dogmatic employment, on the other hand, lands us in dogmatic assertions to which other B 23 assertions, equally specious, can always be opposed—that is, in *scepticism*.

Antinomie

This science cannot be of any very formidable prolixity, since it has to deal not with the objects of reason, the variety of which is inexhaustible, but only with itself and the problems which arise entirely from within itself, and which are imposed upon it by its own nature, not by the nature of things which are distinct from it. When once reason has learnt completely to understand its own power in respect of objects which can be presented to it in experience, it should easily be able to determine, with completeness and certainty, the extent and the limits of its attempted employment beyond the bounds of all experience.

We may, then, and indeed we must, regard as abortive all attempts, hitherto made, to establish a metaphysic *dogmatically*. For the analytic part in any such attempted system, namely, the mere analysis of the concepts that inhere in our reason *a priori*, is by no means the aim of, but only a preparation for, metaphysics proper, that is, the extension of its *a priori* synthetic knowledge. For such a purpose, the analysis of concepts is useless, since it merely shows what is contained in these concepts, not how we arrive at them *a priori*. A solution

[a] Many may still have doubts as regards pure natural science. We have only, however, to consider the various propositions that are to be found at the beginning of (empirical) physics, properly so called, those, for instance, relating to the permanence in the quantity of matter, to inertia, to the equality of action and reaction, etc., in order to be soon convinced that they constitute a *physica pura*, or *rationalis*, which well deserves, as an independent science, to be separately dealt with in its whole extent, be that narrow or wide.

of this latter problem is required, that we may be able to de-
B 24 termine the valid employment of such concepts in regard to the objects of all knowledge in general. Nor is much self-denial needed to give up these claims, seeing that the undeniable, and in the dogmatic procedure of reason also unavoidable, contradictions of reason with itself have long since undermined the authority of every metaphysical system yet propounded. Greater firmness will be required if we are not to be deterred by inward difficulties and outward opposition from endeavouring, through application of a method entirely different from any hitherto employed, at last to bring to a prosperous and fruitful growth a science indispensable to human reason—a science whose every branch may be cut away but whose root cannot be destroyed.[1]

TRANSCENDENTAL DOCTRINE OF ELEMENTS A 19

First Part

TRANSCENDENTAL AESTHETIC

§ 1[1]

In whatever manner and by whatever means a mode of knowledge[2] may relate to objects, *intuition* is that through which it is in immediate relation to them, and to which all thought as a means is directed. But intuition takes place only in so far as the object is given to us. This again is only possible, to man at least,[3] in so far as the mind is affected in a certain way. The capacity (receptivity) for receiving representations through the mode in which we are affected by objects, is entitled *sensibility*. Objects are *given* to us by means of sensibility, and it alone yields us *intuitions*; they are *thought* through the understanding, and from the understanding arise *concepts*. But all thought must, directly or indirectly, by way of certain characters,[4] relate ultimately to intuitions, and therefore, with us, to sensibility, because in no other way can an object be given to us.

The effect of an object upon the faculty of representation, B 34
so far as we are affected by it, is *sensation*. That intuition A 20
which is in relation to the object through sensation, is entitled *empirical*. The undetermined object of an empirical intuition is entitled *appearance*.

That in the appearance which corresponds to sensation

[1] [In A the sub-sections are not numbered.] [2] [*eine Erkenntnis.*]
[3] [*uns Menschen wenigstens* added in B.]
[4] [*vermittelst gewisser Merkmale* added in B. Cf. Kant's *Nachträge zur Kritik* (edited by B. Erdmann, 1881), xi: " if the representation is not in itself the cause of the object."]

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I term its *matter*; but that which so determines[1] the manifold of appearance that it allows of being ordered[2] in certain relations, I term the *form* of appearance. That in which alone the sensations can be posited and ordered in a certain form, cannot itself be sensation; and therefore, while the matter of all appearance is given to us *a posteriori* only, its form must lie ready for the sensations *a priori* in the mind, and so must allow of being considered apart from all sensation.

I term all representations *pure* (in the transcendental sense) in which there is nothing that belongs to sensation. The pure form of sensible intuitions in general, in which all the manifold of intuition is intuited in certain relations, must be found in the mind *a priori*. This pure form of sensibility may
B 35 also itself be called *pure intuition*. Thus, if I take away from the representation of a body that which the understanding thinks in regard to it, substance, force, divisibility, etc., and
A 21 likewise what belongs to sensation, impenetrability, hardness, colour, etc., something still remains over from this empirical intuition, namely, extension and figure. These belong to pure intuition, which, even without any actual object of the senses or of sensation, exists in the mind *a priori* as a mere form of sensibility.

The science of all principles of *a priori* sensibility I call *transcendental aesthetic*.[a] There must be such a science, form-

[a] The Germans are the only people who currently make use of the word 'aesthetic' in order to signify what others call the critique of taste. This usage originated in the abortive attempt made by Baumgarten,[3] that admirable analytical thinker, to bring the critical treatment of the beautiful under rational principles, and so to raise its rules to the rank of a science. But such endeavours are fruitless. The said rules or criteria are, as regards their chief[4] sources, merely empirical, and consequently can never serve as determinate[5] *a priori* laws by which our judgment of taste must be directed. On the contrary, our judgment is the proper test of the correctness
B 36 of the rules. For this reason it is advisable either[6] to give up using the name in this sense of critique of taste, and to reserve it for that doctrine of sensibility which is true science—thus ap-

[1] [*das jenige welches macht dass.*]
[2] [In B: *geordnet werden kann* for *geordnet angeschaut wird.*]
[3] [A. G. Baumgarten (1714–62): *Aesthetica* (1750).]
[4] [*vornehmsten* added in B.]
[5] [*bestimmten* added in B.]
[6] [*entweder* added in B.]

ing the first part of the transcendental doctrine of elements, B 36
in distinction from that part which deals with the principles of pure thought, and which is called transcendental logic.

In the transcendental aesthetic we shall, therefore, first A 22
isolate sensibility, by taking away from it everything which the understanding thinks through its concepts, so that nothing may be left save empirical intuition. Secondly, we shall also separate off from it everything which belongs to sensation, so that nothing may remain save pure intuition and the mere form of appearances, which is all that sensibility can supply *a priori*. In the course of this investigation it will be found that there are two pure forms of sensible intuition, serving as principles of *a priori* knowledge, namely, space and time. To the consideration of these we shall now proceed.

A 137 / B 176 TRANSCENDENTAL DOCTRINE OF JUDGMENT

(OR ANALYTIC OF PRINCIPLES)

CHAPTER I

THE SCHEMATISM OF THE PURE CONCEPTS OF UNDERSTANDING

IN all subsumptions of an object under a concept the representation of the object must be *homogeneous* with the concept; in other words, the concept must contain something which is represented in the object that is to be subsumed under it. This, in fact, is what is meant by the expression, 'an object is contained under a concept'. Thus the empirical concept of a *plate* is homogeneous with the pure geometrical concept of a *circle*. The roundness which is thought in the latter can be intuited in the former.[1]

But pure concepts of understanding being quite heterogeneous from empirical intuitions, and indeed from all sensible intuitions, can never be met with in any intuition. For no one will say that a category, such as that of causality,
B 177 / A 138 can be intuited through sense and is itself contained in appearance. How, then, is the *subsumption* of intuitions under pure concepts, the *application* of a category to appearances, possible? A transcendental doctrine of judgment is necessary just because of this natural and important question. We must be able to show how pure concepts can be applicable to appearances. In none of the other sciences is this necessary. For since in these sciences the concepts through which the object is thought in [its] general [aspects] are not so utterly distinct and heterogeneous from those which represent it *in concreto*,

[1] [Reading, with Vaihinger, *in dem letzteren . . . im ersteren* for *in dem ersteren . . . im letzteren*.]

as given, no special discussion of the applicability of the[1] former to the latter is required.

Obviously there must be some third thing, which is homogeneous on the one hand with the category, and on the other hand with the appearance, and which thus makes the application of the former to the latter possible. This mediating representation must be pure, that is, void of all empirical content, and yet at the same time, while it must in one respect be *intellectual*, it must in another be *sensible*. Such a representation is the *transcendental schema*.

The concept of understanding contains pure synthetic unity of the manifold in general. Time, as the formal condition of the manifold of inner sense, and therefore of the connection of all representations, contains an *a priori* manifold in pure intuition. Now a transcendental determination of time is so far homogeneous with the category, which constitutes its unity, in that it is universal and rests upon an B 178
a priori rule. But, on the other hand, it is so far homogeneous A 139
with appearance, in that time is contained in every empirical representation of the manifold. Thus an application of the category to appearances becomes possible by means of the transcendental determination of time, which, as the schema of the concepts of understanding, mediates the subsumption of the appearances under the category.

After what has been proved in the deduction of the categories, no one, I trust, will remain undecided in regard to the question whether these pure concepts of understanding are of merely empirical or also of transcendental employment; that is, whether as conditions of a possible experience they relate *a priori* solely to appearances, or whether, as conditions of the possibility of things in general, they can be extended to objects in themselves, without any restriction to our sensibility. For we have seen that concepts are altogether impossible,[2] and can have no meaning, if no object is given for them, or at least for the elements of which they are composed. They cannot, therefore, be viewed as applicable to things in themselves, independent of all question as to whether and how these may be given to us. We

[1] [Reading, with Vorländer, *der* for *des*.]
[2] [Altered by Kant (*Nachträge* lviii) to: "are for us without meaning."]

SELECTIONS from KANT on MATHEMATICS p 10

have also proved that the only manner in which objects can be given to us is by modification of our sensibility; and
B 179 finally, that pure *a priori* concepts, in addition to the function
A 140 of understanding expressed in the category, must contain *a priori* certain formal conditions of sensibility, namely, those of inner sense. These conditions of sensibility constitute the universal condition under which alone the category can be applied to any object. This formal and pure condition of sensibility to which the employment of the concept of understanding is restricted, we shall entitle the *schema* of the concept. The procedure of understanding in these schemata we shall entitle the *schematism* of pure understanding.

