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Structure

1 Opening

Every theory, philosophical or otherwise, must take some notions for granted. The philosopher inherits a fully developed language, a U-language in Curry's sense (see the previous chapter). Nevertheless, in both historical and contemporary philosophy, the most basic concepts of discourse are open to articulation and analysis. Every notion or principle in the inherited U-language is up for scrutiny and perhaps revision, at least in principle—even if one cannot revise every notion, all at once. In Neurath's metaphor, the philosopher is at work rebuilding parts of the floating ship of concepts. The notions of existence, object, and identity occur in just about every philosophical work, usually without further ado. Indeed, it is hard to imagine writing philosophy without invoking and presupposing these notions. Should we conclude that everyone already has clear and distinct ideas of them? Is any attempt to articulate such notions a waste of time and effort?

Presumably, one cannot go about articulating basic notions without presupposing and even using them. We have to start somewhere. This part of the book exhibits what is sometimes called a "dialectical" approach. We begin by using certain notions. As we go, some of these notions get refined and even modified. This tempers some of the very statements we use in getting the procedure off the ground. As the notions get further modified, the statements used to make the modifications themselves get modified. In the end, the original statements should be regarded as first approximations.

Structuralism has interesting consequences for the basic building blocks of ontology. Among other things, structuralists have something to say about what an object is and what identity is, at least in mathematics. Along the way, we speculate about how far the structuralist notion of object carries over to ordinary, nonmathematical contexts (see also chapter 8). The problem, however, is that structuralism cannot be articulated without invoking the notion of object, and so I ask for the reader's dialectical indulgence.

My structuralist program is a realism in ontology and a realism in truth-value—once the requisite notions of object and objectivity are on the table. Structuralists hold that a nonalgebraic field like arithmetic is about a realm of objects—numbers—that exist independently of the mathematician, and they hold that arithmetic assertions have nonvacuous, bivalent, objective truth-values in reference to this domain.

Structuralism is usually contrasted with traditional Platonism. Ultimately, the differences may not amount to much when it comes to ontology, but the contrast is a good place to begin our first approximation. Like any realist in ontology, the Platonist holds that the subject matter of a given nonalgebraic branch of mathematics is a collection of objects that have some sort of ontological independence. The natural numbers, for example, exist independently of the mathematician. As I noted in the previous chapter, Resnik [1980, 162] defines an "ontological platonist" to be someone who holds that ordinary physical objects and numbers are "on a par." Numbers are the same kind of thing—objects—as beach balls, only there are more numbers than beach balls and numbers are abstract and eternal.

To pursue this analogy, one might attribute some sort of ontological independence to the individual natural numbers. Just as each beach ball is independent of every other beach ball, each natural number is independent of every other natural number. Just as a given red beach ball is independent of a blue one, the number 2 is independent of the number 6. An attempt to articulate this idea will prove instructive. When we say that the red beach ball is independent of the blue one, we might mean that the red one could have existed without the blue one and vice versa. However, nothing of this sort applies to the natural numbers, as conceived by traditional Platonism. According to the Platonist, numbers exist necessarily. So we cannot say that 2 could exist without 6, because 6 exists of necessity. Nothing exists without 6. To be sure, there is an epistemic independence among the numbers in the sense that a child can learn much about the number 2 while knowing next to nothing about 6 (but having it the other way around does stretch the imagination). This independence is of little interest here, however.

The Platonist view may be that one can state the *essence* of each number without referring to the other numbers. The essence of 2 does not invoke 6 or any other number (except perhaps 0 and 1). If this notion of independence could be made out, we structuralists would reject it. The essence of a natural number is its *relations* to other natural numbers. The subject matter of arithmetic is a single abstract structure, the pattern common to any infinite collection of objects that has a successor relation with a unique initial object and satisfies the (second-order) induction principle. The number 2, for example, is no more and no less than the second position in the natural-number structure; 6 is the sixth position. Neither of them has any independence from the structure in which they are positions, and as places in this structure, neither number is independent of the other. The essence of 2 is to be the successor of the successor of 0, the predecessor of 3, the first prime, and so on.

Plato himself distinguishes two studies involving natural numbers. *Arithmetic* "deals with the even and the odd, with reference to how much each happens to be" (*Gorgias* 451A–C). If "one becomes perfect in the arithmetical art," then "he knows also all of the numbers" (*Theatetus* 198A–B; see also *Republic VII* 522C). The study called *logistic* deals also with the natural numbers but differs from arithmetic "in so far as it studies the even and the odd with respect to the multitude they make both with themselves and with each other" (*Gorgias 451*A–C; see also *Charmides* 165E–166B). So arithmetic deals with the natural numbers, and logistic concerns the relations among the numbers. In ancient works, logistic is usually understood as the theory of *calculation*. Most writers take it to be a practical discipline, concerning measurement, business dealings, and so forth (e.g., Proclus [485, 39]; see Heath [1921, chapter 1]). For Plato, however, logistic is every bit as theoretical as arithmetic. As Jacob Klein [1968, 23] puts it, theoretical logistic "raises to an explicit science that knowledge of relations among numbers which . . . precedes, and indeed must precede, all calculation."

