Selections from Wittgenstein's Remarks on the Foundations of Mathematics

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Russell Marcus

From Part I

3. *How do I know* that in working out the series + 2 I must write

"20004, 20006"

and not

"20004, 20008"?

—(The question: "How do I know that this colour is 'red'?" is similar.)

"But you surely know for example that you must always write the *same* sequence of numbers in the units: 2, 4, 6, 8, 0, 2, 4, etc."—Quite true: the problem must already appear in this sequence, and even in *this* one: 2, 2, 2, 2, etc.—For how do I know that I am to write "2" after the five hundredth "2"? i.e. that 'the same figure' in that place is "2"? And if I know it *in advance*, what use is this knowledge to me later on? I mean: how do I know what to do with this earlier knowledge when the step actually has to be taken?

(If intuition is needed to continue the series + 1, then it is also needed to continue the series + 0.)

"But do you mean to say that the expression '+ 2' leaves you in doubt what you are to do e.g. after 2004?"—No; I answer "2006" without hesitation. But just for that reason it is superfluous to suppose that this was determined earlier on. My having no doubt in face of the question does *not* mean that it has been answered in advance.

"But I surely also know that whatever number I am given I shall be able, straight off and with certainty, to give the next one.—Certainly my dying first is excluded, and a lot of other things too. But my being so certain of being able to go on is naturally very important.—

I. 4. "But then what does the peculiar inexorability of mathematics consist in?"—Would not the inexorability with which two follows one and three two be a good example?—But presumably this means: follows in the *series of cardinal numbers*; for in a different series something different follows. And isn't *this* series just *defined* by this sequence?—"Is that supposed to mean that it is equally correct whichever way a person counts, and that anyone can count as he pleases?"—We should presumably not call it "counting" if everyone said the numbers one after the other *anyhow*; but of course it is not simply a question of a name. For what we call "counting" is an important part of our life's activities. Counting and calculating are not—e.g.—simply a pastime. Counting (and that means: counting like *this*) is a technique that is employed daily in the most various operations of our lives. And that is why we learn to count as we do: with endless practice, with merciless exactitude; that is why it is inexorably insisted that we shall all say "two" after "one", "three" after "two" and so on.—But is this counting only a *use*, then; isn't there also some truth corresponding to this sequence?" The *truth* is that counting has proved to pay.—"Then do you want to say that 'being true' means: being usable (or useful)?"—No, not that; but that it can't be said of the series of natural numbers—any more than of our language—that it is true, but: that it is usable, and, above all, *it is used*.

I. 5. "But doesn't it follow with logical necessity that you get two when you add one to one, and three when you add one to two? and isn't this inexorability the same as that of logical inference?"—Yes! it is the same.—"But isn't there a truth corresponding to logical inference? Isn't it *true* that this follows from that?"—The proposition: "It is true that this follows from that" means simply: this follows from that. And how do we use this proposition?—What would happen if we made a different inference—*how* should we get into conflict with truth?

How should we get into conflict with truth, if our footrules were made of very soft rubber instead of wood and steel?—"Well, we shouldn't get to know the correct measurement of the table."—You mean: we should not get, or could not be sure of getting, *that* measurement which we get with our rigid rulers. So if you had measured the table with the elastic rulers and said it measured five feet by our usual way of measuring, you would be wrong; but if you say that it measured five feet by your way of measuring, that is correct.—"But surely that isn't measuring at all!"—It is similar to our measuring and capable, in certain circumstances, of fulfilling 'practical purposes'. (A shopkeeper might use it to treat different customers differently.)

If a ruler expanded to an extraordinary extent when slightly heated, we should say—in normal circumstances—that that made it *unusable*. But we could think of a situation in which this was just what was wanted. I am imagining that we perceive the expansion with the naked eye; and we ascribe the same numerical measure of length to bodies in rooms of different temperatures, if they measure the same by the ruler which to the eye is now longer, now shorter.

It can be said: What is here called "measuring" and "length" and "equal length", is something different from what we call those things. The use of these words is different from ours; but it is *akin* to it; and we too use these words in a variety of ways.

I. 33. When I say "This proposition follows from that one", that is to accept a rule. The acceptance is *based* on the proof. That is to say, I find this chain (this figure) acceptable as a *proof*.——"But could I do otherwise? Don't I *have* to find it acceptable?"—Why do you say you have to? Because at the end of the proof you say e.g.: "Yes—I have to accept this conclusion". But that is after all only the expression of your unconditional acceptance.

I.e. (I believe): the words "I have to admit this" are used in *two kinds* of case: when we have got a proof—and also with reference to the individual steps of the proof.

I. 34. And how does it come out that the proof *compels* me? Well, in the fact that once I have got it I go ahead in such-and-such a way, and refuse any other path. All I should further say as a final argument against someone who did not want to go that way, would be: "Why, don't you see . . .!"—and that is no *argument*.

I. 35. "But, if you are right, how does it come about that all men (or at any rate all normal men) accept these patterns as proofs of these propositions?"—It is true, there is great—and interesting—agreement here.

I. 61. "*This* follows inexorably from *that*."—True, in this demonstration this issues from that.

This is a demonstration for whoever acknowledges it as a demonstration. If anyone *doesn't* acknowledge it, doesn't go by it as a demonstration, then he has parted company with us even before anything is said.

I. 113. "But am I not compelled, then, to go the way I do in a chain of inferences?"—Compelled? After all I can presumably go as I choose!—"But if you want to remain in accord with the rules you *must* go this way."—Not at all, I call *this* 'accord'.—"Then you have changed the meaning of the word 'accord', or the meaning of the rule."—No;—who says what 'change' and 'remaining the same' mean here?

However many rules you give me—I give a rule which justifies *my* employment of your rules.

I. 116. "Then according to you everybody could continue the series as he likes; and so infer *any*how!" In that case we shan't call it "continuing the series" and also presumably not "inference". And thinking and inferring (like counting) is of course bounded for us, not by an arbitrary definition, but by natural limits corresponding to the body of what can be called the role of thinking and inferring in our life.

For we are at one over this, that the laws of inference do not compel him to say or to write such and such like rails compelling a locomotive. And if you say that, while he may indeed *say* it, still he can't *think* it, then I am only saying that that means, not: try as he may he can't think it, but: it is for us an essential part of 'thinking' that—in talking, writing, etc.—he makes *this sort* of transition. And I say further that the line between what we include in 'thinking' and what we no longer include in 'thinking' is no more a hard and fast one than the line between what is still and what is no longer called "regularity".

Nevertheless the laws of inference can be said to compel us; in the same sense, that is to say, as other laws in human society. The clerk who infers as in (17) *must* do it like that; he would be punished if he inferred differently. If you draw different conclusions you do indeed get into conflict, e.g. with society; and also with other practical consequences.