The schema is in itself always a product of imagination. Since, however, the synthesis of imagination aims at no special intuition, but only at unity in the determination of sensibility, the schema has to be distinguished from the image. If five points be set alongside one another, thus, ..., I have an image of the number five. But if, on the other hand, I think only a number in general, whether it be five or a hundred, this thought is rather the representation of a method whereby a multiplicity, for instance a thousand, may be represented in an image in conformity with a certain concept, than the image itself. For with such a number as a thousand the image can hardly be surveyed and compared with the concept. This representation of a universal procedure of
B 180 imagination in providing an image for a concept, I entitle the schema of this concept.

Indeed it is schemata, not images of objects, which underlie
A 141 our pure sensible concepts. No image could ever be adequate to the concept of a triangle in general. It would never attain that universality of the concept which renders it valid of all triangles, whether right-angled, obtuse-angled, or acute-angled; it would always be limited to a part only of this sphere. The schema of the triangle can exist nowhere but in thought. It is a rule of synthesis of the imagination, in respect to pure figures in space. Still less is an object of experience or its image ever adequate to the empirical concept; for this latter always stands in immediate relation to the schema of imagination, as a rule for the determination of our intuition, in accordance with some specific universal concept. The concept 'dog' signifies a rule according to which my imagination can delineate the figure of a four-footed animal in a general manner, without limitation to any single determinate figure such as experience, or any possible image that I can represent *in concreto*, actually presents. This schematism of our understanding, in its application to appearances and their mere form, is an art concealed in the depths of the human soul, whose real modes of activity nature is hardly likely ever B 181
to allow us to discover, and to have open to our gaze. This much only we can assert: the *image* is a product of the empirical faculty of reproductive[1] imagination; the *schema* of sensible concepts, such as of figures in space, is a product and, A 142
as it were, a monogram, of pure *a priori* imagination, through which, and in accordance with which, images themselves first become possible. These images can be connected with the concept only by means of the schema to which they belong.[2] In themselves they are never completely congruent with the concept. On the other hand, the schema of a *pure* concept of understanding can never be brought into any image whatsoever. It is simply the pure synthesis, determined by a rule of that unity, in accordance with concepts, to which the category gives expression. It is a transcendental product of imagination, a product which concerns the determination of inner sense in general according to conditions of its form (time), in respect of all representations, so far as these representations are to be connected *a priori* in one concept in conformity with the unity of apperception.

That we may not be further delayed by a dry and tedious analysis of the conditions demanded by transcendental schemata of the pure concepts of understanding in general, we shall now expound them according to the order of the categories and in connection with them.

The pure image of all magnitudes (*quantorum*) for[3] outer B 182
sense is space; that of all objects of the senses in general is time. But the pure *schema* of magnitude (*quantitatis*), as a concept of the understanding, is *number*, a representation which comprises the successive addition of homogeneous

[1] [Reading, with Vaihinger, *reproduktiven* for *produktiven*.]
[2] [*welches sie bezeichnen*.]
[3] [Reading, with Grillo, *für den* for *vor dem*.]

A 143 units. Number is therefore simply the unity of the synthesis of the manifold of a homogeneous intuition in general, a unity due to my generating time itself in the apprehension of the intuition.

Reality, in the pure concept of understanding, is that which corresponds to a sensation in general; it is that, therefore, the concept of which in itself points to being (in time). Negation is that the concept of which represents not-being (in time). The opposition of these two thus rests upon the distinction of one and the same time as filled and as empty. Since time is merely the form of intuition, and so of objects as appearances, that in the objects which corresponds to sensation is not[1] the transcendental matter of all objects as things in themselves (thinghood,[2] reality). Now every sensation has a degree or magnitude whereby, in respect of its representation of an object otherwise remaining the same, it can fill out one and the same time, that is, occupy inner sense more or less completely, down to its cessation in nothingness (=0=*negatio*). There therefore exists a relation and connection between reality and negation, or rather a
B 183 transition from the one to the other, which makes every reality representable as a quantum. The schema of a reality, as the quantity of something in so far as it fills time, is just this continuous and uniform production of that reality in time as we successively descend from a sensation which has a certain degree to its vanishing point, or progressively ascend from its negation to some magnitude of it.

The schema of substance is permanence of the real in time, that is, the representation of the real as a substrate of empirical determination of time in general, and so as abiding while all else changes. (The existence of what is transitory[3] passes away in time but not time itself. To time, itself non-transitory[4] and abiding, there corresponds in the [field of] appearance what is non-transitory in its existence, that is, substance. Only in [relation to] substance can the succession and coexistence of appearances be determined in time.)

[1] [Reading, with Wille, *nicht die* for *die*. This seems, on the whole, preferable to taking, with Erdmann, the second part of the sentence as: "that in the objects [as things in themselves] which corresponds to sensation is the transcendental matter . . ."]
[2] [*Sachheit.*] [3] [*des Wandelbaren.*] [4] [*unwandelbar.*]

The schema of cause,[1] and of the causality[2] of a thing in A 144
general, is the real upon which, whenever posited, something else always follows. It consists, therefore, in the succession of the manifold, in so far as that succession is subject to a rule.

The schema of community or reciprocity, the reciprocal causality of substances in respect of their accidents, is the co-existence, according to a universal rule, of the determinations B 184
of the one substance with those of the other.

The schema of possibility is the agreement of the synthesis of different representations with the conditions of time in general. Opposites, for instance, cannot exist in the same thing at the same time, but only the one after the other. The schema is therefore the determination of the representation of a thing at some time or other.

The schema of actuality is existence in some determinate A 145
time.

The schema of necessity is existence of an object at all times.

We thus find that the schema of each category contains and makes capable of representation only a determination of time.[3] The schema of magnitude is the generation (synthesis) of time itself in the successive apprehension of an object. The schema of quality is the synthesis of sensation or perception with the representation of time; it is the filling of time. The schema of relation is the connecting of perceptions with one another at all times according to a rule of time-determination. Finally the schema of modality and of its categories is time itself as the correlate of the determination whether and how an object belongs to time. The schemata are thus nothing but *a priori* determinations of time in accordance with rules. These rules relate in the order of the categories to the *time-series*, the *time-content*, the *time-order*, and lastly to the *scope of time*[4] in respect of all possible objects. B 185

It is evident, therefore, that what the schematism of understanding effects by means of the transcendental synthesis of

[1] [*Ursache.*] [2] [*Kausalität.*]
[3] [Reading, with Adickes, *einer jeden Kategorie nur eine Zeitbestimmung, als* for *einer jeden Kategorie, als.*]
[4] [*Zeitinbegriff.*]

imagination is simply the unity of all the manifold of intuition in inner sense, and so indirectly the unity of apperception which as a function corresponds to the receptivity of inner sense.
A 146 The schemata of the pure concepts of understanding are thus the true and sole conditions under which these concepts obtain relation to objects and so possess *significance*. In the end, therefore, the categories have no other possible employment than the empirical. As the grounds of an *a priori* necessary unity that has its source in the necessary combination of all consciousness in one original apperception, they serve only to subordinate appearances to universal rules of synthesis, and thus to fit them for thoroughgoing connection in one experience.

All our knowledge falls within the bounds of possible experience, and just in this universal relation to possible experience consists that transcendental truth which precedes all empirical truth and makes it possible.

But it is also evident that although the schemata of sensi-
B 186 bility first realise the categories, they at the same time restrict them, that is, limit them to conditions which lie outside the understanding, and are due to sensibility. The schema is, properly, only the phenomenon, or sensible concept, of an object in agreement with the category. (Numerus *est quantitas phaenomenon*, sensatio *realitas phaenomenon*, constans et perdurabile rerum *substantia phaenomenon*, aeternitas *necessitas phaenomenon*,[1] *etc.*) If we omit a restricting condition, we would seem
A 147 to extend the scope of the concept that was previously limited. Arguing from this assumed fact, we conclude that the categories in their pure significance, apart from all conditions of sensibility, ought to apply to things in general, *as they are*, and not, like the schemata, represent them only *as they appear*. They ought, we conclude, to possess a meaning independent of all schemata, and of much wider application. Now there certainly does remain in the pure concepts of understanding, even after elimination of every sensible condition, a meaning; but it is purely logical, signifying only the bare unity of the representations. The pure concepts can find no object, and so

[1 In the text the words *et perdurabile rerum* are in italics, and there are commas after aeternitas and *necessitas*. I also read, with Erdmann, *phaenomenon* for *phaenomena*.]

can acquire no meaning which might yield a concept[1] of some object. Substance, for instance, when the sensible determination of permanence is omitted, would mean simply a something which can be thought only as subject, never as a predicate of something else. Such a representation I can put to no use, for it tells me nothing as to the nature of that which is thus to B 187 be viewed as a primary subject. The categories, therefore, without schemata, are merely functions of the understanding for concepts; and represent no object. This [objective] meaning they acquire from sensibility, which realises the understanding in the very process of restricting it.

[1 [Altered by Kant (*Nachträge* lxi) to: *eine Erkenntnis*.]

from the TRANSCENDENTAL DOCTRINE of METHOD

CHAPTER I

Section 1

THE DISCIPLINE OF PURE REASON IN ITS DOGMATIC EMPLOYMENT

Mathematics presents the most splendid example of the successful extension of pure reason, without the help of experience. Examples are contagious, especially as they quite naturally flatter a faculty which has been successful in one field, [leading it] to expect the same good fortune in other fields. Thus pure
A 713 / B 741 reason hopes to be able to extend its domain as successfully and securely in its transcendental as in its mathematical em-

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ployment, especially when it resorts to the same method as has been of such obvious utility in mathematics. It is therefore highly important for us to know whether the method of attaining apodeictic certainty which is called *mathematical* is identical with the method by which we endeavour to obtain the same certainty in philosophy, and which in that field would have to be called *dogmatic*.