The structuralist rejects this distinction between Plato's arithmetic and theoretical logistic. There is no more to the individual numbers "in themselves" than the relations they bear to each other. Klein [1968, 20] wonders what is to be studied in arithmetic, as opposed to logistic. Presumably, the art of counting—reciting the numerals—is arithmetic par excellence. Yet "addition and also subtraction are only an extension of counting." Moreover, "counting itself already presupposes a continual relating and distinguishing of the numbered things as well as of the numbers." In the *Republic* (525C–D), Plato said that guardians should pursue *logistic* for the sake of knowing. It is through this study of the *relations* among numbers that their soul is able to grasp the nature of numbers as they are in themselves. We structuralists agree.[1]

The natural-number structure is exemplified by the strings on a finite alphabet in lexical order, an infinite sequence of strokes, an infinite sequence of distinct moments of time, and so on. Similarly, group theory studies not a single structure but a type of structure, the pattern common to collections of objects with a binary operation, an identity element thereon, and inverses for each element. Euclidean geometry studies Euclidean-space structure; topology studies topological structures, and so forth.[2]

One lesson we have learned from Plato is that one cannot delineate a philosophical notion just by giving a list of examples. Nevertheless, the examples point in a certain direction. To continue the dialectic, I define a *system* to be a collection of objects with certain relations. An extended family is a system of people with blood and marital

1. Klein [1968, 24] tentatively concludes that logistic concerns ratios among pure units, whereas arithmetic concerns counting, addition, and subtraction. In line with the later dialogues, it might be better to think of logistic as what we would call "arithmetic," with Plato's "arithmetic" being a part of higher philosophy. I am indebted to Peter King for useful conversations on this historical material.

2. Sometimes mathematicians use phrases like "the group structure" and "the ring structure" when speaking loosely about groups and rings. Taken literally, these locutions presuppose that there is a single structure common to all groups and a single structure common to all rings. Here, I prefer to use "structure" to indicate the subject of nonalgebraic theories, those that mathematicians call "concrete." To say that two systems have the same structure is to say that they share something like an isomorphism type. This is what allows us to speak of numbers as individual objects. So, in the present sense, group theory is not about a single structure, but rather a class of similar structures. My fellow structuralist, Michael Resnik (e.g. [1996]) denies the importance of this distinction. For Resnik, it seems, all mathematical theories are algebraic, none are "concrete" (or to be precise, there is no "fact of the matter" whether a given structure is "concrete").

relationships, a chess configuration is a system of pieces under spatial and "possible-move" relationships, a symphony is a system of tones under temporal and harmonic relationships, and a baseball defense is a collection of people with on-field spatial and "defensive-role" relations. A *structure* is the abstract form of a system, highlighting the interrelationships among the objects, and ignoring any features of them that do not affect how they relate to other objects in the system.

Although epistemology is treated in the next chapter, it will help here to mention a few ways that structures are grasped. One way to apprehend a particular structure is through a process of pattern recognition, or abstraction. One observes a system, or several systems with the same structure, and focuses attention on the relations among the objects—ignoring those features of the objects that are not relevant to these relations. For example, one can understand a baseball defense by going to a game (or several games) and noticing the spatial relations among the players who wear gloves, ignoring things like height, hair color, and batting average, because these have nothing to do with the defense system. It is similar to how one comes to grasp the type of a letter, such as an "E," by observing several tokens of the letter and focusing on the typographical pattern, while ignoring the color of the tokens, their height, and the like.

I do not offer much to illuminate the psychological mechanisms involved in pattern recognition. They are interesting and difficult problems in psychology and in the young discipline of cognitive science. Nevertheless, it is reasonably clear that humans do have an ability to recognize patterns.[3] Sometimes, ostension is at work. One points to the system and somehow indicates that it is the pattern being ostended, and not the particular people or objects. That is, one points to a system that exemplifies the structure in order to ostend the structure itself. Similarly, one can point to a capital "E," not to ostend that particular token but to ostend the type, the abstract pattern. Ordinary discourse clearly has the resources to distinguish between pattern and patterned, the psychological problems with pattern recognition and the philosophical problems with *abstracta* notwithstanding.

A second way to understand a structure is through a direct description of it. Thus, one might say that a baseball defense consists of four infielders, arranged thus and so, three outfielders, and so on. One can also describe a structure as a variation of a previously understood structure. A "lefty shift defense" occurs when the shortstop plays to the right of second base and the third baseman moves near the shortstop position. Or a "softball defense" is like a baseball defense, except that there is one more outfielder. In either case, most competent speakers of the language will understand what is meant and can then go on to discuss the structure itself, independent of any particular exemplification of it.[4]

3. Dieterle [1994, chapter 3] contains a brief survey of some of the psychological literature on pattern recognition, relating the process to structuralism. She argues that pattern recognition is the central component of a reliabilist epistemology of abstract patterns. I return to this in chapter 4.

4. An anecdote: Several years ago, I was called on to observe a remedial mathematics class. The students were among the worst prepared in mathematics. While waiting for the class to begin, I overheard a conversation between two of them concerning the merits of a certain basketball defense. The discussion was at a rather high level of abstraction and complexity, at least as great as that of the subject matter of the class that day (the addition of fractions). It seems to me that there is not much difference in kind between abstract discussions of basketball defenses and the addition of fractions (but see section 5).

For our first (or second) approximation, then, pure mathematics is the study of structures, independently of whether they are exemplified in the physical realm, or in any realm for that matter. The mathematician is interested in the internal relations of the places of these structures, and the methodology of mathematics is, for the most part, deductive. As Resnik puts it:

> In mathematics, I claim, we do not have objects with an 'internal' composition arranged in structures, we have only structures. The objects of mathematics, that is, the entities which our mathematical constants and quantifiers denote, are structureless points or positions in structures. As positions in structures, they have no identity or features outside a structure. ([1981, 530])

> Take the case of linguistics. Let us imagine that by using the abstractive process . . . a grammarian arrives at a complex structure which he calls *English*. Now suppose that it later turns out that the English corpus fails in significant ways to instantiate this pattern, so that many of the claims which our linguist made concerning his structure will be falsified. Derisively, linguists rename the structure *Tenglish*. Nonetheless, much of our linguist's knowledge about *Tenglish qua* pattern stands; for he has managed to describe *some* pattern and to discuss some of its properties. Similarly, I claim that we know much about Euclidean space despite its failure to be instantiated physically. ([1982, 101])

Of course, some of the examples mentioned above are too simple to be worthy of the mathematician's attention. What can we prove about an infield structure, or about the type of the letter "E"? There are, however, nontrivial theorems about chess games. For example, it is not possible to force a checkmate with a king and two knights against a lone king. This holds no matter what the pieces are made of, and even whether or not chess has ever been played. This fact about chess is a more or less typical mathematical theorem about a certain structure. Here, it is the structure of a certain game.