And there is even something in saying: he can't *think* it. One is trying e.g. to say: he can't fill it with personal content; he can't really *go along with it*—personally, with his intelligence. It is like when one says: this sequence of notes makes no sense, I can't sing it with expression. I cannot *respond* to it. Or, what comes to the same thing here: I don't respond to it.

"If he says it"—one might say—"he can only say it without thinking". And here it merely needs to be noticed that 'thoughtless' talk and other talk do indeed sometimes differ as regards what goes on in the talker, his images, sensations and so on while he is talking, but that this accompaniment does not constitute the thinking, and the lack of it is not enough to constitute 'thoughtlessness'.

I. 117. In what sense is logical argument a compulsion?—"After all you grant *this* and *this*; so you must also grant *this*!" That is the way of compelling someone. That is to say, one can in fact compel people to admit something in this way.—Just as one can e.g. compel someone to go over there by pointing over there with a bidding gesture of the hand.

Suppose in such a case I point with two fingers at once in different directions, thus leaving it open to the man to go in which of the two directions he likes,—and another time I point in only *one* direction; then this can also be expressed by saying: my first order did not compel him to go just in *one* direction, while the second one did. But this is a statement to tell us what kind of orders I gave; not the way they operate, not whether they do in fact compel such-and-such a person, i.e. whether he obeys them.

I. 118. It looked at first as if these considerations were meant to shew that 'what seems to be a logical compulsion is in reality only a psychological one'—only here the question arose: am I acquainted with both kinds of compulsion, then?!

Imagine that people used the expression: "The law § . . . punishes a murderer with death". Now this could only mean: this law runs so and so. That form of expression, however, might force itself on us, because the law is an instrument when the guilty man is brought to punishment.—Now we talk of 'inexorability' in connexion with people who punish. And here it might occur to us to say: "The law is *inexorable*—men can let the guilty go, the law executes him". (And even: "the law *always* executes him".)—What is the use of such a form of expression?—In the first instance, this proposition only says that such-and-such is to be found in the law, and human beings sometimes do not go by the law. Then, however, it does give us a picture of a single inexorable judge, and many lax judges. That is why it serves to express respect for the law. Finally, the expression can also be so used that a law is called inexorable when it makes no provision for a possible act of grace, and in the opposite case it is perhaps called 'discriminating'.

Now we talk of the 'inexorability' of logic; and think of the laws of logic as inexorable, still more inexorable than the laws of nature. We now draw attention to the fact that the word "inexorable" is used in a variety of ways. There correspond to our laws of logic very general facts of daily experience. They are the ones that make it possible for us to keep on demonstrating those laws in a very simple way (with ink on paper for example). They are to be compared with the facts that make measurement with a yardstick easy and useful. This suggests the use of precisely these laws of inference, and now it is *we* that are inexorable in applying these laws. Because we '*measure*'; and it is part of measuring for everybody to have the same measures. Besides this, however, inexorable, i.e. *unambiguous* rules of inference can be distinguished from ones that are not unambiguous, I mean from such as leave an alternative open to us.

I. 143. We teach someone a method of sharing out nuts among people; a part of this method is multiplying two numbers in the decimal system.

We teach someone to build a house; and at the same time how he is to obtain a sufficient quantity of material, boards, say; and for this purpose a technique of calculation. The technique of calculation is part of the technique of house-building.

People pile up logs and sell them, the piles are measured with a ruler, the measurements of length, breadth and height multiplied together, and what comes out is the number of pence which have to be asked and given. They do not know 'why' it happens like this; they simply do it like this: that is how it is done.—Do these people not calculate?

I. 148. Those people—we should say—sell timber by cubic measure——but are they right in doing so? Wouldn't it be more correct to sell it by weight—or by the time that it took to fell the timber—or by the labour of felling measured by the age and strength of the woodsman? And why should they not hand it over for a price which is independent of all this: each buyer pays the same however much he takes (they have found it possible to live like that). And is there anything to be said against simply giving the wood away?

I. 149. Very well; but what if they piled the timber in heaps of arbitrary, varying height and then sold it at a price proportionate to the area covered by the piles?

And what if they even justified this with the words: "Of course, if you buy more timber, you must pay more"?

I. 150. How could I shew them that—as I should say—you don't really buy more wood if you buy a pile covering a bigger area?—I should, for instance, take a pile which was small by their ideas and, by laying the logs around, change it into a 'big' one. This *might* convince them—but perhaps they would say: "Yes, now it's a *lot* of wood and costs more"—and that would be the end of the matter.—We should presumably say in this case: they simply do not mean the same by "a lot of wood" and "a little wood" as we do; and they have a quite different system of payment from us.

I. 156. Isn't it like this: so long as one thinks it can't be otherwise, one draws logical conclusions. This presumably means: so long as *such-and-such is not brought in question at all.*

The steps which are not brought in question are logical inferences. But the reason why they are not brought in question is not that they 'certainly correspond to the truth'—or something of the sort,—no, it is just this that is called 'thinking', 'speaking', 'inferring', 'arguing'. There is not any question at all here of some correspondence between what is said and reality; rather is logic *antecedent* to any such correspondence; in the same sense, that is, as that in which the establishment of a method of measurement is *antecedent* to the correctness or incorrectness of a statement of length.

I. 168. The mathematician is an inventor, not a discoverer.

from PART III

II. 16. It is not logic—I should like to say—that compels me to accept a proposition of the form (∃) (∃) ⊃ (∃), when there are a million variables in the first two pairs of brackets and two million in the third. I want to say: logic would not compel me to accept any proposition at all in this case. Something *else* compels me to accept such a proposition as in accord with logic.

Logic compels me only so far as the logical calculus compels me.

But surely it is essential to the calculus with 1000000 that this number must be capable of resolution into a sum 1 + 1 + 1 . . . , and in order to be certain that we have the right number of units before us, we can number the units: $\underset{1}{1} + \underset{2}{1} + \underset{3}{1} + \underset{4}{1} + \ldots + \underset{1000000.}{1}$ This notation would be like: '100,000.000,000' which also makes the numeral surveyable. And I can surely imagine someone's having a great sum of money in pennies entered in a book in which perhaps they appear as numbers of 100 places, with which I have to calculate. I should now begin to translate them into a surveyable notation, but still I should call them 'numerals', should treat them as a record of numbers. For I should even regard it as the record of a number if someone were to tell me that *N* has as many shillings as this vessel will hold peas. Another case again: "He has as many shillings as the Song of Songs has letters".

II. 25. The proof convinces us of something——though what interests us is, not the mental state of conviction, but the applications attaching to this conviction.

For this reason the assertion that the proof convinces us of the truth of this proposition leaves us cold,—since this expression is capable of the most various constructions.

When I say: "the proof convinces me of something", still the proposition expressing this conviction need not be constructed in the proof. As e.g. we multiply, but do not necessarily write down the result in the form of the proposition '. . . × . . . = ...' So we shall presumably say: the multiplication gives us this conviction without our ever uttering the *sentence* expressing it.