Philosophical knowledge is the *knowledge gained by reason from concepts*; mathematical knowledge is the knowledge gained by reason from the *construction* of concepts. To *construct* a concept means to exhibit *a priori* the intuition which corresponds to the concept. For the construction of a concept we therefore need a *non-empirical* intuition. The latter must, as intuition, be a *single* object, and yet none the less, as the construction of a concept (a universal representation), it must in its representation express universal validity for all possible intuitions which fall under the same concept. Thus I construct a triangle by representing the object which corresponds to this concept either by imagination alone, in pure intuition, or in accordance therewith also on paper, in empirical intuition—in both cases completely *a priori*, without having borrowed the pattern from any experience. The single figure which we draw is empirical, and yet it serves to express the concept, without impairing its universality. For in this empirical intuition we consider only the act whereby we construct the concept, and abstract from the many determinations (for instance, the magnitude of the sides and of the angles), which are quite indifferent, as not altering the concept 'triangle'. A 714 / B 742

Thus philosophical knowledge considers the particular only in the universal, mathematical knowledge the universal in the particular, or even in the single instance, though still always *a priori* and by means of reason. Accordingly, just as this single object is determined by certain universal conditions of construction, so the object of the concept, to which the single object corresponds merely as its schema, must likewise be thought as universally determined.

The essential difference between these two kinds of knowledge through reason consists therefore in this formal difference, and does not depend on difference of their material or objects. Those who propose to distinguish philosophy from mathematics by saying that the former has as its object *quality* only and the latter *quantity* only, have mistaken the effect for the cause. The form of mathematical knowledge is the cause why it is limited exclusively to quantities. For it is the concept of quantities only that allows of being constructed, that is, exhibited *a priori* in intuition; whereas qualities cannot be presented in any intuition that is not empirical. Consequently reason can obtain a knowledge of qualities only through concepts. No one can obtain an intuition corresponding to the concept of reality otherwise than from experience; we can never come into possession of it *a priori* out of our own resources, and prior to the empirical consciousness of reality. The shape of a cone we can form for ourselves in intuition, unassisted by any experience, according to its concept alone, but the colour of this cone must be previously given in some experience or other. I cannot represent in intuition the concept of a cause in general except in an example supplied by experience; and similarly with other concepts. Philosophy, as well as mathematics, does indeed treat of quantities, for instance, of totality, infinity, etc. Mathematics also concerns itself with qualities, for instance, the difference between lines and surfaces, as spaces of different quality, and with the continuity of extension as one of its qualities. But although in such cases they have a common object, the mode in which reason handles that object is wholly different in philosophy and in mathematics. Philosophy confines itself to universal concepts; mathematics can achieve nothing by concepts alone but hastens at once to intuition, in which it considers the concept *in concreto*, though not empirically, but only in an intuition which it presents *a priori*, that is, which it has constructed, and in which whatever follows from the universal conditions of the construction must be universally valid of the object of the concept thus constructed. A 715 / B 743 A 716 / B 744

Suppose a philosopher be given the concept of a triangle and he be left to find out, in his own way, what relation the sum of its angles bears to a right angle. He has nothing but the concept of a figure enclosed by three straight lines, and possessing three angles. However long he meditates on this concept, he will never produce anything new. He can analyse and clarify the concept of a straight line or of an angle or of the number three, but he can never arrive at any proper-

ties not already contained in these concepts. Now let the geometrician take up these questions. He at once begins by constructing a triangle. Since he knows that the sum of two right angles is exactly equal to the sum of all the adjacent angles which can be constructed from a single point on a straight line, he prolongs one side of his triangle and obtains two adjacent angles, which together are equal to two right angles. He then divides the external angle by drawing a line parallel to the opposite side of the triangle, and observes that he has thus obtained an external adjacent angle which is equal to an internal angle—and so on. In this fashion, through a chain of inferences guided throughout by intuition, he arrives at a fully evident and universally valid solution of the problem. {A 717 B 745

But mathematics does not only construct magnitudes (*quanta*) as in geometry; it also constructs magnitude as such (*quantitas*), as in algebra In this it abstracts completely from the properties of the object that is to be thought in terms of such a concept of magnitude. It then chooses a certain notation for all constructions of magnitude as such (numbers),[1] that is, for addition, subtraction, extraction of roots, etc. Once it has adopted a notation for the general concept of magnitudes so far as their different relations are concerned, it exhibits in intuition, in accordance with certain universal rules, all the various operations through which the magnitudes are produced and modified. When, for instance, one magnitude is to be divided by another, their symbols are placed together, in accordance with the sign for division, and similarly in the other processes; and thus in algebra by means of a symbolic construction, just as in geometry by means of an ostensive construction (the geometrical construction of the objects themselves), we succeed in arriving at results which discursive knowledge could never have reached by means of mere concepts.

Now what can be the reason of this radical difference in the fortunes of the philosopher and the mathematician, both of whom practise the art of reason, the one making his way by means of concepts, the other by means of intuitions which he exhibits *a priori* in accordance with concepts? The cause is evident from what has been said above, in our exposition of the {A 718 B 746

[1] [Reading, with Hartenstein and Erdmann (*Zahlen*), *als . . . Wurzeln usw.* for (*Zahlen, als . . . Subtraktion usw*).]

fundamental transcendental doctrines. We are not here concerned with analytic propositions, which can be produced by mere analysis of concepts (in this the philosopher would certainly have the advantage over his rival), but with synthetic propositions, and indeed with just those synthetic propositions that can be known *a priori*. For I must not restrict my attention to what I am actually thinking in my concept of a triangle (this is nothing more than the mere definition); I must pass beyond it to properties which are not contained in this concept, but yet belong to it. Now this is impossible unless I determine my object in accordance with the conditions either of empirical or of pure intuition. The former would only give us an empirical proposition (based on the measurement of the angles), which would not have universality, still less necessity; and so would not at all serve our purpose. The second method of procedure is the mathematical one, and in this case is the method of geometrical construction, by means of which I combine in a pure intuition (just as I do in empirical intuition) the manifold which belongs to the schema of a triangle in general, and therefore to its concept. It is by this method that universal synthetic propositions must be constructed.

It would therefore be quite futile for me to philosophise upon the triangle, that is, to think about it discursively. I should not be able to advance a single step beyond the mere definition, which was what I had to begin with. There is indeed a transcendental synthesis [framed] from concepts alone, a synthesis with which the philosopher is alone competent to deal; but it relates only to a thing in general, as defining the conditions under which the perception of it can belong to possible experience. But in mathematical problems there is no question of this, nor indeed of existence at all, but only of the properties of the objects in themselves, [that is to say], solely in so far as these properties are connected with the concept of the objects. {A 719 B 747

In the above example we have endeavoured only to make clear the great difference which exists between the discursive employment of reason in accordance with concepts and its intuitive employment by means of the construction of concepts. This naturally leads on to the question, what can be the cause

which necessitates such a twofold employment of reason, and how we are to recognise whether it is the first or the second method that is being employed.

All our knowledge relates, finally, to possible intuitions, for it is through them alone that an object is given. Now an *a priori* concept, that is, a concept which is not empirical, either already includes in itself a pure intuition (and if so, it can be constructed), or it includes nothing but the synthesis of possible intuitions which are not given *a priori*. In this latter case we can indeed make use of it in forming synthetic *a priori* judgments, but only discursively in accordance with concepts, never intuitively through the construction of the concept. {A 720 B 748

The only intuition that is given *a priori* is that of the mere form of appearances, space and time. A concept of space and time, as quanta, can be exhibited *a priori* in intuition, that is, constructed, either in respect of the quality (figure) of the quanta, or through number in their quantity only (the mere synthesis of the homogeneous manifold). But the matter of appearances, by which *things* are given us in space and ti ne, can only be represented in perception, and therefore *a posteriori*. The only concept which represents *a priori* this empirical content of appearances is the concept of a *thing* in general, and the *a priori* synthetic knowledge of this thing in general can give us nothing more than the mere rule of the synthesis of that which perception may give *a posteriori* It can never yield an *a priori* intuition of the real object, since this must necessarily be empirical.

Synthetic propositions in regard to *things* in general, the intuition of which does not admit of being given *a priori*, are transcendental. Transcendental propositions can never be given through construction of concepts, but only in accordance with concepts that are *a priori*. They contain nothing but the rule according to which we are to seek empirically for a certain synthetic unity of that which is incapable of intuitive representation *a priori* (that is, of perceptions). But these synthetic principles cannot exhibit *a priori* any one of their concepts in a specific instance; they can only do this *a posteriori*, by means of experience, which itself is possible only in conformity with these principles {A 721 B 749

If we are to judge synthetically in regard to a concept, we must go beyond this concept and appeal to the intuition in which it is given. For should we confine ourselves to what is contained in the concept, the judgment would be merely analytic, serving only as an explanation of the thought, in terms of what is actually contained in it. But I can pass from the concept to the corresponding pure or empirical intuition, in order to consider it in that intuition *in concreto*, and so to know, either *a priori* or *a posteriori*, what are the properties of the object of the concept. The *a priori* method gives us our rational and mathematical knowledge through the construction of the concept, the *a posteriori* method our merely empirical (mechanical) knowledge, which is incapable of yielding necessary and apodeictic propositions. Thus I might analyse my empirical concept of gold without gaining anything more than merely an enumeration of everything that I actually think in using the word, thus improving the logical character of my knowledge but not in any way adding to it. But I take the material body, familiarly known by this name, and obtain perceptions by means of it; and these perceptions yield various propositions which are synthetic but empirical. When the concept is mathematical, as in the concept of a triangle, I am in a position to construct the concept, that is, to give it *a priori* in intuition, and in this way to obtain knowledge which is at once synthetic and rational. But if what is given me is the *transcendental* concept of a reality, substance, force, etc., it indicates neither an empirical nor a pure intuition, but only the synthesis of empirical intuitions, which, as being empirical, cannot be given *a priori*. And since the synthesis is thus unable to advance *a priori*, beyond the concept, to the corresponding intuition, the concept cannot yield any determining synthetic proposition, but only a principle of the synthesis[a] of possible A 722 B 750

[a] With the concept of cause I do really go beyond the empirical concept of an event (something happening), yet I do not pass to the intuition which exhibits the concept of cause *in concreto*, but to the time-conditions in general, which in experience may be found to be in accord with this concept. I therefore proceed merely in accordance with concepts; I cannot proceed by means of the construction of concepts, since the concept is a rule of the synthesis of perceptions, and the latter are not pure intuitions, and so do not permit of being *given a priori*.