Most of the structures studied in mathematics have an infinite, indeed uncountable, number of positions. The set-theoretic hierarchy has a proper class of positions. It is contentious to suggest that we can come to understand structures like this by abstraction, or pattern recognition, from perceptual experience. That would require a person to view (or hear) a system that consists of infinitely many objects. There is thus an interesting epistemological problem for structuralism, which will be dealt with in due course (chapter 4). My present purpose is to point to the notion of structure and to characterize mathematics as the science of structure.

There is a revealing error in Hartry Field's *Science without numbers* [1980]. The purpose of that book is to articulate a view, now called "nominalism," that there are no abstract objects. According to Field, everything is concrete (in the philosophers' sense of that word). Because, presumably, numbers are *abstracta* par excellence, the nominalist rejects the existence of numbers. A central item on Field's agenda is to show how science can proceed, at least in principle, without presupposing the existence of numbers and other abstract objects. He develops one example, Newtonian gravitational theory, in brilliant detail. The ontology of Field's nominalistic theory includes points and regions of space-time, but he argues that points and regions are concrete, not abstract, entities. There is no need to dispute the last claim here (but see Resnik [1985]). Whether abstract or concrete, Field's Newtonian space-time is Euclidean, consisting of continuum-many points and even more regions. Space-time exemplifies most (but not all) of the structure of $\mathbb{R}^4$, the system of quadruples of real numbers. Field himself insightfully exploits the fact that any model of space-time can be extended to a model of $\mathbb{R}^4$ by adding a reference frame and units for the metrics. Each line of space-time is then isomorphic to $\mathbb{R}$, and so addition and multiplication can be defined on a line. So something like addition and multiplication, as well as the calculus of real-valued functions, can be carried out in this nominalistic theory. All of this is supposed to be consistent with nominalistic rejection of *abstracta*.

Field considers the natural objection that "there doesn't seem to be a very significant difference between postulating such a rich physical space and postulating the real numbers." He replies, "[T]he nominalistic objection to using real numbers was not on the grounds of their uncountability or of the structural assumptions (e.g., Cauchy completeness) typically made about them. Rather, the objection was to their abstractness: even postulating *one* real number would have been a violation of nominalism. . . . Conversely, postulating uncountably many *physical* entities . . . is not an objection to nominalism; nor does it become any more objectionable when one postulates that these physical entities obey structural assumptions analogous to the ones that platonists postulate for the real numbers" (p. 31). The structuralist balks at this point. For us, a real number *is* a place in the real-number structure. It makes no sense to "postulate one real number," because each number is part of a large structure. It would be like trying to imagine a shortstop independent of an infield, or a piece that plays the role of the black queen's bishop independent of a chess game. Where would it stand? What would its moves be? One can, of course, ask whether the real-number structure is exemplified by a given system (like a collection of points). Then one could locate objects that have the *roles* of individual numbers, just as on game day one can identify the people who have the roles of shortstop on each team, or in a game of chess one can identify the pieces that are the bishops. But it is nonsense to contemplate *numbers* independent of the structure they are part of.

It is common for mathematicians to claim that mathematics has not really been eliminated from Field's system. Even if the title, *Science without numbers*, is an accurate description of the enterprise, it is not science without mathematics.[5] Some philosophers might be inclined to let the response of the mathematicians settle the matter. After all, mathematicians should be able to recognize their subject when they see it. In response, Field could point out that these mathematicians are simply not interested in questions of ontology, or that they do not understand or care about the distinction between abstract and concrete. This, of course, may be true. But observations about the typical interests of mathematicians miss the point. Field concedes that nominalistic physics makes substantial "structural assumptions" about space-time, and he articulates these assumptions with admirable rigor. Although Field would not put it this way, the "structural assumptions" characterize a structure, an uncountable one. This is a consequence of the fact that (the second-order version of) Field's theory of space-time is categorical—all of its models are isomorphic. Field's nominalistic physicist would study this structure *as such*, at least sometimes. Field himself proves theorems about this structure. As I see it, he thereby engages in mathematics. The activity of proving things about space-time is the same kind of activity as proving theorems about real numbers. Both are the deductive study of a structure, no more and certainly no less.

5. This response was made by the mathematicians who attended an interdisciplinary seminar I once gave on *Science without numbers*. In correspondence, Field himself reported similar observations from colleagues in mathematics departments. Of course, mathematicians are not the only ones to balk at the claim that Field's system does not significantly reduce the mathematical presuppositions of Newtonian gravitational theory. A number of philosophers and prominent logicians have joined the chorus.

Field might reply that he is interested in one particular (concrete) exemplification of the structure, not the structure itself. This is fair, but it misses the point. As far as mathematics goes, it does not matter where, how, or even if the relevant structure is exemplified. The substructure of $\mathbb{R}^4$ is in the purview of mathematics, and both Field and his nominalistic physicist use typical mathematical methods to illuminate this structure, along with the concrete system that exemplifies it. I suggest that these observations underlie the mathematicians' response to Field. They are correct.

2 Ontology: Object

On the ontological front, there are two groups of issues. One is the status of whole structures, such as the natural-number structure, the real-number structure, and the set-theoretic hierarchy, as well as more mundane structures like a symphony, a chess configuration, and a baseball defense. The other issue concerns the status of mathematical objects, the places within structures: natural numbers, real numbers, points, sets, and so on.