A psychological disadvantage of proofs that construct *propositions* is that they easily make us forget that the *sense* of the result is not to be read off from this by itself, but from the *proof*. In this respect the intrusion of the Russellian symbolism into the proofs has done a great deal of harm.

The Russellian signs veil the important forms of proof as it were to the point of unrecognizability, as when a human form is wrapped up in a lot of cloth.

III. 26. Let us remember that in mathematics we are convinced of *grammatical* propositions; so the expression, the result, of our being convinced is that we *accept a rule*.

Nothing is more likely than that the verbal expression of the result of a mathematical proof is calculated to delude us with a myth.

III. 27. I am trying to say something like this: even if the proved mathematical proposition seems to point to a reality outside itself, still it is only the expression of acceptance of a new measure (of reality).

Thus we take the constructability (provability) of this symbol (that is, of the mathematical proposition) as a sign that we are to transform symbols in such and such a way.

We have won through to a piece of knowledge in the proof? And the final proposition expresses this knowledge? Is this knowledge now independent of the proof (is the navel string cut)?—Well, the proposition is now used by itself and without having the proof attached to it.

Why should I not say: in the proof I have won through to a *decision*?

The proof places this decision in a system of decisions.

(I might of course also say: "the proof convinces me that this rule serves my purpose". But to say this might easily be misleading.)

III. 39. *What* is unshakably certain about what is proved?

To accept a proposition as unshakably certain—I want to say—means to use it as a grammatical rule: this removes uncertainty from it.

"Proof must be capable of being taken in" really means nothing but: a proof is not an experiment. We do not accept the result of a proof because it results once, or because it often results. But we see in the proof the reason for saying that this *must* be the result.

What *proves* is not that this correlation leads to this result—but that we are persuaded to take these appearances (pictures) as models for what it is like if. . . . .

The proof is our new model for what it is like if nothing gets added and nothing taken away when we count correctly etc.. But these words shew that I do not quite know what the proof is a model of.

I want to say: with the logic of *Principia Mathematica* it would be possible to justify an arithmetic in which $1000 + 1 = 1000$; and all that would be necessary for this purpose would be to doubt the sensible correctness of calculations. But if we do not doubt it, then it is not our conviction of the truth of logic that is responsible.

When we say in a proof: "This *must* come out"—then this is not for reasons that we do not *see*.

It is not our getting this result, but its being the end of this route, that makes us accept it.

What convinces us—*that* is the proof: a configuration that does not convince us is not the proof, even when it can be shewn to exemplify the proved proposition.

That means: it must not be necessary to make a physical investigation of the proof-configuration in order to shew us what has been proved.

III, 66. The prophecy does *not* run, that a man will get *this* result when he follows this rule in making a transformation—but that he will get this result, when we *say* that he is following the rule.

What if we said that mathematical propositions were prophecies in *this* sense: they predict what result members of a society who have learnt this technique will get in agreement with other members of the society? '$25 \times 25 = 625$' would thus mean that men, if we judge them to obey the rules of multiplication, will reach the result 625 when they multiply 25×25.—That this is a correct prediction is beyond doubt; and also that calculating is in essence founded on such predictions. That is to say, we should not call something 'calculating' if we could not make such a prophecy with certainty. This really means: calculating is a technique. And what we have said pertains to the essence of a technique.

III. 67. This consensus belongs to the essence of *calculation*, so much is certain. I.e.: this consensus is part of the phenomenon of our calculating.

In a technique of *calculating* prophecies must be possible.
And that makes the technique of calculating similar to the technique of a *game*, like chess.

But what about this consensus—doesn't it mean that *one* human being by himself could not calculate? Well, *one* human being could at any rate not calculate just *once* in his life.

It might be said: all *possible* positions in chess can be conceived as propositions saying that they (themselves) are *possible* positions, or again as prophecies that people will be able to reach these positions by moves which they agree in saying are in accordance with the rules. A position *reached* in this way is then a proved proposition of this kind.

"A calculation is an experiment."——A calculation can be an experiment. The teacher makes the pupil do a calculation in order to see whether he can calculate; that is an experiment.

When the stove is lit in the morning, is that an experiment? But it could be one.
And in the same way moves in chess are *not* proofs either, and chess positions are not propositions. And mathematical propositions are not positions in a game. And in *this* way they are not prophecies either.

III. 82. Earlier I was not certain that, among the kinds of multiplication corresponding to *this* description, there was none yielding a result different from the accepted one. But say my uncertainty is such as only to arise at a certain distance from calculation of the normal kind; and suppose that we said: there it does no harm; for if I calculate in a very abnormal way, then I must just reconsider everything. Wouldn't this be all right?

I want to ask: *must* a proof of consistency (or of non-ambiguity) necessarily give me greater certainty than I have without it? And, if I am really out for adventures, *may* I not go out for ones where this proof no longer offers me any certainty?

My aim is to alter the *attitude* to contradiction and to consistency proofs. (*Not* to shew that this proof shews something unimportant. How *could* that be so?)

If for example I were anxious to produce contradictions, say for aesthetic purposes, then I should now unhesitatingly accept the inductive proof of consistency and say: it is hopeless to try and produce a contradiction in this calculus; the proof shews that it won't work. (Proof in theory of harmony.)———

III. 85. Could I imagine our fearing a possibility of constructing the heptagon, like the construction of a contradiction; and that the proof that the construction of the heptagon is impossible should have a settling effect, like a consistency proof?

How does it come about that we are at all tempted (or at any rate come near it) to divide through by $(3-3)$ in $(3-3)\times 2=(3-3)\times 5$? How does it come about that by the rules this step looks plausible, and that even so it is still unusable?
When one tries to describe this situation it is enormously easy to make a mistake in the description. (So it is very difficult to describe.) The descriptions which immediately suggest themselves are all misleading—that is how our language in this field is arranged.

And there will be constant lapses from description into explanation here.

It was, or appears to be, *roughly* like this: we have a calculus, let us say, with the beads of an abacus; we then replace it by a calculus with written signs; this calculus suggests to us an extension of the method of calculating which the first calculus did not suggest—or perhaps better: the second calculus *obliterates* a distinction which was not to be overlooked in the first one. Now if it was the point of the first calculus that this distinction was made, and it is not made in the second one then the latter thereby lost its usability as an equivalent of the former. And now—it seems—the problem might arise: *where* did we depart from the original calculus, what frontiers in the new one correspond to the natural frontiers of the old?

formed a system of rules of calculation which were modelled on ose of another calculus. I took the latter as a model. But exceeded limits. This was even an advantage; but now the new calculus came unusable in certain parts (at least for the former purposes). I erefore seek to alter it: that is, to replace it by one that is *to some tent* different. And by one that has the advantages without the disvantages of the new one. But is that a clearly *defined* task?

Is there such a thing—it might also be asked—as *the right* logical lculus, only without the contradictions?