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empirical intuitions. A transcendental proposition is therefore synthetic knowledge through reason, in accordance with mere concepts; and it is discursive, in that while it is what alone makes possible any synthetic unity of empirical knowledge, it yet gives us no intuition *a priori*.

There is thus a twofold employment of reason; and while A 723 / B 751 the two modes of employment resemble each other in the universality and *a priori* origin of their knowledge, in outcome they are very different. The reason is that in the [field of] appearance, in terms of which[1] all objects are given us, there are two elements, the form of intuition (space and time), which can be known and determined completely *a priori*, and the matter (the physical element) or content—the latter signifying something which is met with in space and time and which therefore contains an existent[2] corresponding to sensation. In respect to this material element, which can never be given in any determinate fashion otherwise than empirically, we can have nothing *a priori* except indeterminate concepts of the synthesis of possible sensations, in so far as they belong, in a possible experience, to the unity of apperception. As regards the formal element, we can determine our concepts in *a priori* intuition, inasmuch as we create for ourselves, in space and time, through a homogeneous synthesis, the objects themselves —these objects being viewed simply as *quanta*. The former method is called the employment of reason in accordance with concepts; in so employing it[3] we can do nothing more than bring appearances under concepts, according to their actual content. The concepts cannot be made determinate in this manner,[4] save only empirically, that is, *a posteriori* (although always in accordance with these concepts as rules of an empirical synthesis). The other method is the employment of reason through the construction of concepts; and since the concepts here re- A 724 / B 752 late to an *a priori* intuition, they are for this very reason themselves *a priori* and can be given in a quite determinate fashion in pure intuition, without the help of any empirical data. The consideration of everything which exists in space or time, in regard to the questions, whether and how far it is a quantum

[1] [*als wodurch.*] [2] [*Dasein.*]
[3] [Reading, with Erdmann, *in dem* for *indem.*]
[4] [Reading, with Erdmann, *dadurch* for *darauf.*]

or not, whether we are to ascribe to it positive being or the absence of such, how far this something occupying space or time is a primary substratum or a mere determination [of substance], whether there be a relation of its existence to some other existence, as cause or effect, and finally in respect of its existence whether it is isolated or is in reciprocal relation to and dependence upon others—these questions, as also the question of the possibility of this existence, its actuality and necessity, or the opposites of these, one and all belong altogether to knowledge obtained by reason from concepts, such knowledge being termed *philosophical*. But the determination of an intuition *a priori* in space (figure), the division of time (duration), or even just the knowledge of the universal element in the synthesis of one and the same thing in time and space, and the magnitude of an intuition that is thereby generated (number),—all this is the work of reason through construction of concepts, and is called *mathematical*.

The great success which attends reason in its mathematical employment quite naturally gives rise to the expectation that it, or at any rate its method, will have the same success in other fields as in that of quantity. For this method has the advantage A 725 / B 753 of being able to realise all its concepts in intuitions, which it can provide *a priori*, and by which it becomes, so to speak, master of nature; whereas pure philosophy is all at sea when it seeks through *a priori* discursive concepts to obtain insight in regard to the natural world, being unable to intuit *a priori* (and thereby to confirm) their reality. Nor does there seem to be, on the part of the experts in mathematics, any lack of self-confidence as to this procedure—or on the part of the vulgar of great expectations from their skill—should they apply themselves to carry out their project. For, since they have hardly ever attempted to philosophise in regard to their mathematics (a hard task!), the specific difference between the two employments of reason has never so much as occurred to them. Current, empirical rules, which they borrow from ordinary consciousness, they treat as being axiomatic. In the question as to the source of the concepts of space and time they are not in the least interested, although it is precisely with these concepts (as the only original quanta) that they are themselves occupied. Similarly, they think it unnecessary to investigate

the origin of the pure concepts of understanding and in so doing to determine the extent of their validity; they care only to make use of them. In all this they are entirely in the right, provided only they do not overstep the proper limits, that is, the limits of the natural world. But, unconsciously, they pass from the field of sensibility to the precarious ground of pure and even transcendental concepts, a ground (*instabilis tellus, innabilis unda*) that permits them neither to stand nor to swim, and where their hasty tracks are soon obliterated. In mathematics, on the other hand, their passage gives rise to a broad highway, which the latest posterity may still tread with confidence. A 726 / B 754

We have made it our duty to determine, with exactitude and certainty, the limits of pure reason in its transcendental employment. But the pursuit of such transcendental knowledge has this peculiarity, that in spite of the plainest and most urgent warnings men still allow themselves to be deluded by false hopes, and therefore to postpone the total abandonment of all proposed attempts to advance beyond the bounds of experience into the enticing regions of the intellectual world. It therefore becomes necessary to cut away the last anchor of these fantastic hopes, that is, to show that the pursuit of the mathematical method cannot be of the least advantage in this kind of knowledge (unless it be in exhibiting more plainly the limitations of the method); and that mathematics[1] and philosophy, although in natural science they do, indeed, go hand in hand, are none the less so completely different, that the procedure of the one can never be imitated by the other.

The exactness of mathematics rests upon definitions, axioms and demonstrations. I shall content myself with showing that none of these, in the sense in which they are understood by the mathematician, can be achieved or imitated by the philosopher. I shall show that in philosophy the geometrician can by his method build only so many houses of cards, just as in mathematics the employment of a philosophical method results only in mere talk. Indeed it is precisely in knowing its limits that philosophy consists; and even the mathematician, unless his talent is of such a specialised character that it naturally confines itself to its proper field, cannot afford to ignore the warnings of philosophy, or to behave as if he were superior to them. A 727 / B 755

[1] [*Messkunst.*]

1. *Definitions.*—To *define*, as the word itself indicates, really only means to present the complete, original concept of a thing within the limits of its concept.[a] If this be our standard, an *empirical* concept cannot be defined at all, but only *made explicit*. For since we find in it only a few characteristics of a certain species of sensible object, it is never certain that we are not using the word, in denoting one and the same object, sometimes so as to stand for more, and sometimes so as to stand for fewer characteristics. Thus in the concept of *gold* one man may think, in addition to its weight, colour, malleability, also its property of resisting rust, while another will perhaps know nothing of this quality. We make use of certain characteristics only so long as they are adequate for the purpose of making distinctions; new observations remove some properties and add others; and thus the limits of the concept are never assured. And indeed what useful purpose could be served by defining an empirical concept, such, for instance, as that of water? When we speak of water and its properties, we do not stop short at what is thought in the word, water, but proceed to experiments. The word, with the few characteristics which we attach to it, is more properly to be regarded as merely a designation than as a concept of the thing; the so-called definition is nothing more than a determining of the word. In the second place, it is also true that no concept given *a priori*, such as substance, cause, right, equity, etc., can, strictly speaking, be defined. For I can never be certain that the clear representation of a given concept, which as given may still be confused, has been completely effected, unless I know that it is adequate to its object. But since the concept of it may, as given, include many obscure representations, which we overlook in our analysis, although we are constantly making use of them in our application of the concept, the completeness of the analysis of my concept is always in doubt, and a multiplicity A 728 / B 756

[a] *Completeness* means clearness and sufficiency of characteristics; by *limits* is meant the precision shown in there not being more of these characteristics than belong to the complete concept; by *original* is meant that this determination of these limits is not derived from anything else, and therefore does not require any proof; for if it did, that would disqualify the supposed explanation from standing at the head of all the judgments regarding its object.

of suitable examples suffices only to make the completeness *probable*, never to make it *apodeictically* certain. Instead of the term, definition, I prefer to use the term, *exposition*, as being a more guarded term, which the critic can accept as being up to a certain point valid, though still entertaining doubts as to the completeness of the analysis. Since, then, neither empirical concepts nor concepts given *a priori* allow of definition, the only remaining kind of concepts, upon which this mental operation[1] can be tried, are arbitrarily invented concepts. A concept which I have invented I can always define; for since it is not given to me either by the nature of understanding or by experience, but is such as I have myself deliberately made it to be, I must know what I have intended to think in using it. I cannot, however, say that I have thereby defined a true object.[2] For if the concept depends on empirical conditions, as *e.g.* the concept of a ship's clock, this arbitrary concept of mine does not assure me of the existence or of the possibility of its object. I do not even know from it whether it has an object at all, and my explanation may better be described as a declaration of my project than as a definition of an object. There remain, therefore, no concepts which allow of definition, except only those which contain an arbitrary synthesis that admits of *a priori* construction. Consequently, mathematics is the only science that has definitions. For the object which it thinks it exhibits *a priori* in intuition, and this object certainly cannot contain either more or less than the concept, since it is through the definition[3] that the concept of the object is given—and given originally, that is, without its being necessary to derive the definition[3] from any other source. The German language has for the [Latin] terms *exposition*, *explication*, *declaration*, and *definition* only one word, *Erklärung*,[4] and we need not, therefore, be so stringent in our requirements as altogether to refuse to philosophical explanations[5] the honourable title, definition. We shall confine ourselves simply to remarking that while philosophical definitions are never more than expositions of given concepts, mathematical definitions are constructions of con-

A 729 / B 757

A 730 / B 758

[1] [*dieses Kunststück.*]

[2] [*einen wahren Gegenstand.*]

[3] [*Erklärung.*]

[4] [This term Kant usually employs in the sense of explanation; but, as above indicated, it is used in the preceding sentence in the sense of definition.]