We begin with the issue concerning mathematical objects. The existence of structures will be addressed directly later, but because of the interconnections, we will go back and forth between the issues. Once again, a natural number is a place in the natural-number structure, a particular infinite pattern. The pattern may be exemplified by many different systems, but it is the same pattern in each case. The number 2 is the second place in that pattern. Individual numbers are analogous to particular offices within an organization. We distinguish the office of vice president, for example, from the person who happens to hold that office in a particular year, and we distinguish the white king's bishop from the piece of marble that happens to play that role on a given chess board. In a different game, the very same piece of marble might play another role, such as that of white queen's bishop or, conceivably, black king's rook. Similarly, we can distinguish an object that plays the role of 2 in an exemplification of the natural-number structure from the number itself. The number is the office, the place in the structure. The same goes for real numbers, points of

Euclidean geometry, members of the set-theoretic hierarchy, and just about every object of a nonalgebraic field of mathematics. Each mathematical object is a place in a particular structure. There is thus a certain priority in the status of mathematical objects. The structure is prior to the mathematical objects it contains, just as any organization is prior to the offices that constitute it. The natural-number structure is prior to 2, just as "baseball defense" is prior to "shortstop" and "U.S. Government" is prior to "vice president."

Structuralism resolves one problem taken seriously by at least some Platonists—or realists in ontology—and which has been invoked by its opponents as an argument against realism. Frege [1884], who has been called an "arch-Platonist," argued that numbers are objects. This conclusion was based in part on the grammar of number words. Numerals, for example, exhibit the trappings of singular terms. Frege went on to give an insightful and eminently plausible account of the use of number terms in certain contexts, typically forms like "the number of *F* is *y*," where *F* stands for a predicate like "moons of Jupiter" or "cards on this table." But then Frege noted that this preliminary account does not sustain the conclusion that numbers are objects. For this, we need a criterion to decide whether any given number, like 2, is the same or different from any other object, say Julius Caesar. That is, Frege's preliminary account does not have anything to say about the truth-value of the identity "Julius Caesar = 2." This quandary has come to be called the *Caesar problem.* A solution to it should determine how and why each number is the same or different from *any object whatsoever.* The Caesar problem is related to the Quinean dictum that we need criteria to individuate the items in our ontology. If we do not have an identity relation, then we do not have bona fide objects. The slogan is "no entity without identity." Frege attempted to solve this problem with the use of extensions. He proposed that the number 2 is a certain extension, the collection of all pairs. Thus, 2 is not Julius Caesar because, presumably, persons are not extensions. This turned out to be a tragic maneuver, because Frege's account of extensions (in [1903]) is inconsistent. With the wisdom that hindsight brings, Frege should have quit while he was ahead.[6]

Paul Benacerraf's celebrated [1965] and Philip Kitcher [1983, chapter 6] raise a variation of this problem. After the discovery that virtually every field of mathematics can be reduced to (or modeled in) set theory, the foundationally minded came to think of the set-theoretic hierarchy as the ontology for all of mathematics. An economy in regimentation suggests that there should be a single type of object. Why have sets and numbers when sets alone will do? But there are several reductions of arithmetic to set theory, an embarrassment of riches. If numbers are mathematical objects and all mathematical objects are sets, then we need to know *which* sets the natural numbers are. According to one account, due to von Neumann, it is correct to say that 1 is a member of 4. According to Zermelo's account, 1 is not a member of 4. Moreover, there seems to be no principled way to decide between the reductions. Each serves whatever purpose a reduction is supposed to serve. So we are left without an answer to the question, "Is 1 really a member of 4, or not?" What, after all, are the natural numbers? Are they finite von Neumann ordinals, Zermelo numerals, or some other sets altogether? From these observations and questions, Benacerraf and Kitcher conclude, against Frege, that numbers are not objects. This conclusion, I believe, is not warranted. It all depends on what it is to *be an object*, a matter that is presently under discussion.[7]

6. A number of writers have shown that the essence of Frege's account of *arithmetic* is consistent (e.g., Boolos [1987]). The idea is to speak of numbers directly, not mediated by extensions. But then the Caesar problem remains unsolved. One can also consistently identify numbers with extensions, as long as one does not maintain that every open formula determines an extension and that two formulas determine the same extension if and only if they are coextensive (cf. Frege's infamous Principle V). I return to Frege's notion of "object" in chapter 5.

I would think that a good philosophy of mathematics need not answer questions like "Is Julius Caesar = 2?" and "Is $1 \in 4$?" Rather, a philosophy of mathematics should show why these questions need no answers, even if the questions are intelligible. It is not that we just do not care about the answers; we want to see why there is no answer to be discovered—even for a realist in ontology. Again, a number is a place in the natural-number structure. The latter is the pattern common to all of the models of arithmetic, whether they be in the set-theoretic hierarchy or anywhere else. One can form coherent and determinate statements about the identity of two numbers: $1 = 1$ and $1 \neq 4$. And one can look into the identity between numbers denoted by different descriptions *in the language of arithmetic*. For example, 7 is the largest prime that is less than 10. And one can apply arithmetic in the Fregean manner and assert, for example, that the number of cards in a deck is 52. But it makes no sense to pursue the identity between a place in the natural-number structure and some other object, expecting there to be a fact of the matter. Identity between natural numbers is determinate; identity between numbers and other sorts of objects is not, and neither is identity between numbers and the positions of other structures.

Along similar lines, one can ask about *numerical* relations between numbers, relations definable in the language of arithmetic, and one can expect determinate answers to these questions. Thus, $1 < 4$ and 1 evenly divides 4. These are questions internal to the natural-number structure. But if one inquires whether 1 is an element of 4, there is no answer waiting to be discovered. It is similar to asking whether 1 is braver than 4, or funnier.

Similar considerations hold for our more mundane structures. It is determinate that the shortstop position is not the catcher position and that a queen's bishop cannot capture the opposing queen's bishop, but there is something odd about asking whether positions in patterns are identical to other objects. It is nonsense to ask whether *the shortstop* is identical to Ozzie Smith—whether the person is identical to the po-

7. In chapter 2, we encountered an argument of Putnam [1981, 72–74] against metaphysical realism: "[C]ouldn't there be some kind of abstract isomorphism, or . . . mapping of concepts onto things in the (mind-independent) world? . . . The trouble with this suggestion is . . . that *too many* correspondences exist. To pick out just *one* correspondence between words or mental signs and mind-independent things we would have to already have referential access to the mind-independent things." Again, it all depends on what it is to be a "thing," and it depends on what "reference" is.

sition. Ozzie Smith is, of course, *a* shortstop and, arguably, he is (or was) the quint-essential shortstop, but is he the position? There is also something odd about asking whether the shortstop position is taller or faster or a better hitter than the catcher position. Shortness, tallness, and batting average do not apply to positions.