Could it be said, e.g., that while Russell's Theory of Types avoids he contradiction, still Russell's calculus is not THE universal logical lculus but perhaps an artificially restricted, mutilated one? Could it be said that the *pure*, *universal* logical calculus has yet to be found?

I was playing a game and in doing so I followed certain rules: but as for *how* I followed them, that depended on circumstances and the way it so depended was not laid down in black and white. (This is to some extent a misleading account.) Now I wanted to play this game in such a way as to follow rules 'mechanically' and I 'formalized' the game. But in doing this I reached positions where the game lost *all* point; I therefore wanted to avoid these positions 'mechanically'.—The formalization of logic did not work out satisfactorily. But what was the attempt made for at all? (What was it useful for?) Did not this need, and the idea that it must be capable of satisfaction, arise from a lack of clarity in another place?

The question "what was it useful for?" was a quite *essential* question. For the calculus was not invented for some practical purpose, but in order 'to give arithmetic a foundation'. But who says that arithmetic is logic, or what has to be done with logic to make it in some sense into a substructure for arithmetic? If we had e.g. been led to attempt this by aesthetic considerations, who says that it can succeed? (Wh says that this English poem can be translated into German to our satisfaction?!)

(Even *if* it is clear that there is in *some* sense a translation of any English sentence into German.)

Philosophical dissatisfaction disappears by our seeing *more*.

By my allowing the cancelling of (3 — 3) this type of calculation loses its point. But suppose that, for example, I were to introduce a new sign of equality which was supposed to express: 'equal after *this* operation'? Would it, however, make sense to say: "Won in *this* sense", if in this sense I should win *every* game?

At certain places the calculus led me to its own abrogation. Now I want a calculus that does not do this and that excludes these places.—Does this mean, however, that any calculus in which such an exclusion does not occur is an uncertain one? "Well, the discovery of these places was a warning to us."—But did you not *misunderstand* this 'warning'?

III. 87. Where it is enough for me to get a proof that a contradiction or trisection of the angle cannot be constructed in *this* way, the recursive proof achieves what is required of it. But if I had to fear that something somehow might at some time be interpreted as the construction of a contradiction, then no proof can take this indefinite fear from me.

The fence that I put round contradiction is not a super-fence.

How can a proof have put the calculus right in principle?

How can it have failed to be a proper calculus until this proof was found?

"This calculus is purely mechanical; a machine could carry it out." What sort of machine? One constructed of the usual materials—or a super-machine? Are you not confusing the hardness of a rule with the hardness of a material?

We shall see contradiction in a quite different light if we look at its occurrence and its consequences as it were anthropologically—and when we look at it with a mathematician's exasperation. That is to say, we shall look at it differently, if we try merely to *describe* how the contradiction influences language-games, and if we look at it from the point of view of the mathematical law-giver.

from PART IV

IV. 56. Contradiction. Why just this *one* bogy? That is surely very suspicious.

Why should not a calculation made for a practical purpose, with a contradictory result, tell me: "Do as you please, I, the calculation, do not decide the matter"?

The contradiction might be conceived as a hint from the gods that I am to act and *not* consider.

IV. 57. "Why should contradiction be disallowed in mathematics?" Well, why is it not allowed in our simple language-games? (There is certainly a connexion here.) Is this then a fundamental law governing all thinkable language-games?

Let us suppose that a contradiction in an order, e.g. produces astonishment and indecision—and now we say: that is just the purpose of contradiction in this language-game.

from PART V

V. 9. We only see how queer the question is whether the pattern ϕ (a particular arrangement of digits e.g. '770') will occur in the infinite expansion of π, when we try to formulate the question in a quite common or garden way: men have been trained to put down signs according to certain rules. Now they proceed according to this training and we say that it is a problem whether they will *ever* write down the pattern ϕ in following the given rule.

But what are you saying if you say that one thing is clear: either one will come on ϕ in the infinite expansion, or one will not?

It seems to me that in saying this you are yourself setting up a rule or postulate.

What if someone were to reply to a question: 'So far there is no such thing as an answer to this question'?

So, e.g., the poet might reply when asked whether the hero of his poem has a sister or not—when, that is, he has not yet decided anything about it.

The question—I want to say—changes its status, when it becomes decidable. For a connexion is made then, which formerly *was not there.*

Of someone who is trained we can ask 'How *will* he interpret the rule for this case?', or again 'How *ought* he to interpret the rule for this case?'—but what if no decision about this question has been made?—Well, then the answer is, not: 'he ought to interpret it in such a way that ϕ occurs in the expansion' or: 'he ought to interpret it in such a way that it does not occur', but: 'nothing has so far been decided about this'.

However queer it sounds, the further expansion of an irrational number is a further expansion of mathematics.

We do mathematics with concepts.—And with certain concepts more than with other ones.

I want to say: it *looks* as if a ground for the decision were already there; and it has yet to be invented.

Would this come to the same thing as saying: in thinking about the technique of expansion, which we have learnt, we use the false picture of a completed expansion (of what is ordinarily called a "row") and this forces us to ask unanswerable questions?

For after all in the end every question about the expansion of $\sqrt{2}$ must be capable of formulation as a practical question concerning the technique of expansion.

And what is in question here is of course not merely the case of the expansion of a real number, or in general the production of mathematical signs, but every analogous process, whether it is a game, a dance, etc., etc..

V. 10. When someone hammers away at us with the law of excluded middle as something which cannot be gainsaid, it is clear that there is something wrong with his question.

When someone sets up the law of excluded middle, he is as it were putting two pictures before us to choose from, and saying that one must correspond to the fact. But what if it is questionable whether the pictures can be applied here?

And if you say that the infinite expansion must contain the pattern ϕ or not contain it, you are so to speak shewing us the picture of an unsurveyable series reaching into the distance.

But what if the picture began to flicker in the far distance?

V. 12. In the law of excluded middle we think that we have already got something solid, something that at any rate cannot be called in doubt. Whereas in truth this tautology has just as shaky a sense (if I may put it like that), as the question whether p or $\sim p$ is the case.

Suppose I were to ask: what is meant by saying "the pattern . . . occurs in this expansion"? The reply would be: "you surely *know* what it means. It occurs as the pattern . . . in fact occurs in the expansion." —So *that* is the way it occurs?—But *what way* is that?

Imagine it were said: "Either it occurs in that way, or it does not occur in that way"!

"But don't you really understand what is meant?"—But may I not believe I understand it, and be wrong?—

For how do I know what it means to say: the pattern . . . occurs in the expansion? Surely by way of examples—which shew me what it is like for. . . . But these examples do not shew me what it is like for this pattern to occur in the expansion!

Might one not say: if I really had a right to say that these examples tell me what it is like for the pattern to occur in the expansion, then they would also have to shew me what the opposite means.

14. Suppose children are taught that the earth is an infinite flat surface; or that God created an infinite number of stars; or that a star keeps on moving uniformly in a straight line, without ever stopping.