[5] [*Erklärungen.*]

cepts, originally framed by the mind itself; and that while the former can be obtained only by analysis (the completeness of which is never apodeictically certain), the latter are produced synthetically. Whereas, therefore, mathematical definitions *make* their concepts, in philosophical definitions concepts are only *explained*. From this it follows:

(*a*) That in philosophy we must not imitate mathematics by beginning with definitions, unless it be by way simply of experiment. For since the definitions are analyses of given concepts, they presuppose the prior presence of the concepts, although in a confused state; and the incomplete exposition must precede the complete. Consequently, we can infer a good deal from a few characteristics, derived from an incomplete analysis, without having yet reached the complete exposition, that is, the definition. In short, the definition in all its precision and clarity ought, in philosophy, to come rather at the end than at the beginning of our enquiries.[a] In mathematics, on the other hand, we have no concept whatsoever prior to the definition, through which the concept itself is first given. For this reason mathematical science must always begin, and it can always begin, with the definition.

A 731 / B 759

(*b*) That mathematical definitions can never be in error. For since the concept is first given through the definition, it includes nothing except precisely what the definition intends should be understood by it. But although nothing incorrect can be introduced into its content, there may sometimes, though rarely, be a defect in the form in which it is clothed, namely as regards precision. Thus the common explanation of the circle that it is a *curved* line every point in which is equidistant

[a] Philosophy is full of faulty definitions, especially of definitions which, while indeed containing some of the elements required, are yet not complete. If we could make no use of a concept till we had defined it, all philosophy would be in a pitiable plight. But since a good and safe use can still be made of the elements obtained by analysis so far as they go, defective definitions, that is, propositions which are properly not definitions, but are yet true, and are therefore approximations to definitions, can be employed with great advantage. In mathematics definition belongs *ad esse*, in philosophy *ad melius esse*. It is desirable to attain an adequate definition, but often very difficult. The jurists are still without a definition of their concept of right.

from one and the same point (the centre), has the defect that {A 732 / B 760} the determination, curved, is introduced unnecessarily. For there must be a particular theorem, deduced from the definition and easily capable of proof, namely, that if all points in a line are equidistant from one and the same point, the line is curved (no part of it straight). Analytic definitions, on the other hand, may err in many ways, either through introducing characteristics which do not really belong to the concept, or by lacking that completeness which is the essential feature of a definition. The latter defect is due to the fact that we can never be quite certain of the completeness of the analysis. For these reasons the mathematical method of definition does not admit of imitation in philosophy.

2. *Axioms.*—These, in so far as they are immediately certain, are synthetic *a priori* principles. Now one concept cannot be combined with another synthetically and also at the same time immediately, since, to be able to pass beyond either concept, a third something is required to mediate our knowledge. Accordingly, since philosophy is simply what reason knows by means of concepts, no principle deserving the name of an axiom is to be found in it. Mathematics, on the other hand, can have axioms, since by means of the construction of concepts in the intuition of the object it can combine the predicates of the object both *a priori* and immediately, as, for instance, in the proposition that three points always lie in a {A 733 / B 761} plane. But a synthetic principle derived from concepts alone can never be immediately certain, for instance, the proposition that everything which happens has a cause. Here I must look round for a third something, namely, the condition of time-determination in an experience; I cannot obtain knowledge of such a principle directly and immediately from the concepts alone. Discursive principles are therefore quite different from intuitive principles, that is, from axioms; and always require a deduction. Axioms, on the other hand, require no such deduction, and for the same reason are evident—a claim which the philosophical principles can never advance, however great their certainty. Consequently, the synthetic propositions of pure, transcendental reason are, one and all, infinitely removed from being as evident—which is yet so often arrogantly claimed on their behalf—as the proposition that *twice two make four.*

In the Analytic I have indeed introduced some axioms of intuition into the table of the principles of pure understanding; but the principle[1] there applied is not itself an axiom, but serves only to specify the principle[2] of the possibility of axioms in general, and is itself no more than a principle[1] derived from concepts. For the possibility of mathematics must itself be demonstrated in transcendental philosophy. Philosophy has therefore no axioms, and may never prescribe its *a priori* {A 734 / B 762} principles in any such absolute manner, but must resign itself to establishing its authority in their regard by a thorough deduction.

3. *Demonstrations.*—An apodeictic proof can be called a demonstration, only in so far as it is intuitive. Experience teaches us what is, but does not teach us that it could not be other than what it is. Consequently, no empirical grounds of proof can ever amount to apodeictic proof. Even from *a priori* concepts, as employed in discursive knowledge, there can never arise intuitive certainty, that is, [demonstrative] evidence, however apodeictically certain the judgment may otherwise be. Mathematics alone, therefore, contains demonstrations, since it derives its knowledge not from concepts but from the construction of them, that is, from intuition, which can be given *a priori* in accordance with the concepts. Even the method of algebra with its equations, from which the correct answer, together with its proof, is deduced by reduction, is not indeed geometrical in nature, but is still constructive in a way characteristic of the science.[3] The concepts attached to the symbols, especially concerning the relations of magnitudes, are presented in intuition; and this method, in addition to its heuristic advantages, secures all inferences against error by setting each one before our eyes. While philosophical knowledge must do without this advantage, inasmuch as it has always to consider the universal *in abstracto* (by means of concepts), mathematics can consider {A 735 / B 763} the universal *in concreto* (in the single intuition) and yet at the same time through pure *a priori* representation, whereby all errors are at once made evident. I should therefore prefer to

[1] [*Grundsatz.*] [2] [*Prinzipium.*]
[3] [*charakteristische Konstruction.* The meaning in which Kant uses this phrase is doubtful. It might also be translated 'construction by means of symbols'.]

SELECTIONS from KANT on MATHEMATICS p 20

call the first kind *acroamatic* (discursive) *proofs*, since they may be conducted by the agency of words alone (the object in thought), rather than *demonstrations* which, as the term itself indicates, proceed in and through the intuition of the object.

From all this it follows that it is not in keeping with the nature of philosophy, especially in the field of pure reason, to take pride in a dogmatic procedure, and to deck itself out with the title and insignia of mathematics, to whose ranks it does not belong, though it has every ground to hope for a sisterly union with it. Such pretensions are idle claims which can never be satisfied, and indeed must divert philosophy from its true purpose, namely, to expose the illusions of a reason that forgets its limits, and by sufficiently clarifying our concepts to recall it from its presumptuous speculative pursuits to modest but thorough self-knowledge. Reason must not, therefore, in its transcendental endeavours, hasten forward with sanguine expectations, as though the path which it has traversed led directly to the goal, and as though the accepted premisses could be so securely relied upon that there can be no need of constantly returning to them and of considering whether we may not perhaps, in the course of the inferences, discover defects which have been overlooked in the principles, and which render it necessary either to determine these principles more fully or to change them entirely. A 736 / B 764

I divide all apodeictic propositions, whether demonstrable or immediately certain, into *dogmata* and *mathemata*. A synthetic proposition directly derived from concepts is a *dogma*; a synthetic proposition, when directly obtained through the construction of concepts, is a *mathema*. Analytic judgments really teach us nothing more about the object than what the concept which we have of it already contains; they do not extend our knowledge beyond the concept of the object, but only clarify the concept. They cannot therefore rightly be called dogmas (a word which might perhaps be translated *doctrines*).[1] Of the two kinds of synthetic *a priori* propositions only those belonging to philosophical knowledge can, according to the ordinary usage of words, be entitled dogmas; the propositions of arithmetic or geometry would hardly be so named. The customary use of words thus confirms our interpretation of the term, namely. that only judgments derived from concepts can be called dogmatic, not those based on the construction of concepts.

[1] [*Lehrsprüche.*]

Now in the whole domain of pure reason, in its merely speculative employment, there is not to be found a single synthetic judgment directly derived from concepts. For, as we have shown, ideas cannot form the basis of any objectively valid synthetic judgment. Through concepts of understanding pure reason does, indeed, establish secure principles, not however directly from concepts alone, but always only indirectly through relation of these concepts to something altogether contingent, namely, *possible experience*. When such experience (that is, something as object of possible experiences) is presupposed, these principles are indeed apodeictically certain; but in themselves, directly, they can never be known *a priori*. Thus no one can acquire insight into the proposition that everything which happens has its cause, merely from the concepts involved. It is not, therefore, a dogma, although from another point of view, namely, from that of the sole field of its possible employment, that is, experience, it can be proved with complete apodeictic certainty. But though it needs proof, it should be entitled a *principle*, not a *theorem*, because it has the peculiar character that it makes possible the very experience which is its own ground of proof, and that in this experience it must always itself be presupposed. A 737 / B 765

Now if in the speculative employment of pure reason there are no dogmas, to serve as its special subject-matter,[1] all *dogmatic* methods, whether borrowed from the mathematician or specially invented, are as such inappropriate. For they only serve to conceal defects and errors, and to mislead philosophy, whose true purpose is to present every step of reason in the clearest light. Nevertheless its method can always be *systematic*. For our reason is itself, subjectively, a system, though in its pure employment, by means of mere concepts, it is no more than a system whereby our investigations can be conducted in accordance with principles of unity, the material being provided by *experience* alone. We cannot here discuss the method peculiar to transcendental philosophy; we are at present con- A 738 / B 766

[1] [*auch dem Inhalte nach.*]

A 738 / B 766

cerned only with a critical estimate of what may be expected from our faculties—whether we are in a position to build at all; and to what height, with the material at our disposal (the pure *a priori* concepts), we may hope to carry the edifice.

from the ANALYTIC, AGAIN

I

AXIOMS OF INTUITION *

A 162
B 202

Their principle is: All intuitions are extensive magnitudes.