Similar, less philosophical questions are asked on game day, about a particular lineup, but those questions concern the people who occupy the positions of shortstop and catcher that day, not the positions themselves. When a fan asks whether Ozzie Smith is the shortstop or whether the shortstop is a better hitter than the catcher, she is referring to the people in a particular lineup.[8] Virtually any person prepared to play ball *can* be a shortstop—anybody can occupy that role in an infield system (some better than others). Any small, moveable object can play the role of (i.e., can be) black queen's bishop. Similarly, and more generally, anything at all can "be" 2—anything can occupy that place in a system exemplifying the natural-number structure. The Zermelo 2 ({{ϕ}}), the von Neumann 2 ({ϕ, {ϕ}}), and even Julius Caesar can each play that role. The Frege–Benacerraf questions do not have determinate answers, and they do not need them.

One can surely *ask* the Frege–Benacerraf questions. Are Julius Caesar or {ϕ, {ϕ}} places in the natural-number structure? Do the monarch and the ordinal have essential properties relating them to other places in the natural-number structure? If the question is taken seriously, the answer will surely be "of course not." The retort is "How do you know?" or, to paraphrase Frege, "Of course these items are not places in the natural-number structure, but this is no thanks to structuralism." A structuralist could reply that Julius Caesar and {ϕ, {ϕ}} have essential properties other than those relating to other places in the natural-number structure, but that would miss the point.

We point toward a relativity of ontology, at least in mathematics. Roughly, mathematical objects are tied to the structures that constitute them. Benacerraf [1965, §III.A] himself espoused a related view, at least temporarily. In order to set up his dilemma, he "treated expressions of the form $n = s$, where n is a number expression and s a set expression as if . . . they made perfectly good sense, and . . . it was our job to sort out the true from the false. . . . I did this to dramatize the kind of answer that a Fregean might give to the request for an analysis of number. . . . To speak from Frege's standpoint, there is a world of objects . . . in which the identity relation [has] free reign." Benacerraf's suggestion is to hold that at least some identity statements are meaningless: "Identity statements make sense only in contexts where there exist possible individuating conditions. . . . [Q]uestions of identity contain the presupposition that the 'entities' inquired about both belong to some general category." We need not go this far, but notice that items from the same structure are certainly in the same "general category," and there are "individuating conditions" among them. Whether Benacerraf has given the only ways to construe identity statements remains to be seen. He concluded, "What constitutes an entity is category or theory-dependent. . . . There are . . . two correlative ways of looking at the problem. One might conclude that identity is systematically ambiguous, or else one might agree with Frege, that identity is unambiguous, always meaning sameness of object, but that (contra-Frege now) the notion of *object* varies from theory to theory, category to category." In mathematics, at least, the notions of "object" and "identity" are unequivocal but thoroughly relative. Objects are tied to the structures that contain them. It is thus strange that Benacerraf should eventually conclude that natural numbers are not objects. Arithmetic is surely a coherent theory, "natural number" is surely a legitimate category, and numbers are its objects.[9]

Suppose that mathematicians develop a new field. Call its objects "hypernumbers." Analogues in (reconstructed) history are the study of negative, irrational, and complex numbers, and quaternions. It would surely be pompous of the philosopher to suggest that the field of hyperarithmetic is somehow illegitimate and is destined to remain so until we know how to individuate hypernumbers. The mathematicians do not have to tell us, once and for all, how to figure out whether, say, the additive identity of the hypernumbers is the same thing as the zero of arithmetic or the zero of analysis or the empty set. It is enough for them to differentiate hypernumbers from each other.

As hinted earlier, there is an important caveat to this relativity. I do not wish to go as far as Benacerraf in holding that identifying positions in different structures (or positions in a structure with other objects) is always meaningless. On the contrary, mathematicians sometimes find it convenient, and even compelling, to identify the positions of different structures. This occurs, for example, when set theorists settle on the finite von Neumann ordinals as the natural numbers. They stipulate that 2 is {ϕ, {ϕ}}, and so it follows that 2 ≠ {{ϕ}}. For a more straightforward example, it is surely wise to identify the positions in the natural-number structure with their counterparts in the integer-, rational-, real-, and complex-number structures. Accordingly, the natural number 2 is identical to the integer 2, the rational number 2, the real number 2, and the complex number 2 (i.e., $2 + 0i$). Hardly anything could be more straightforward. For an intermediate case, mathematicians occasionally look for the "natural settings" in which a structure is best studied. An example is the embedding of the complex numbers in the Euclidean plane, which illuminates both structures. It is not an exaggeration to state that some structures grow and thrive in certain environments. This phenomenon will occupy us several times, in chapters 4 and 5. The point here is that cross-identifications like these are matters of *decision*, based on convenience, not matters of discovery.

Parsons [1990, 334] puts the relativity into perspective:

> [O]ne should be cautious in making such assertions as that identity statements involving objects of different structures are meaningless or indeterminate. There is an obvi-

8. In professional baseball, shortstops are generally faster than catchers, and they are better hitters (even though there are notable exceptions). Statements like this have to do with the particular skills needed to play each position well.