Queer: when one takes something of this sort as a matter of course, as it were in one's stride, it loses its whole paradoxical aspect. It is as if I were to be told: Don't worry, this series, or movement, goes on without ever stopping. We are as it were excused the labour of thinking of an end.

'We won't bother about an end.'

It might also be said: 'for us the series is infinite'.

'We won't worry about an end to this series; for us it is always beyond our ken.'

16. The comparison with alchemy seems natural. We might speak of a kind of alchemy in mathematics.

Is it already mathematical alchemy, that mathematical propositions are regarded as statements about mathematical objects,—and mathematics as the exploration of these objects?

In a certain sense it is not possible to appeal to the meaning of the signs in mathematics, just because it is only mathematics that gives them their meaning.

What is typical of the phenomenon I am talking about is that a *mysteriousness* about some mathematical concept is not *straight away* interpreted as an erroneous conception, as a mistake of ideas; but rather as something that is at any rate not to be despised, is perhaps even rather to be respected.

All that I can do, is to shew an easy escape from this obscurity and this glitter of the concepts.

Strangely, it can be said that there is so to speak a solid core to all these glistening concept-formations. And I should like to say that that is what makes them into mathematical productions.

It might be said: what you see does of course look more like a gleaming Fata Morgana; but look at it from another quarter and you can see the solid body, which only looks like a gleam without a corporeal substrate when seen from that other direction.

from PART VI

VI. 7. The spectator sees the whole impressive procedure. And he becomes convinced of something; that is the special impression that he gets. He goes away from the performance convinced of something. Convinced that (for example) he will end up the same way with other numbers. He will be ready to express what he is convinced of in such-and-such a way. Convinced of what? Of a psychological fact?—

He will say that he has drawn a conclusion from what he has seen.—*Not*, however as one does from an experiment. (Think of periodic division.)

Could he say: "What I have seen was very impressive. I have drawn a conclusion from it. In future I shall . . ."?

(E.g.: In future I shall always calculate like *this*.)
He tells us: "I saw that it must be like that."

"I realised that it must be like that"—that is his report.

He will now perhaps run through the proof procedure in his mind.

But he does not say: I realised that *this* happens. Rather: that it must be like that. This "must" shews what kind of lesson he has drawn from the scene.
The "must" shews that he has gone in a circle.

I decide to see things like *this*. And so, to act in such-and-such a way.

I imagine that whoever sees the process also draws a moral from it.

'It must be so' means that this outcome has been defined to be essential to this process.

VI. 8. This *must* shews that he has adopted a concept.

This *must* signifies that he has gone in a circle.

He has read off from the process, not a proposition of natural science but, instead of that, the determination of a concept.
Let concept here mean method. In contrast to the application of the method.

VI. 16. . . . And this series is defined by a rule. Or again by the training in proceeding according to the rule. And the inexorable proposition is that according to this rule this number is the successor of this one.[1]

And this proposition is not an empirical one. But why not an empirical one? A rule is surely something that we go by, and we produce one numeral out of another. Is it not matter of experience, that this rule takes someone from here to there?

And if the rule $+1$ carries him one time from 4 to 5, perhaps another time it carries him from 4 to 7. Why is that impossible?

The question arises, what we take as criterion of going according to the rule. Is it for example a feeling of satisfaction that accompanies the act of going according to the rule? Or an intuition (intimation) that tells me I have gone right? Or is it certain practical consequences of proceeding that determine whether I have really followed the rule?—In that case it would be possible that $4 + 1$ sometimes made 5 and sometimes something else. It would be thinkable, that is to say, that an experimental investigation would shew whether $4 + 1$ always makes 5.

If it is not supposed to be an empirical proposition that the rule leads from 4 to 5, then *this*, the result, must be taken as the criterion for one's having gone by the rule.

Thus the truth of the proposition that $4 + 1$ makes 5 is, so to speak, *overdetermined*. Overdetermined by this, that the result of the operation is defined to be the criterion that this operation has been carried out.

The proposition rests on one too many feet to be an empirical proposition. It will be used as a determination of the concept 'applying the operation $+1$ to 4'. For we now have a new way of judging whether someone has followed the rule.

Hence $4 + 1 = 5$ is now itself a rule, by which we judge proceedings. This rule is the result of a proceeding that we assume as *decisive* for the judgment of other proceedings. The rule-grounding proceeding is the proof of the rule.

VI. 21. The application of the concept 'following a rule' presupposes a custom. Hence it would be nonsense to say: just once in the history of the world someone followed a rule (or a signpost; played a game, uttered a sentence, or understood one; and so on).

Here there is nothing more difficult than to avoid pleonasms and only to say what really describes something.

For here there is an overwhelming temptation to say something more, when everything has already been described.

It is of the greatest importance that a dispute hardly ever arises between people about whether the colour of this object is the same as the colour of that, the length of this rod the same as the length of that, etc. This peaceful agreement is the characteristic surrounding of the use of the word "same".

And one must say something analogous about proceeding according to a rule.

No dispute breaks out over the question whether a proceeding was according to the rule or not. It doesn't come to blows, for example.

This belongs to the framework, out of which our language works (for example, gives a description).

VI. 24. "I have a particular concept of the rule. If in this sense one follows it, then from that number one can only arrive at this one". That is a spontaneous decision.

But why do I say "I *must*", if it is my decision? Well, may it not be that I must decide?

Doesn't its being a spontaneous decision merely mean: that's how I act; ask for no reason!

You say you must; but cannot say what compels you.

I have a definite concept of the rule. I know what I have to do in any particular case. I know, that is I am in no doubt: it is obvious to me. I say "Of course". I can give no reason.

When I say "I decide spontaneously", naturally that does not mean: I consider which number would really be the best one here and then plump for . . .

We say: "First the calculations must be done right, and then it will be possible to pass some judgment on the facts of nature."

VI. 30. It might however be asked: if all humans that are educated like this also calculate like *this*, or at least agree to *this* calculation as the right one; then what does one need the *law* for?

"$25^2 = 625$" cannot be the empirical proposition that people calculate like that, because $25^2 \neq 626$ would in that case not be the proposition that people get not this but another result; and also it could be true if people did not calculate at all.

The agreement of people in calculation is not an agreement in opinions or convictions.

Could it be said: "In calculating, the rules strike you as inexorable; you feel that you can only do that and nothing else if you want to follow the rule"?

"As I see the rule, *this* is what it requires." It does not depend on whether I am disposed this way or that.

I feel that I have given the rule an interpretation before I have followed it; and that this interpretation is enough to *determine* what I have to do in order to follow it in the particular case.

If I take the rule as I have taken it, then only doing *this* will correspond to it.

"Have you understood the rule?"—Yes, I have understood—"Then apply it now to the numbers . . . . ." If I want to follow the rule, have I now any choice left?