Proof[1]

Appearances, in their formal aspect,[2] contain an intuition in space and time, which conditions them, one and all, *a*

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* [In A:]

The Axioms of Intuition.

Principle of the pure understanding: All appearances are, in their intuition, extensive magnitudes.

[a] [Note added in B.] All combination (*conjunctio*) is either composition (*compositio*) or connection (*nexus*). The former is the synthesis of the manifold where its constituents do not necessarily belong to one another. For example, the two triangles into which a square is divided by its diagonal do not necessarily belong to one another. Such also is the synthesis of the *homogeneous* in everything which can be *mathematically* treated. This synthesis can itself be divided into that of *aggregation* and that of *coalition*, the former

[1] [This heading and the first paragraph added in B.]
[2] [*der Form nach.*]

priori. They cannot be apprehended, that is, taken up into empirical consciousness, save through that synthesis of the manifold whereby the representations of a determinate space or time are generated, that is, through combination of the
B 203 homogeneous manifold and consciousness of its synthetic unity. Consciousness of the synthetic unity[1] of the manifold [and] homogeneous in intuition in general, in so far as the representation of an object first becomes possible by means of it, is, however, the concept of a magnitude (*quantum*). Thus even the perception of an object, as appearance, is only possible through the same synthetic unity of the manifold of the given sensible intuition as that whereby the unity of the combination of the manifold [and] homogeneous is thought in the concept of a *magnitude*. In other words, appearances are all without exception *magnitudes*, indeed *extensive magnitudes*. As intuitions in space or time, they must be represented through the same synthesis whereby space and time in general are determined.

I entitle a magnitude extensive when the representation of the parts makes possible, and therefore necessarily precedes, the representation of the whole. I cannot represent to myself a line, however small, without drawing it in thought, that
A 163 is, generating from a point all its parts one after another. Only in this way can the intuition be obtained. Similarly with all times, however small. In these I think to myself only that successive advance from one moment to another, whereby through the parts of time and their addition a determinate time-magnitude is generated. As the [element of]

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applying to *extensive* and the latter to *intensive* quantities. The second mode of combination (*nexus*) is the synthesis of the manifold so far as its constituents *necessarily belong to one another*, as, for example, the accident to some substance, or the effect to the cause. It is therefore synthesis of that which, though *heterogeneous*, is yet represented as combined *a priori*. This combination, as not being arbitrary and as concerning the connection of the *existence* of the
B 202 manifold, I entitle *dynamical*. Such connection can itself, in turn, be divided into the *physical* connection of the appearances with one another, and their *metaphysical* connection in the *a priori* faculty of knowledge.

[1] [Adding, with Vaihinger, *der synthetischen Einheit*].

SELECTIONS from KANT on MATHEMATICS p 22

pure intuition in all appearances is either space or time, every appearance is as intuition an extensive magnitude; only B 204
through successive synthesis of part to part in [the process of] its apprehension can it come to be known. All appearances are consequently intuited as aggregates, as complexes of previously given parts. This is not the case with magnitudes of every kind, but only with those magnitudes which are represented and apprehended by us in this *extensive* fashion.

The mathematics of space[1] (geometry) is based upon this successive synthesis of the productive imagination in the generation of figures. This is the basis of the axioms which formulate the conditions of sensible *a priori* intuition under which alone the schema of a pure concept of outer appearance can arise—for instance, that between two points only one straight line is possible, or that two straight lines cannot enclose a space, etc. These are the axioms which, strictly, relate only to magnitudes (*quanta*) as such.

As regards magnitude (*quantitas*), that is, as regards the answer to be given to the question, 'What is the magnitude of a thing?' there are no axioms in the strict meaning of the A 164
term, although there are a number of propositions which are synthetic and immediately certain (*indemonstrabilia*). The propositions, that if equals be added to equals the wholes are equal, and if equals be taken from equals the remainders are equal, are analytic propositions; for I am immediately conscious of the identity of the production of the one magnitude with the production of the other. [Consequently, they B 205
are not] axioms, [for these] have to be *a priori synthetic* propositions. On the other hand, the evident propositions of numerical relation are indeed synthetic, but are not general like those of geometry, and cannot, therefore, be called axioms but only numerical formulas. The assertion that $7 + 5$ is equal to 12 is not an analytic proposition. For neither in the representation of 7, nor in that of 5, nor in the representation of the combination of both, do I think the number 12. (That I must do so in the *addition* of the two numbers is not to the point, since in the analytic proposition the question is only whether I actually think the predicate in the representation of the subject.) But although the proposition is synthetic, it is also only singular. So far as we are here attending merely to the synthesis of the homogeneous (of units), that synthesis can take place only in one way, although the *employment* of these numbers is general. If I assert that through three lines, two of which taken together are greater than the third, a triangle can be described, I have expressed merely
A 165 the function of productive imagination whereby the lines can be drawn greater or smaller, and so can be made to meet at any and every possible angle. The number 7, on the other hand, is possible only in one way. So also is the number 12, as thus generated through the synthesis of 7 with 5. Such propositions must not, therefore, be called
B 206 axioms (that would involve recognition of an infinite number of axioms), but numerical formulas.

This transcendental principle of the mathematics of appearances greatly enlarges our *a priori* knowledge. For it alone can make pure mathematics, in its complete precision, applicable to objects of experience. Without this principle, such application would not be thus self-evident; and there has indeed been much confusion of thought in regard to it. Appearances are not things in themselves. Empirical intuition is possible only by means of the pure intuition of space and of time. What geometry asserts of pure intuition is therefore undeniably valid of empirical intuition. The idle objections, that objects of the senses may not conform to such rules of construction in space as that of the infinite divisibility of lines or angles, must[1] be given up. For if these objections hold good, we deny the objective validity of space, and consequently of all mathematics, and no longer know why and how far mathematics can be applicable to appearances. The synthesis of spaces and times, being a synthesis of the essential forms[2]
A 166 of all intuition, is what makes possible the apprehension of appearance, and consequently every outer experience and all knowledge of the objects of such experience. Whatever pure mathematics establishes in regard to the synthesis of the form of apprehension is also necessarily valid of the objects apprehended. All objections are only the chicanery of a falsely

[1] [*Ausdehnung*.]

[1] [Reading, with Kehrbach, *dürfen*, *mussen* for *dürfe*, *muss*.]
[2] [Reading, with Erdmann, *der wesentlichen Formen* for *der wesentlichen Form*.]

SELECTIONS from KANT on MATHEMATICS p 23

instructed reason, which, erroneously professing to isolate the B 207
objects of the senses from the formal condition of our sensibility, represents them, in spite of the fact that they are mere appearances, as objects in themselves, given to the understanding. Certainly, on that assumption, no synthetic knowledge of any kind could be obtained of them *a priori*, and nothing therefore could be known of them synthetically through pure concepts of space. Indeed, the science which determines these concepts, namely geometry, would not itself be possible.

2

ANTICIPATIONS OF PERCEPTION*

In all appearances, the real that is an object of sensation has intensive magnitude, that is, a degree.

Proof[1]

Perception is empirical consciousness, that is, a consciousness in which sensation is to be found. Appearances, as objects of perception, are not pure, merely formal, intuitions, like space and time. For in and by themselves these latter cannot be perceived. Appearances contain in addition to intuition the matter for some object in general (whereby something existing in space or time is represented); they contain, that is to say, the real of sensation as merely subjective representation, which gives us only the consciousness that the subject is affected, and which we relate to an object in general. Now from empirical B 208
consciousness to pure consciousness a graduated transition is possible, the real in the former completely vanishing and a merely formal *a priori* consciousness of the manifold in space and time remaining. Consequently there is also possible a

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* [In A:]

The Anticipations of Perception

The principle which anticipates all perceptions, as such, is as follows: In all appearances sensation, and the *real* which corresponds to it in the object (*realitas phaenomenon*), has an *intensive magnitude*, that is, a degree.

[1] [This heading and the first paragraph added in B.]

synthesis in the process of generating the magnitude of a sensation from its beginning in pure intuition = 0, up to any required magnitude. Since, however, sensation is not in itself an objective representation, and since neither the intuition of space nor that of time is to be met with in it, its magnitude is not extensive but *intensive*. This magnitude is generated in the act of apprehension whereby the empirical consciousness of it can in a certain time increase from nothing = 0 to the[1] given measure. Corresponding to this intensity of sensation, an *intensive magnitude*, that is, a degree of influence on the sense [*i.e.* on the special sense involved], must be ascribed to all objects of perception, in so far as the perception contains sensation.

All knowledge by means of which I am enabled to know and determine *a priori* what belongs to empirical knowledge may be entitled an anticipation; and this is undoubtedly the A 167
sense in which Epicurus employed the term προλήψις. But as B 209
there is an element in the appearances (namely, sensation, the matter of perception) which can never be known *a priori*, and which therefore constitutes the distinctive difference between empirical and *a priori* knowledge, it follows that sensation is just that element which cannot be anticipated. On the other hand, we might very well entitle the pure determinations in space and time, in respect of shape as well as of magnitude, anticipations of appearances, since they represent *a priori* that which may always be given *a posteriori* in experience. If, however, there is in every sensation, as sensation in general (that is, without a particular sensation having to be given), something that can be known *a priori*, this will, in a quite especial sense, deserve to be named anticipation. For it does indeed seem surprising that we should forestall experience, precisely in that which concerns what is only to be obtained through it, namely, its matter. Yet, none the less, such is actually the case.