9. I will return to this relativity throughout the book, notably in section 6 of chapter 4 on epistemology, section 3 of chapter 5 on Frege, and chapter 8. I am indebted to Crispin Wright and Bob Hale for pressing the Caesar issue.

ous sense in which identity of natural numbers and sets is indeterminate, in that different interpretations of number theory and set theory are possible which give different answers about the truth of identities of numbers and sets. In a lot of ordinary, mathematical discourse, where different structures are involved, the question of identity or non-identity of elements of one with elements of another just does not arise (even to be rejected). But of course some discourse about numbers and sets makes identity statements between them meaningful, and some of that . . . makes commitments as to the truth value of such identities. Thus it would be quite out of order to say (without reference to context) that identities of numbers and sets are meaningless or that they lack truth-values.

Even with this caveat, the ontological relativity threatens the semantic uniformity between mathematical discourse and ordinary or scientific discourse. Of course, this depends on what is required for uniformity, of which more later (section 9 of chapter 4, on reference). The threat also depends on the extent to which ordinary objects are not relative. I briefly return to this in chapter 8.

On a related matter, Azzouni [1994, 7–8, 146–147] accuses structuralists of being "ontologically radical" in the sense that we "replace the traditional metaphysically inert mathematical object with something else." It depends on what one thinks was there to be accepted or replaced. Structuralism is a view about what the objects of, say, arithmetic *are*, not what they should be, and we claim to make sense of what goes on in mathematics. Mathematicians do not usually use phrases like "metaphysically inert." Perhaps Azzouni's view is that we structuralists are being radical with respect to traditional *philosophies* of mathematics, Platonism in particular. I leave it to the reader to determine the extent to which I am proposing a replacement or further articulation of prior realist philosophies of mathematics.

One slogan of structuralism is that mathematical objects are places in structures. We must be careful here, however, because there is an intuitive difference between an object and a place in a structure, between an officeholder and an office. We can accommodate this intuition and yet maintain that numbers and sets are objects by invoking a distinction in linguistic practice. There are, in effect, two different orientations involved in discussing structures and their places (although the border between them is not sharp). Sometimes the places of a structure are discussed in the context of one or more systems that exemplify the structure. We might say, for example, that the shortstop today was the second baseman yesterday, or that the current vice president is more intelligent than his predecessor. Similarly, we might say that the von Neumann 2 has one more element than the Zermelo 2. Call this the *places-are-offices* perspective. This office orientation presupposes a background ontology that supplies objects that fill the places of the structures. In the case of baseball defense and that of government, the background ontology is people; in the case of chess games, the background ontology is small, movable objects—pieces with certain colors and shapes. In the case of arithmetic, sets—or anything else—will do for the background ontology. With mathematics, the background ontology can even consist of places from other structures, when we say, for example, that the negative, whole real numbers exemplify the natural-number structure, or that a Euclidean line exemplifies the real-number structure. Indeed, the background ontology for the places-are-offices perspective can even consist of the places of *the very structure under discussion*, when it is noted, for example, that the even natural numbers exemplify the natural-number structure. We will have occasion later to consider structures whose places are occupied by other structures. One consequence of this is that, in mathematics at least, the distinction between office and officeholder is a relative one. What is an object from one perspective is a place in a structure from another.

In contrast to this office orientation, there are contexts in which the places of a given structure are treated as objects in their own right, at least grammatically. That is, sometimes items that denote places are bona fide singular terms. We say that the vice president is president of the Senate, that the chess bishop moves on a diagonal, or that the bishop that is on a black square cannot move to a white square. Call this the *places-are-objects* perspective. Here, the statements are about the respective structure as such, independent of any exemplifications it may have. Arithmetic, then, is about the natural-number structure, and its domain of discourse consists of the places in this structure, treated from the places-are-objects perspective. The same goes for the other nonalgebraic fields, such as real and complex analysis, Euclidean geometry, and perhaps set theory.

It is common to distinguish the "is" of identity from the "is" of predication. The sentence "Cicero is Tully" does not have the same form as "Cicero is Roman." When, in the places-are-objects perspective, we say that 7 is the largest prime less than 10, and that the number of outfielders is 3, we use the "is" of identity. We could just as well write "=" or "is identical to." In contrast, when we invoke the places-are-offices perspective and say that $\{\{\phi\}\}$ is 2 and that $\{\phi, \{\phi\}\}$ is 2, we use something like the "is" of predication, but here it is predication *relative to a system* that exemplifies a structure. Let us call this the "is" of office occupancy. We are saying that $\{\{\phi\}\}$ plays the role of 2 in the system of Zermelo numerals and that $\{\phi, \{\phi\}\}$ plays the role of 2 in the system of finite von Neumann ordinals. When we say that Ozzie Smith is the shortstop, or that Al Gore is the vice president, we also invoke the "is" of office occupancy.

This does not exhaust the uses of the copula in mathematics. I noted earlier that for convenience, mathematicians sometimes identify places from different structures. For example, when set theorists settle on the von Neumann account of arithmetic, and thereby declare that 2 is $\{\phi, \{\phi\}\}$, they invoke what may be called the "is of identity by fiat."

The point here is that sometimes we use the "is" of identity when referring to offices, or places in a structure. This is to treat the positions *as objects*, at least when it comes to surface grammar. When the structuralist asserts that numbers are objects, this is what is meant. The places-are-objects perspective is thus the background for the present realism in ontology toward mathematics. Places in structures are bona fide objects.

My perspective thus presupposes that statements in the places-are-objects perspective are to be taken literally, at face value. Bona fide singular terms, like "vice president," "shortstop," and "2" denote bona fide objects. This reading might be questioned. Notice, for example, that places-are-objects statements entail generalizations over all systems that exemplify the structure in question. Everyone who is vice president—whether it be Gore, Quayle, Bush, or Mondale—is president of the Senate in that government. Every chess bishop moves on a diagonal, and none of those on black squares ever move to white squares (in the same game). No person can be shortstop and catcher simultaneously; and anything playing the role of 3 in a natural-number system is the successor of whatever plays the role of 2 in that system. In short, places-are-objects statements apply to the particular objects or people that happen to occupy the positions with respect to any system exemplifying the structure. Someone might hold, then, that places-are-objects statements are no more than a convenient rephrasing of corresponding generalizations over systems that exemplify the structure in question. If successful, a maneuver like this would eliminate the places-are-objects perspective altogether. The apparent singular terms mask implicit bound variables. This rephrasing plan, however, depends on being able to generalize over all systems that exemplify the structure in question. To assess this idea, we turn to our other main ontological question, the status of structures themselves.