Assuming that he orders me to follow the rule and that I am frightened not to obey him: am I now not compelled?

But that is surely so too if he orders me: "Bring me this stone." Am I compelled less by *these* words?

VI. 38. "I know how I have to go" means: I am in no doubt how I have to go.

"How can one follow a rule?" That is what I should like to ask.

But how does it come about that I want to ask that, when after all I find no kind of difficulty in following a rule?

Here we obviously misunderstand the facts that lie before our eyes.

How can the word "Slab" indicate what I have to do, when after all I can bring any action into accord with any interpretation?

How can I follow a rule, when after all whatever I do can be interpreted as following it?

What must I know, in order to be able to obey the order? Is there some *knowledge*, which makes the rule followable only in *this* way?

Sometimes I must *know* something, *sometimes* I must *interpret* the rule before I apply it.

Now, *how* was it possible for the rule to have been given an interpretation during instruction, an interpretation which reaches as far as to any arbitrary step?

And if this step was not named in the explanation, how then *can* we agree about what has to happen at this step, since after all whatever happens can be brought into accord with the rule and the examples?

Thus, you say, nothing definite has been said about these steps.

Interpretation comes to an end.

VI. 39. It is true that *anything* can be somehow justified. But the phenomenon of language is based on regularity, on agreement in action.

Here it is of the greatest importance that all or the enormous majority of us agree in certain things. I can, e.g., be quite sure that the colour of this object will be called 'green' by far the most of the human beings who see it.

It would be imaginable that humans of different stocks possessed languages that all had the same vocabulary, but the meanings of the words were different. The word that meant green among one tribe, meant same among another, table for a third and so on. We could even imagine that the same sentences were used by the tribes, only with entirely different senses.

Now in this case I should not say that they spoke the same language.

We say that, in order to communicate, people must agree with one another about the meanings of words. But the criterion for this agreement is not just agreement with reference to definitions, e.g., ostensive definitions—but *also* an agreement in judgments. It is essential for communication that we agree in a large number of judgments.

VI. 41. The word "agreement" and the word "rule" are *related*, they are cousins. The phenomena of agreement and of acting according to a rule hang together.

There might be a cave-man who produced *regular* sequences of marks for himself. He amused himself, e.g., by drawing on the wall of the cave:

— · — — · — — · — — ·

or

— · — · · — · · · — · · · · —

But he is not following the general expression of a rule. And when we say that he acts in a regular way that is not because we can form such and expression.

But suppose he now developed π! (I mean without a general expression of the rule.)

Only in the practice of a language can a word have meaning.

Certainly I can give myself a rule and then follow it. But is it not a rule only for this reason, that it is analogous to what is called 'rule' in human dealings?

When a thrush always repeats the same phrase several times in its song, do we say that perhaps it gives itself a rule each time, and then follows the rule?

VI. 46. When I say: "If you follow the rule, this *must* come out," that doesn't mean: it must, because it always has. Rather, that it comes out is one of my *foundations*.

What *must* come out is a foundation of judgment, which I do not touch.

On what occasion will it be said: "If you follow the rule this *must* come out"?

This may be a mathematical definition given in the train of a proof that a particular route branches. It may also be that one says it to someone in order to impress the nature of a rule upon him, in order to tell him something like: "You are *not* making an experiment here".

VI. 47. "But at every step I know absolutely what I have to do; what the rule demands of me." The rule, as I conceive it. I don't reason. The picture of the rule makes it clear how the picture of the series is to be continued.

"But I know at every step what I have to do. I see it quite clear before me. It may be boring, but there is no doubt what I have to do."

Whence this certainty? But why do I ask that question? Is it not enough that this certainty exists? What for should I look for a source of it? (And I can indeed give *causes* of it.)

When someone, whom we fear to disobey, orders us to follow the rule . . . which we understand, we shall write down number after number without any hesitation. And that is a typical kind of reaction to a rule.

"You already know how it is"; "You already know how it goes on."

I can now determine to follow the rule (–•–) →.

Like this: – • – – • – – • – – • –

But it is remarkable that I don't lose the meaning of the rule as I do it. For how do I hold it fast?

But—how do I know that I do hold it fast, that I do not lose it?! It makes no sense at all to say I have held it fast unless there is such a thing as an outward mark of this. (If I were falling through space I might hold something, but not hold it still.)

Language just is a phenomenon of human life.

VI. 48. One person makes a bidding gesture, as if he meant to say "Go!" The other slinks off with a frightened expression. Might I not call this procedure "order and obedience", even if it happened only once?

What is this supposed to mean: "Might I not call the proceeding ——"? Against any such naming the objection could naturally be made, that among human beings other than ourselves a quite different gesture corresponds to "Go away!" and that perhaps our gesture for this order has among them the significance of our extending the hand in token of friendship. And whatever interpretation one has to give to a gesture depends on other actions, which precede and follow the gesture.

As we employ the word "order" and "obey", gestures no less than words are intertwined in a net of multifarious relationships. If I am now construing a simplified case, it is not clear whether I ought still to call the phenomenon "ordering" and "obeying".

We come to an alien tribe whose language we do not understand. Under what circumstances shall we say that they have a chief? What will occasion us to say that this man is the chief even if he is more poorly clad than others? The one whom the others obey—is he without question the chief?

What is the difference between inferring wrong and not inferring? between adding wrong and not adding? Consider this.

VI. 49. What you say seems to amount to this, that logic belongs to the natural history of man. And that is not combinable with the hardness of the logical "must".

But the logical "must" is a component part of the propositions of logic, and these are not propositions of human natural history. If what a proposition of logic said was: Human beings agree with one another in such and such ways (and that would be the form of the natural-historical proposition), then its contradictory would say that there is here a *lack* of agreement. Not, that there is an agreement of another kind.

The agreement of humans that is a presupposition of logic is not an agreement in *opinions*, much less in opinions on questions of logic.

from PART VII

VII. 11. Suppose that people calculated with numbers, and sometimes did divisions by expressions of the form $(n - n)$, and in this way occasionally got results different from the normal results of multiplying etc. But that nobody minded this.—Compare with this: lists, rolls, of people are prepared, but not alphabetically as we do it; and in this way it happens that in some lists the same name appears more than once.——But now it can be supposed that this does not strike anyone; or that people see it, but accept it without worrying. As we could imagine people of a tribe who, when they dropped coins on the ground, did not think it worth while to pick them up. (They have, say, an idiom for these occasions: "It belongs to the others" or the like.)

But now times have changed and people (at first only a few) begin to demand exactness. Rightly, wrongly?—Were the earlier lists *not* really lists?—

Say we quite often arrived at the results of our calculations through a hidden contradiction. Does that make them illegitimate?——But suppose that we now absolutely refuse to accept such results, but still are afraid that some might slip through.—Well then, in that case we have an idea which might serve as a model for a new calculus. As one can have the idea of a new game.