Apprehension by means merely of sensation occupies only an instant,[2] if, that is, I do not take into account the succession of different sensations. As sensation is that element in

[1] [Taking, with Erdmann, *ihrem* as referring, not to *Bewusstsein*, but to *Empfindung*.]

[2] [*Augenblick*.]

SELECTIONS from KANT on MATHEMATICS p 24

the [field of] appearance the apprehension of which does not involve a successive synthesis proceeding from parts to the whole representation, it has no extensive magnitude. The absence of sensation at that instant would involve the representation of the instant as empty, therefore as = 0. Now A 168
what corresponds in empirical intuition to sensation is reality (*realitas phaenomenon*); what corresponds to its absence is negation = 0. Every sensation, however, is capable of diminu- B 210
tion, so that it can decrease and gradually vanish. Between reality in the [field of] appearance and negation there is therefore a continuity[1] of many possible intermediate sensations, the difference between any two of which is always smaller than the difference between the given sensation and zero or complete negation. In other words, the real in the [field of] appearance has always a magnitude. But since its apprehension by means of mere sensation takes place in an instant and not through successive synthesis of different sensations, and therefore does not proceed from the parts to the whole, the magnitude is to be met with only in the apprehension.[2] The real has therefore magnitude, but not extensive magnitude.

A magnitude which is apprehended only as unity, and in which multiplicity can be represented only through approximation to negation = 0, I entitle an *intensive* magnitude. Every reality in the [field of] appearance has therefore intensive magnitude or degree. If this reality is viewed as cause, either of sensation or of some other reality in the [field of] appearance, such as change, the degree of the reality as cause is then entitled a moment,[3] the moment of gravity. It is so named for the reason that degree signifies only that magnitude A 169
the apprehension of which is not successive, but instantaneous.[4] This, however, I touch on only in passing; for with causality I am not at present dealing.

Every sensation, therefore, and likewise every reality in B 211
the [field of] appearance, however small it may be, has a degree, that is, an intensive magnitude which can always be diminished. Between reality and negation there is a continuity of possible realities and of possible smaller perceptions.

[1] [*ein kontinuierlicher Zusammenhang.*]
[2] [Reading, with Wille, *welche aber nur in der Apprehension* for *welche aber nicht in der Apprehension.* Cf. proof added in B, 207-8.]
[3] [*ein Moment.*] [4] [*augenblicklich.*]

Every colour, as for instance red, has a degree which, however small it may be, is never the smallest; and so with heat, the moment of gravity, etc.

The property of magnitudes by which no part of them is the smallest possible, that is, by which no part is simple, is called their continuity. Space and time are *quanta continua*, because no part of them can be given save as enclosed between limits (points or instants), and therefore only in such fashion that this part is itself again a space or a time. Space therefore consists solely of spaces, time solely of times. Points and instants are only limits, that is, mere positions which limit space and time. But positions always presuppose the intuitions which they limit or are intended to limit; and out of mere positions,
A 170 viewed as constituents capable of being given prior to space or time, neither space nor time can be constructed. Such magnitudes may also be called *flowing*, since the synthesis of productive imagination involved in their production is a pro-
B 212 gression in time, and the continuity of time is ordinarily designated by the term flowing or flowing away.

All appearances, then, are continuous magnitudes, alike in their intuition, as extensive, and in their mere perception (sensation, and with it reality) as intensive. If the synthesis of the manifold of appearance is interrupted, we have an aggregate of different appearances, and not appearance as a genuine quantum. Such an aggregate[1] is not generated by continuing without break productive synthesis of a certain kind, but through repetition of an ever-ceasing synthesis. If I called thirteen thalers a quantum of money, I should be correct, provided my intention is to state the value of a mark of fine silver. For this is a continuous magnitude, in which no part is the smallest, and in which every part can constitute a piece of coin that always contains material for still smaller pieces. But if I understand by the phrase thirteen round thalers, so many coins, quite apart from the question of what their silver standard may be, I then use the phrase, quantum of thalers, inappropriately. It ought to be entitled an aggregate, that is,
A 171 a number of pieces of money. But as unity must be presupposed in all number, appearance as unity is a quantum, and as a quantum is always a continuum.

[1] [Reading, with Kehrbach, *welches Aggregat* for *welches.*]

Since all appearances, alike in their extensive and in their intensive aspect, are thus continuous magnitudes, it might seem to be an easy matter to prove with mathematical conclusiveness the proposition that all alteration (transition of a B 213
thing from one state to another), is continuous. But the causality of an alteration in general, presupposing, as it does, empirical principles, lies altogether outside the limits of a transcendental philosophy. For upon the question as to whether a cause capable of altering the state of a thing, that is, of determining it to the opposite of a certain given state, may be possible, the *a priori* understanding casts no light; and this not merely because it has no insight into its possibility (such insight is lacking to us in many other cases of *a priori* knowledge), but because alterableness is to be met with only in certain determinations of appearances, and because, whereas [in fact] the cause of these determinations lies in the unalterable, experience alone can teach what they are. Since in our present enquiry we have no data of which we can make use save only the pure fundamental concepts of all possible experience, in which there must be absolutely nothing that is empirical, we cannot, without destroying the unity of our system, anticipate general natural science, which is based A 172
on certain primary experiences.[1]

At the same time, there is no lack of proofs of the great value of our principle in enabling us to anticipate perceptions, and even to some extent to make good their absence, by placing a check upon all false inferences which might be drawn from their absence.

If all reality in perception has a degree, between which B 214
and negation there exists an infinite gradation of ever smaller degrees, and if every sense must likewise[2] possess some particular degree[3] of receptivity of sensations, no perception, and consequently no experience, is possible that could prove, either immediately or mediately (no matter how far-ranging the reasoning may be), a complete absence of all reality in the [field of] appearance. In other words, the proof of an empty space or of an empty time can never be derived from experience. For, in the first place, the complete absence of reality

[1] [*Grunderfahrungen.*]
[2] [Reading, with Erdmann, *ebensowohl* for *gleichwohl.*] [3] [*i.e.* limit.]

from a sensible intuition can never be itself perceived; and, secondly, there is no appearance whatsoever and no difference in the degree of reality of any appearance from which it can be inferred. It is not even legitimate to postulate it in order to explain any difference. For even if the whole intuition of a certain determinate space or time is real through and through, that is, though no part of it is empty, none the less, since every
A 173 reality has its degree, which can diminish to nothing (the void) through infinite gradations without in any way altering the extensive magnitude of the appearance, there must be infinite different degrees in which space and time may be filled. Intensive magnitude can in different appearances be smaller or greater, although the extensive magnitude of the intuition remains one and the same.

B 215 Let us give an example. Almost all natural philosophers, observing—partly by means of the moment of gravity or weight, partly by means of the moment of opposition to other matter in motion—a great difference in the quantity of various kinds of matter in bodies that have the same volume, unanimously conclude that this volume, which constitutes the extensive magnitude of the appearance, must in all material bodies be empty in varying degrees. Who would ever have dreamt of believing that these students of nature, most of whom are occupied with problems in mathematics and mechanics, would base such an inference solely on a metaphysical presupposition—the sort of assumption they so stoutly profess to avoid? They assume that the real in space (I may not here name it impenetrability or weight, since these are empirical concepts) is everywhere uniform and varies only in extensive magnitude, that is, in amount. Now to this presupposition, for which they could find no support in experi-
A 174 ence, and which is therefore purely metaphysical, I oppose a transcendental proof, which does not indeed explain the difference in the filling of spaces, but completely destroys the supposed necessity of the above presupposition, that the difference is only to be explained on the assumption of empty space. My proof has the merit at least of freeing the understanding, so that it is at liberty to think this difference in
B 216 some other manner, should it be found that some other hypothesis is required for the explanation of the natural

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appearances. For we then recognise that although two equal spaces can be completely filled with different kinds of matter, so that there is no point in either where matter is not present, nevertheless every reality has, while keeping its quality unchanged, some specific degree (of resistance or weight) which can, without diminution of its extensive magnitude or amount, become smaller and smaller *in infinitum*, before it passes into the void and [so] vanishes [out of existence]. Thus a radiation which fills a space, as for instance heat, and similarly every other reality in the [field of] appearance, can diminish in its degree *in infinitum*, without leaving the smallest part of this space in the least empty. It may fill the space just as completely with these smaller degrees as another appearance does with greater degrees. I do not at all intend to assert that this is what actually occurs when material bodies differ in specific gravity, but only to establish from a principle of pure understanding that the nature of our perceptions allows of such a mode of explanation, that we are not justified in assuming the real in appearances to be uniform in degree, differing only in aggregation and extensive magnitude, and that we are especially in error when we claim that such interpretation can be based on an *a priori* principle of the understanding. A 175

This anticipation of perception must always, however, appear somewhat strange to anyone trained in transcendental reflection,[1] and to any student of nature who by such teaching has been trained to circumspection. The assertion that the understanding anticipates[2] such a synthetic principle, ascribing a degree to all that is real in the appearances, and so asserting the possibility of an internal distinction in sensation itself (abstraction being made of its empirical quality), awakens doubts and difficulties. It is therefore a question not unworthy of solution, how the understanding can thus in *a priori* fashion pronounce synthetically upon appearances, and can indeed anticipate in that which in itself is merely empirical and concerns only sensation. B 217

The *quality* of sensation, as for instance in colours, taste, etc., is always merely empirical, and cannot be represented *a priori*. But the real, which corresponds to sensations in general, as opposed to negation = 0, represents only that something the very concept of which includes being, and signifies nothing but the synthesis in an empirical consciousness in general. Empirical consciousness can in inner sense be raised from 0 to any higher degree, so that a certain extensive magnitude of intuition, as for instance of illuminated surface, may excite as great a sensation as the combined aggregate of many[1] such surfaces less illuminated. [Since the extensive magnitude of the appearance thus varies independently], we can completely abstract from it, and still represent in the mere sensation in any one of its moments a synthesis that advances uniformly from 0 to the given empirical consciousness. Consequently, though all sensations as such are given only *a posteriori*,[2] their property of possessing a degree can be known *a priori*. It is remarkable that of magnitudes in general we can know *a priori* only a single *quality*, namely, that of continuity, and that in all quality (the real in appearances) we can know *a priori* nothing save [in regard to] their intensive *quantity*, namely that they have degree. Everything else has to be left to experience. A 176 B 218

[1] [Adding, with Erdmann, *Überlegung*.]
[2] [Adding, with Hartenstein, *antizipiert*.]

