VAN HEIJENOORT, JEAN
FROM FREGE TO GODEL: A SOURCE BOOK IN MATHEMATICAL LOGIC, 1879–1931
Cambridge, Harvard U.P. 1967

Letter to Russell

GOTTLOB FREGE

(*1902*)

This is Frege's prompt answer to Russell's letter published above. Frege first calls Russell's attention to an error in *Begriffsschrift*; it is a mere oversight, without any consequence (see above, p. 15, footnote 12). He then describes his reaction to the paradox that Russell has just communicated to him, and he begins to look for the source of the predicament. He incriminates the "transformation of the generalization of an equality into an equality of courses-of-values". For Frege a function is something incomplete, "unsaturated". When it is written $f(x)$, x is something extraneous that merely serves to indicate the kind of supplementation that is needed; we might just as well write $f(\ )$. Consider now two functions that, for the same argument, always have the same value: $(x)(f(x) = g(x))$. (This is not Frege's notation, but its modern equivalent.) Since f and g, or rather $f(\ )$ and $g(\ )$, are something incomplete, we cannot simply write $f = g$. Functions are not objects, and in order to treat them, in some respect, as objects Frege introduces their *Werthverlauf*. The *Werthverlauf* of a function $f(x)$ is denoted by $\grave{\varepsilon}f(\varepsilon)$ (where ε is a dummy; we can also write $\grave{\alpha}f(\alpha), \ldots$). The expression "the function $f(x)$ has the same *Werthverlauf* as the function $g(x)$" is taken to mean "for the same argument the function $f(x)$ always has the same value as the function $g(x)$", and we can write (in modern notation)

$$(*)\quad (x)(f(x) = g(x)) \equiv (\grave{\varepsilon}f(\varepsilon) = \grave{\alpha}g(\alpha)).$$

This is the "transformation of the generalization of an equality into an equality of courses-of-values". Whereas the function is unsaturated and is not an object, its *Werthverlauf* is "something complete in itself", an object, in particular so far as substitution is concerned. There Frege sees the origin of the paradox.

Frege soon made his point more specific. He received Russell's letter while the second volume of his *Grundgesetze der Arithmetik* was at the printshop, and he barely had the time to add an appendix in which he shows how the schema (*) above (or rather half of it, the implication from right to left) allows the derivation of the paradox; he also proposed a restriction in the schema to prevent that. Russell, whose *Principles of mathematics* was at the printshop when he received Frege's volume, added to his book an appendix in which he endorsed Frege's emendation. But soon thereafter he tried out various other solutions (*1905a*); he finally proposed his theory of types (*1908a*).

Russell's paradox has been leaven in modern logic, and countless works have dealt with it. For a late and thorough study of Frege's "way out", see *Quine 1955*.

When Lord Russell was asked whether he would consent to the publication of his letter to Frege (*1902*), he replied with the following letter, in which the reader will find a stirring tribute to Frege.

I should be most pleased if you would publish the correspondence between Frege and myself, and I am grateful to you for suggesting this. As I think about acts of integrity and grace, I realise that there is nothing in my knowledge to compare with Frege's dedication to truth. His entire life's work was on the verge of completion, much of his work had been ignored to the benefit of men infinitely less capable, his second volume was about to be published, and upon finding that his fundamental assumption was in error, he responded with intellectual pleasure clearly submerging any feelings of dedication is to creative work and knowledge instead of cruder efforts to dominate and be known.

Yours sincerely,

Bertrand Russell

The translation of Frege's letter is by Beverly Woodward, and it is printed here with the kind permission of Verlag Felix Meiner and the Institut für mathematische Logik und Grundlagenforschung in Münster, who are preparing an edition of Frege's scientific correspondence and hitherto unpublished writings; this edition will include the German text of the letter.

Jena, 22 June 1902

Dear colleague,

Many thanks for your interesting letter of 16 June. I am pleased that you agree with me on many points and that you intend to discuss my work thoroughly. In response to your request I am sending you the following publications:

1. "Kritische Beleuchtung" ⟦*1895*⟧,
2. "Ueber die Begriffsschrift des Herrn Peano" ⟦*1896*⟧,
3. "Ueber Begriff und Gegenstand" ⟦*1892*⟧,
4. "Über Sinn und Bedeutung" ⟦*1892a*⟧,
5. "Ueber formale Theorien der Arithmetik" ⟦*1885*⟧.

I received an empty envelope that seems to be addressed by your hand. I surmise that you meant to send me something that has been lost by accident. If this is the case, I thank you for your kind intention. I am enclosing the front of the envelope.

When I now read my *Begriffsschrift* again, I find that I have changed my views on many points, as you will see if you compare it with my *Grundgesetze der Arithmetik*. I ask you to delete the paragraph beginning "Nicht minder erkennt man" on page 7 of my *Begriffsschrift* ⟦"It is no less easy to see", p. 15 above⟧, since it is incorrect; incidentally, this had no detrimental effects on the rest of the booklet's contents.

Your discovery of the contradiction caused me the greatest surprise and, I would almost say, consternation, since it has shaken the basis on which I intended to build arithmetic. It seems, then, that transforming the generalization of an equality into an equality of courses-of-values ⟦die Umwandlung der Allgemeinheit einer Gleichheit in eine Werthverlaufsgleichheit⟧ (§ 9 of my *Grundgesetze*) is not always permitted, that my Rule V (§ 20, p. 36) is false, and that my explanations in § 31 are not sufficient to ensure that my combinations of signs have a meaning in all cases. I must reflect further on the matter. It is all the more serious since, with the loss of my Rule V, not only the foundations of my arithmetic, but also the sole possible foundations of arithmetic, seem to vanish. Yet, I should think, it must be possible to set up conditions for the transformation of the generalization of an equality into an equality of courses-of-values such that the essentials of my proofs remain intact. In any case your discovery is very remarkable and will perhaps result in a great advance in logic, unwelcome as it may seem at first glance.

Incidentally, it seems to me that the expression "a predicate is predicated of itself" is not exact. A predicate is as a rule a first-level function, and this function requires an object as argument and cannot have itself as argument (subject). Therefore I would prefer to say "a concept is predicated of its own extension". If the function $\Phi(\xi)$ is a concept, I denote its extension (or the corresponding class) by "$\grave{\varepsilon}\Phi(\varepsilon)$" (to be sure, the justification for this has now become questionable to me). In "$\Phi(\grave{\varepsilon}\Phi(\varepsilon))$" or "$\grave{\varepsilon}\Phi(\varepsilon) \cap \grave{\varepsilon}\Phi(\varepsilon)$"[1] we then have a case in which the concept $\Phi(\xi)$ is predicated of its own extension.

The second volume of my *Grundgesetze* is to appear shortly. I shall no doubt have to add an appendix in which your discovery is taken into account. If only I already had the right point of view for that!

Very respectfully yours,

G. Frege