The Russellian contradiction is disquieting, not because it is a contradiction, but because the whole growth culminating in it is a cancerous growth, seeming to have grown out of the normal body aimlessly and senselessly.

Now can we say: "We want a calculus which more certainly tells us the truth"?

But you can't allow a contradiction to stand!—Why not? We do sometimes use this form in our talk, of course not often—but one could imagine a technique of language in which it was a regular instrument.

It might for example be said of an object in motion that it existed and did not exist in this place; change might be expressed by means of contradiction.

Take a theme like that of Haydn's (St. Antony Chorale), take the part of one of Brahms's variations corresponding to the first part of the theme, and set the task of constructing the second part of the variation in the style of its first part. That is a problem of the same kind as mathematical problems are. If the solution is found, say as Brahms gives it, then one has no doubt;—that is the solution.

We are agreed on this route. And yet, it is obvious here that there may easily be different routes, on each of which we can be in agreement, each of which we might call consistent.

'We take a number of steps, all legitimate—i.e. allowed by the rules—and suddenly a contradiction results. So the list of rules, as it is, is of no use, for the contradiction wrecks the whole game!' Why do you have it wreck the game?

But what I want is that one should be able to go on inferring *mechanically* according to the rule without reaching any contradictory results. Now, what kind of provision do you want? One that your present calculus does not allow? Well, that does not make that calculus a bad piece of mathematics,—or not mathematics in the fullest sense. The meaning of the word "mechanical" misleads you.

VII. 15. If the calculation lost its point for me as soon as I knew I could work out any arbitrary result—did it have none so long as I did *not* know that?

I may of course now declare all these calculations to be null—for I have given up doing them now——but does that mean that they weren't calculations?

I at one time inferred *via* a contradiction without realizing it. Is my result then wrong, or at any rate wrongly got?

If the contradiction is so well hidden that no one notices it, why shouldn't we call what we do now proper calculation?

We say that the contradiction would *destroy* the calculus. But suppose it only occurred in tiny doses in lightning flashes as it were, not as a constant instrument of calculation, would it nullify the calculus?

Imagine people had fancied that $(a + b)^2$ must be equal to $a^2 + b^2$. (Is this a fancy of the same kind as that there must be a trisection of the angle by ruler and compass?) Is it possible, then, to fancy that two ways of calculating had to yield the same result, if it is not the same?

I add up a column, doing it in a variety of ways (e.g. I take the numbers in a different order), and I keep on getting random different results.—I shall perhaps say: "I am in a complete muddle, either I am making random mistakes in calculating, or I am making certain mistakes in particular connexions: e.g. always saying '$7 + 7 = 15$' after '$6 + 3 = 9$'."

Or I might imagine that suddenly, once in the sum, I subtract instead of adding, but don't think I am doing anything different.

Now it might be that I didn't find the mistake and thought I had lost my wits. But this would not have to be my reaction.

'Contradiction destroys the calculus'—what gives it this special position? With a little imagination, I believe, it can certainly be shaken.

To resolve these philosophical problems one has to compare things which it has never seriously occurred to anyone to compare.

In this field one can ask all sorts of things which, while they belong to the topic, still do not lead through its centre.

A particular series of questions leads through the centre and out into the open. The rest get answered incidentally.

It is enormously difficult to find the path through the centre.

It goes *via new* examples and comparisons. The hackneyed ones don't shew us it.

Let us suppose that the Russellian contradiction had never been found. Now—is it quite clear that in that case we should have possessed a false calculus? For aren't there various possibilities here?

And suppose the contradiction had been discovered but we were not excited about it, and had settled e.g. that no conclusions were to be drawn from it. (As no one does draw conclusions from the 'Liar'.) Would this have been an obvious mistake?

"But in that case it isn't a proper calculus! It loses all *strictness*!" Well, not *all*. And it is only lacking in full strictness, if one has a particular ideal of rigour, wants a particular style in mathematics.

'But a contradiction in mathematics is incompatible with its application.

'If it is consistently applied, i.e. applied to produce arbitrary results, it makes the application of mathematics into a farce, or some kind of superfluous ceremony. Its effect is e.g. that of non-rigid rulers which permit various results of measuring by being expanded and contracted.' But was measuring by pacing not measuring at all? And if people worked with rulers made of dough, would that of itself have to be called wrong?

Couldn't reasons be easily imagined, on account of which a certain elasticity in rulers might be desirable?

"But isn't it right to manufacture rulers out of ever harder, more unalterable material?" Certainly it is right; if that is what one wants!

'Then are you in favour of contradiction?' Not at all; any more than of soft rulers.

There is *one* mistake to avoid: one thinks that a contradiction *must* be senseless: that is to say, if e.g. we use the signs 'p', '$\sim$', '.' *consistently*, then '$p . \sim p$' cannot say anything.—But think: what does it mean to continue such and such a use 'consistently'? ('A consistent continuation of this bit of a curve.')

J. 29. I am defining a game and I say: "If you move like this, then I move like *this*, and if you do that, then I do *this*.—Now play." And now he makes a move, or something that I have to accept as a move and when I want to reply according to my rules, whatever I do proves to conflict with the rules. How can this have come about? When I set the rules up, I *said* something: I was following a certain use. I did not foresee what we should go on to do, or I saw only a particular possibility. It was just as if I had said to somebody: "Give up the game; you can't mate with these pieces" and had overlooked an existing possibility of mating.

The various half joking guises of logical paradox are only of interest in so far as they remind anyone of the fact that a serious form of the paradox is indispensable if we are to understand its function properly. The question is: what part can such a logical mistake play in a language-game?

You may instruct someone what to do in such-and-such a case; and these instructions later prove *nonsensical*.

J. 34. —There is a contradiction here. But we don't see it and we draw conclusions from it. E.g. we infer mathematical propositions; and wrong ones. But we accept these inferences.—And now if a bridge collapses, which we built on the basis of these calculations, we find some other cause for it, or we call it an Act of God. Now was our calculation wrong; or was it not a calculation?

Certainly, if we are explorers observing the people who do this we shall perhaps say: these people don't calculate at all. Or: there is an element of arbitrariness in their calculations, which distinguishes the nature of their mathematics from ours. And yet we should not be able to deny that these people have a mathematics.

What kind of rules must the king[2] give so as to escape henceforward from the awkward position, which his prisoner has put him in?—What sort of problem is this?—It is surely like the following one: how must I change the rules of this game, so that such-and-such a situation cannot occur? And that is a mathematical problem.

But can it be a mathematical problem to make mathematics into mathematics?

Can one say: "After this mathematical problem was solved, human beings began really to calculate"?

~~[1] I.e. the Axiom of Choice. (Eds.)~~

[2] Presumably the king who made the law that all who came to his city must state their business and be hanged if they lied. A sophist said he came to be hanged under that law.—(Eds.)