3 Ontology: Structure

Because the same structure can be exemplified by more than one system, a structure is a one-over-many. Entities like this have received their share of philosophical attention throughout the ages. The traditional exemplar of one-over-many is a universal, a property, or a Form. In more recent philosophy, there is the type/token dichotomy. In philosophical jargon, one says that several tokens *have* a particular type, or *share* a particular type; and we say that an object *has* a universal or, as Plato put it, an object *has a share of*, or *participates in* a Form. As defined above, a structure is a pattern, the form of a system. A system, in turn, is a collection of related objects. Thus, structure is to structured as pattern is to patterned, as universal is to subsumed particular, as type is to token.

The nature and status of types and universals is a deep and controversial matter in philosophy. There is no shortage of views on such issues. Two of the traditional views stand out. One, due to Plato, is that universals exist prior to and independent of any items that may instantiate them. Even if there were no red objects, the Form of Redness would still exist. This view is sometimes called "*ante rem* realism," and universals so construed are "*ante rem* universals." The main alternative, attributed to Aristotle, is that universals are ontologically dependent on their instances. There is no more to redness than what all red things have in common. Get rid of all red things, and redness goes with them. Destroy all good beings, all good things, and all good actions, and you destroy goodness itself. A sobering thought. Forms so construed are called "in re universals," and the view is sometimes called "in re realism." Advocates of this view may admit that universals exist, after a fashion, but they deny that universals have any existence independent of their instances.

Of course, there are other views on universals. Conceptualism entails that universals are mental constructions, and nominalism entails that they are linguistic constructions or that they do not exist at all. For present purposes, I lump these alternate

views with in re realism. The important distinction is between *ante rem* realism and the others. Our question is whether, and in what sense, structures exist independently of the systems that exemplify them. Is it reasonable to speak of the natural-number structure, the real-number structure, or the set-theoretic hierarchy on the off chance that there are no systems that exemplify these structures?

One who thinks that there is no more to structures than the systems that exemplify them—an advocate of an in re view of structures—might be attracted to the program suggested at the very end of the previous section. Recall that from the structuralist perspective, it is the places-are-objects perspective that sanctions the thesis that numbers are objects. On the program in question, however, places-are-objects statements are not taken at face value but are understood as generalizations in the places-are-offices perspective. So "3 + 9 = 12" would come to something like "in any natural-number system *S*, the object in the 3 place of *S* *S*-added to the object in the 9 place of *S* results in the object in the 12 place of *S*." When paraphrased like this, seemingly bold ontological statements become harmless—analytic truths if you will. For example, "3 exists" comes to "every natural-number system has an object in its 3 place," and "numbers exist" comes to "every natural-number system has objects in its places."

In sum, the program of rephrasing mathematical statements as generalizations is a manifestation of structuralism, but it is one that does not countenance mathematical objects, or structures for that matter, as bona fide objects. Talk of numbers is convenient shorthand for talk about all systems that exemplify the structure. Talk of structures generally is convenient shorthand for talk about systems. A slogan for the program might be "structuralism without structures."[10]

Dummett [1991, chapter 23] makes the same distinction concerning the nature of structures. According to "mystical" structuralism, "mathematics relates to *abstract structures*, distinguished by the fact that their elements have no non-structural properties" (p. 295). Thus, for example, the zero place of the natural-number structure "has no other properties than those which follow from its being the zero" of that structure. It is not a set, or anything else whose nature is extrinsic to the structure. Dummett's mystical structuralist is thus an *ante rem* realist about structures. The other version of structuralism takes a "hardheaded" orientation: "According to it, a mathematical theory, even if it be number theory or analysis which we ordinarily take as intended to characterize *one* particular mathematical system, can never properly be so understood: it always concerns all systems with a given structure" (Dummett [1991, 296]). If the hardheaded structuralist countenances structures at all, it is only in an in re sense.

Parsons [1990, §§. 2–7] presents, but eventually rejects, a hardheaded view like this, which he dubs *eliminative structuralism*: "It . . . avoids singling out any one . . . system as the natural numbers. . . . [Eliminative structuralism] exemplifies a very natural response to the considerations on which a structuralist view is based, to see statements about a kind of mathematical objects as general statements about struc-

10. This slogan was adopted by Hellman [1996], after he read a draft of this chapter.

tures of a certain type and to look for a way of eliminating reference to mathematical objects of the kind in question by means of this idea" (p. 307). Benacerraf [1965, 291] settles on a hardheaded, eliminative, in re version of structuralism, when he writes, "Number theory is the elaboration of the properties of all structures of the order type of the numbers." This, of course, is of a piece with his rejection, noted earlier, of the thesis that numbers are objects.[11]

Thus, the eliminative structuralist program paraphrases places-are-objects statements in terms of the places-are-offices perspective. Recall that the places-are-offices orientation requires a background ontology, a domain of discourse. This domain contains objects that fill the places in the requisite (in re) structures. In the case of baseball defenses, the background ontology consists of people ready to play ball; in the case of chess configurations, the ontology consists of pieces of marble, wood, plastic, metal, and so on, manufactured in a certain way. In the case of mathematics, any old objects will do—so long as there are enough of them.