4

THE POSTULATES OF EMPIRICAL THOUGHT IN GENERAL

1. That which agrees with the formal conditions of experience, that is, with the conditions of intuition and of concepts, is *possible*.

2. That which is bound up with the material conditions of experience, that is, with sensation, is *actual*. B 266

3. That which in its connection with the actual is determined in accordance with universal conditions of experience, is (that is, exists as) *necessary*.

Explanation

The categories of modality have the peculiarity that, in determining an object, they do not in the least enlarge the concept to which they are attached as predicates. They only express the relation of the concept to the faculty of knowledge. Even when the concept of a thing is quite complete, I can still enquire whether this object is merely possible or is also actual, or if actual, whether it is not also necessary. No additional determinations are thereby thought in the object itself; the question is only how the object, together with all its determinations, is related to understanding and its empirical employment, to empirical judgment,[1] and to reason in its application to experience. A 219

Just on this account also the principles of modality are nothing but explanations of the concepts of possibility, actuality, and necessity, in their empirical employment; at the same time they restrict all categories to their merely empirical employment, and do not approve or allow their transcendental employment. For if they are not to have a purely logical significance, analytically expressing the form of *thought*, but are to refer to the possibility, actuality, or necessity of *things*, they must concern possible experience and its synthetic unity, in which alone objects of knowledge can be given. B 267

The postulate of the *possibility* of things requires that the concept of the things should agree with the formal conditions of an experience in general. But this, the objective form of experience in general, contains all synthesis that is A 220

[1] [*Urteilskraft.*]

required for knowledge of objects. A concept which contains a synthesis is to be regarded as empty and as not related to any object, if this synthesis does not belong to experience either as being derived from it, in which case it is an *empirical concept*, or as being an *a priori* condition upon which experience in general in its formal aspect rests, in which case it is a *pure concept*. In the latter case it still belongs to experience, inasmuch as its object is to be met with only in experience. For whence shall we derive the character of the possibility of an object which is thought through a synthetic *a priori* concept, if not from the synthesis which constitutes the form of the empirical knowledge of objects? It is, indeed, a necessary logical condition that a concept of the possible must not contain any contradiction; but this is not by any means sufficient to determine the objective reality of the concept, that is, the possibility of such an object as is thought through the concept. Thus there is no contradiction in the concept of a figure which is enclosed within two straight lines, since the concepts of two straight lines and of their coming together contain no negation of a figure. The impossibility arises not from the concept in itself, but in connection with its construction in space, that is, from the conditions of space and of its determination. And since these contain *a priori* in themselves the form of experience in general, they have objective reality; that is, they apply to possible things. B 268 A 221

We shall now proceed to show the far-reaching utility and influence of this postulate of possibility. If I represent to myself a thing which is permanent, so that everything in it which changes belongs only to its state, I can never know from such a concept that a thing of this kind is possible. Or if I represent to myself something which is so constituted that if it is posited something else invariably and inevitably follows from it, this may certainly be so thought without contradiction; but this thought affords no means of judging whether this property (causality) is to be met with in any possible thing. Lastly, I can represent to myself diverse things (substances), which are so constituted that the state of the one carries with it some consequence in the state of the other, and this reciprocally; but I can never determine from these concepts, which contain a merely arbitrary synthesis, whether a relation of this kind B 269

can belong to any [possible] things. Only through the fact that these concepts express *a priori* the relations of perceptions in every experience, do we know their objective reality, that is, A 222
their transcendental truth, and this, indeed, independently of experience, though not independently of all relation to the form of an experience in general, and to the synthetic unity in which alone objects can be empirically known.

But if we should seek to frame quite new concepts of substances, forces, reciprocal actions, from the material which perception presents to us, without experience itself yielding the example of their connection, we should be occupying ourselves with mere fancies, of whose possibility there is no criterion since we have neither borrowed these concepts [directly] from experience, nor have taken experience as our instructress in their formation Such fictitious concepts, unlike the categories, can acquire the character of possibility not in *a priori* fashion, as conditions upon which all experience depends, but only *a posteriori* as being concepts which are given through experience itself. And, consequently, their possibility must either be known *a posteriori* and empirically, or B 270
it cannot be known at all. A substance which would be permanently present in space, but without filling it (like that mode of existence intermediate between matter and thinking being which some would seek to introduce), or a special ultimate mental power of *intuitively* anticipating the future (and not merely inferring it), or lastly a power of standing in community of thought with other men, however distant they may be—are concepts the possibility of which is altogether ground- A 223
less, as they cannot be based on experience and its known laws; and without such confirmation they are arbitrary combinations of thoughts, which, although indeed free from contradiction, can make no claim to objective reality, and none, therefore, as to the possibility of an object such as we here profess to think. As regards reality, we obviously cannot think it *in concreto*, without calling experience to our aid. For reality is bound up with sensation, the matter of experience, not with that form of relation in regard to which we can, if we so choose, resort to a playful inventiveness.[1]

But I leave aside everything the possibility of which can

[1] [*in Erdichtungen spielen.*]

be derived only from its actuality in experience, and have here in view only the possibility of things through *a priori* concepts;
B 271 and I maintain the thesis that their possibility can never be established from such concepts taken in and by themselves, but only when the concepts are viewed as formal and objective conditions of experience in general.

It does, indeed, seem as if the possibility of a triangle could be known from its concept in and by itself (the concept is certainly independent of experience), for we can, as a matter of fact, give it an object completely *a priori*, that is, can construct it. But since this is only the form of an object, it would remain
A 224 a mere product of imagination, and the possibility of its object would still be doubtful. To determine its possibility, something more is required, namely, that such a figure be thought under no conditions save those upon which all objects of experience rest. That space is a formal *a priori* condition of outer experiences, that the formative[1] synthesis through which we construct a triangle in imagination is precisely the same as that which we exercise in the apprehension of an appearance, in making for ourselves an empirical concept of it—these are the considerations that alone enable us to connect the representation of the possibility of such a thing with the concept of it. Similarly, since the concepts of continuous magnitudes, indeed of magnitudes in general, are one and all synthetic, the possibility of such magnitudes is never clear from the concepts them-
B 272 selves, but only when they are viewed as formal conditions of the determination of objects in experience in general. And where, indeed, should we seek for objects corresponding to these concepts if not in experience, through which alone objects are given to us? We can, indeed, prior to experience itself, know and characterise the possibility of things, merely by reference to the formal conditions under which in experience anything whatsoever is determined as object, and therefore can do so completely *a priori*. But, even so, this is possible only in relation to experience and within its limits.

A 225 The postulate bearing on the knowledge of things as *actual* does not, indeed, demand immediate *perception* (and, therefore, sensation of which we are conscious) of the object whose existence is to be known. What we do, however,

[1] [*bildende.*]

require is the connection of the object with some actual perception, in accordance with the analogies of experience, which define[1] all real connection in an experience in general.[2]

In the *mere concept* of a thing no mark[3] of its existence is to be found. For though it may be so complete that nothing which is required for thinking the thing with all its inner determinations is lacking to it, yet existence has nothing to do with all this, but only with the question whether such a thing be so given us that the perception of it can, if need be, precede the concept. For that the concept precedes the perception signifies the concept's mere possibility; the perception which supplies the content to the concept is the sole mark of actuality. We can also, however, know the existence of the thing prior to its perception and, consequently, comparatively speaking, in an *a priori* manner, if only it be bound up with certain perceptions, in accordance with the principles of their empirical connection (the analogies). For the existence of the thing being thus bound up with our perceptions in a possible experience, we are able in the series of possible perceptions and under the guidance of the analogies to make the transition from our actual perception to the thing in question. Thus from the perception of the attracted iron filings we know of the existence of a magnetic matter pervading all bodies, although the constitution of our organs cuts us off from all immediate perception of this medium.[4] For in accordance with the laws of sensibility and the context of our perceptions, we should, were our senses more refined, come also in an experience[5] upon the immediate empirical intuition of it. The grossness of our senses does not in any way decide the form of possible experience in general. Our knowledge of the existence of things reaches, then, only so far as perception and its advance[6] according to empirical laws can extend. If we do not start from experience, or do not proceed in accordance with laws of the empirical connection of appearances, our guessing or enquiring into the existence of anything will only be an idle pretence.[1] Idealism raises, however, what is a serious objection to these rules for proving existence mediately; and this is the proper place for its refutation.

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[1] [*darlegen.*]

[2] [In the opening sentence of this paragraph I adopt a change in the order of the words, as suggested by Valentiner.]

[3] [*Charakter.*]

[4] [*dieses Stoffs.*]

[5] [If a comma be omitted from the text of A and B, we have what is perhaps the more natural reading: "the context of our perceptions in one experience, we should . . . come also upon . . ."]

[6] [Reading, with Wille, *Fortgang* for *Anhang.*]