VII. 35. What sort of certainty is it that is based on the fact that in general there *won't* actually be a run on the banks by all their customers; though they would break if it did happen?! Well, it is a *different* kind of certainty from the more primitive one, but it is a kind of certainty all the same.

I mean: if a contradiction were now actually found in arithmetic—that would only prove that an arithmetic with *such* a contradiction in it could render very good service; and it will be better for us to modify our concept of the certainty required, than to say that it would really not yet have been a proper arithmetic.

"But surely this isn't ideal certainty!"—Ideal for what purpose?

The rules of logical inference are rules of the *language-game*.

VII. 43. The proof of a proposition shews me what I am prepared to stake on its truth. And different proofs can perfectly well cause me to stake the same thing.

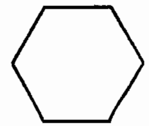
Something surprising, a paradox, is a paradox only in a particular, as it were defective, surrounding. One needs to complete this surrounding in such a way that what looked like a paradox no longer seems one.

If I have proved that 18 × 15 = 270, I have thereby also proved the geometrical proposition that we get the sign '270' by applying certain transformation rules to the sign '18 × 15'.—Now suppose that people, having their vision or memory impaired (as we now put it) by some harmful drug, did not get '270' when they did this calculation.—If we cannot use it to make a correct prediction of the result anyone is going to get under normal circumstances, isn't the calculation useless? Well, even if it is, that does not shew that the proposition '18 × 15 = 270' is the empirical proposition: people in general calculate like *this*.

On the other hand it is not clear that the general agreement of people doing calculations is a characteristic mark of all that is called "calculating". I could imagine that people who had learned to calculate might in particular circumstances, say under the influence of opium, begin to calculate differently from one another, and might make use of these calculations; and that they were not said not to be calculating at all and to be deranged—but that their calculations were accepted as a reasonable procedure.

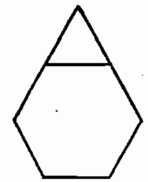
But must they not at least be trained to do the same calculations? Doesn't *this* belong essentially to the concept of calculating? I believe that we could imagine deviations here too.

VII 61. An addition of shapes together, so that some of the edges fuse, plays a very small part in our life.—As when



and

yield the figure



But if this were an *important* operation, our ordinary concept of arithmetical addition would perhaps be different.

It is natural for us to regard it as a geometrical fact, not as a fact of physics, that a square piece of paper can be folded into a boat or hat. But is not geometry, so understood, part of physics? No; we split geometry off from physics. The geometrical possibility from the physical one. But what if we left them together? If we simply said: "If you do this and this and this with the piece of paper then *this* will be the result"? What has to be done might be told in a rhyme. For might it not be that someone did not distinguish at all between the two possibilities? As e.g. a child who learns this technique does not. It does not know and does not consider whether these results of folding are possible only because the paper stretches, is pulled out of shape, when it is folded in such-and-such a way, or because it is *not* pulled out of shape.

And now isn't it like this in arithmetic too? Why shouldn't it be possible for people to learn to calculate without having the concepts of a mathematical and a physical fact? They merely know that this is always the result when they take care and do what they have learnt.

Let us imagine that while we were calculating the figures on paper altered erratically. A 1 would suddenly become a 6 and then a 5 and then again a 1 and so on. And I want to assume that this does not make any difference to the calculation because, as soon as I read a figure in order to calculate with it or to apply it, it once more becomes the one that we have in *our* calculating. At the same time, one would see how the figures change during the calculation; but we are trained not to worry about this.

Of course, even if we do not make the above assumption, this calculation could lead to useful results.

Here we calculate strictly according to rules, yet this result does not *have* to come out.—I am assuming that we see no sort of regularity in the alteration of the figures.

I want to say: this calculating could really be conceived as an experiment, and we might for example say: "Let's try what will come out now if I apply this rule".

Or again: "Let us make the following experiment: we'll write the figures with ink of such-and-such a composition . . . and calculate according to the rule. . . ."

Now you might of course say: "In this case the manipulation of figures according to rules is not calculation."

"We are calculating only when there is a *must* behind the result."—But suppose we don't know this *must*,—is it contained in the calculation all the same? Or are we not calculating, if we do it quite naïvely?

How about the following: You aren't calculating if, when you get now this, now that result, and cannot find a mistake, you accept this and say: this simply shews that certain circumstances which are still unknown have an influence on the result.

This might be expressed: if calculation reveals a causal connexion to you, then you are not calculating.

Our children are not only given practice in calculation but are also trained to adopt a particular attitude towards a mistake in calculating.[1]

What I am saying comes to this, that mathematics is *normative*. But "norm" does not mean the same thing as "ideal".

VII - 66. Why do I always speak of being compelled by a rule; why not of the fact that I can *choose* to follow it? For that is equally important.

But I don't want to say, either, that the rule compels me to act like this; but that it makes it possible for me to hold by it and let it compel me.

And if e.g. you play a game, you keep to its rules. And it is an interesting fact that people set up rules for the fun of it, and then keep to them.

My question really was: "How can one keep to a rule[1]?" And the picture that might occur to someone here is that of a short bit of hand-rail, by means of which I am to let myself be guided further than the rail reaches. [But there *is* nothing there; but there isn't *nothing* there!] For when I ask "How *can* one . . .", that means that something here looks *paradoxical* to me; and so a picture is confusing me.

"I never thought of its being red too; I only saw it as part of a multi-coloured ornament."

Logical inference is a transition that is justified if it follows a particular paradigm and its rightness is not dependent on anything else.

VII. 67. We say: "If you really follow the rule in multiplying, you *must* all get the same result." Now if this is only the somewhat hysterical way of putting things that you get in university talk, it need not interest us overmuch.

It is however the expression of an attitude towards the technique of calculation, which comes out everywhere in our life. The emphasis of the *must* corresponds only to the inexorableness of this attitude both to the technique of calculating and to a host of related techniques.

The mathematical Must is only another expression of the fact that mathematics forms concepts.

And concepts help us to comprehend things. They correspond to a particular way of dealing with situations.

Mathematics forms a network of norms.

VII. 74. Any proof in applied mathematics may be conceived as a proof in pure mathematics which proves that *this* proposition follows from *these* propositions, or can be got from them by means of such and such operations; etc.

The proof is a particular *path*. When we describe it, we do not mention causes.

I act on the proof.—But how?—I act according to the proposition that got proved.

The proof taught me e.g., a technique of approximation. But still it proved *something*, convinced me of something. *That* is expressed by the proposition: It says what I shall now do on the strength of the proof.

The proof belongs to the background of the proposition. To the system in which the proposition has an effect.

See, *this* is how 3 and 2 yield 5. Note this proceeding.

Every empirical proposition may serve as a rule if it is fixed, like a machine part, made immovable, so that now the whole representation turns around it and it becomes part of the coordinate system, independent of facts.

"This is how it is, if this proposition is derived from these ones. That you have to admit."—What I admit is, *this* is what I call such a procedure.