The main stumbling block of the eliminative program is that to make sense of a substantial part of mathematics, the background ontology must be quite robust. The *nature* of the objects in the final ontology does not matter, but there must be a lot of objects there. To see this, let Φ be a sentence in the language of arithmetic. According to eliminative structuralism, Φ amounts to something in the form:

(Φ') for any system S, if S exemplifies the natural-number structure, then $\Phi[S]$,

where $\Phi[S]$ is obtained from Φ by interpreting the nonlogical terminology and restricting the variables to the objects in S. If the background ontology is finite, then there are no systems that exemplify the natural-number structure, and so Φ' and $(\neg\Phi)'$ are both true. Because mathematics is not vacuous, this is unacceptable. We do not end up with a rendering of arithmetic if the background ontology is finite. Similarly, an eliminative-structuralist account of real analysis and Euclidean geometry requires a background ontology whose cardinality is at least that of the continuum, and set theory requires a background ontology that has the size of a proper class (or at least an inaccessible cardinal).

I suppose that one can maintain that there are infinitely many *physical* objects, in which case an eliminative account of arithmetic may get off the ground with a physical ontology. As we have seen (section 1), Field [1980] holds that each space-time point is a physical object. If this claim is plausible, then an eliminative structuralist might follow this lead with an account of analysis and geometry. Nevertheless, it seems reasonable to insist that there is *some* limit to the size of the physical universe. If so, then any branch of mathematics that requires an ontology larger than that of the physical universe must leave the realm of physical objects if these branches are not to be doomed to vacuity. Even with arithmetic, it is counterintuitive for an account of mathematics to be held hostage to the size of the physical universe.[12]

11. Here, Benacerraf uses "structure" as I use "system."

12. See Parsons [1990]. This is why Field [1980] himself does not attempt to reduce analysis and geometry to a theory of space-time points and regions. Incidentally, according to the view developed in Maddy [1990, chapter 5], if the transitive closure of a set s contains only physical objects, then s itself is a physical object. It follows that there is a proper class of physical objects, and there are systems of physical objects that exemplify the set-theoretic hierarchy. Such systems are located where the original objects are. The thesis that sets of physical objects are themselves physical objects is criticized by Chihara [1990, chapter 10] and Balaguer [1994].

There are three structuralist responses to this threat of vacuity. One is to maintain an eliminative program but postulate that enough abstract objects exist for all of the structures under study to be exemplified. That is, for each field of arithmetic, we assume that there are enough objects to keep that field from being vacuous. I call this the *ontological option.*

On this program, if one wants a single account for all (or almost all) of mathematics, then the background ontology of abstract objects must be quite big. As noted earlier, several logicians and philosophers think of the set-theoretic hierarchy as the ontology for all of mathematics. The universe is V. If one assumes that every set in the hierarchy exists, then there will surely be enough objects to exemplify just about any structure one might consider. Because, historically, one purpose of set theory was to provide as many isomorphism types as possible, set theory is rich fodder for eliminative structuralism. A structure, on this account, is an order type of sets, no more and no less.

The crucial feature of this version of eliminative structuralism is that the background ontology is not understood in structuralist terms. If the iterative hierarchy is the background, then set theory is not, after all, the theory of a particular structure. Rather, it is about a particular class of objects, the background ontology V. Perhaps from a different point of view, set theory can be thought of as the study of a particular structure U, but this would require another background ontology to fill the places of U. This new background ontology is not to be understood as the places of another structure or, if it is, we need yet another background ontology for *its* places. On the ontological option, we have to stop the regress of system and structure somewhere. The final ontology is not understood in terms of structures, even if everything else in mathematics is.

To be sure, there is nothing sacrosanct about Zermelo-Fraenkel set theory. Foundationalists have shown that mathematics can be rendered in theories other than that of the iterative hierarchy (e.g., Quine [1937]; Lewis [1991], [1993]). Among these are a dedicated contingent of mathematicians and philosophers who hold that the category of categories is the proper foundation for mathematics (see, for example, Lawvere [1966]).[13] The ultimate background ontology for eliminative structuralism can thus be the domain of any of several set theories or category theories.

A structuralist might be tempted to step back from this competition of background theories and wonder if there is a structure common to all of them. However, on the ontological option, this temptation needs to be resisted. The structures studied in two theories can be compared only in terms of a more inclusive theory.

Of course, eliminative structuralists need not consider their most powerful theory (or theories) to be about the background ontology. They may regard, say, ZFC and

13. McLarty [1993] is a lucid and insightful start of a structuralist program in terms of category theory—topos theory in particular.

topos theory both as theories of specific structures. On the present version of eliminative structuralism, they do need to acknowledge the existence of the ultimate background ontology, but they need not develop a formal theory of this system.

On all versions of structuralism, the nature of the objects in the places of a structure does not matter—only the relations among the objects are significant. On the ontological option, then, the only relevant feature of the background ontology is its *size*. Are there enough objects to exemplify every structure the mathematician might consider? If a theory of this ontology is developed, the only relevant factor is the size of the ontology.

In correspondence and conversation, some nominalists express sympathy with a structuralist account of mathematics, but they quickly add that one should speak of "possible structures" rather than just structures. Our second option pursues this suggestion by modalizing eliminative structuralism. Instead of saying that arithmetic is about all systems of a certain type, one says that arithmetic is about all *possible* systems of a certain type. Again, let Φ be a sentence in the language of arithmetic. Earlier, on behalf of eliminative structuralism, I rendered Φ as "for any system S, if S exemplifies the natural-number structure, then $\Phi[S]$." With the present option, Φ is understood as

for any *possible* system S, if S exemplifies the natural-number structure, then $\Phi[S]$,

or

necessarily, for any system S, if S exemplifies
the natural-number structure, then $\Phi[S]$.

The problem, of course, is to keep arithmetic from being vacuous without assuming that there is a system that exemplifies the structure. The solution here is to merely assume that such a system is possible. The same goes for real analysis and even set theory. Unlike the ontological option, here we do not require an actual, rich background ontology. Instead, we need a rich background ontology to be *possible*. I call this the *modal option*.

Hellman [1989] carries out a program like this in meticulous detail. The title of the book, *Mathematics without numbers*, sums things up nicely. It is a structuralist account of mathematics that does not countenance the existence of structures—or any other mathematical objects for that matter. Statements in a nonalgebraic branch of mathematics are understood as generalizations inside the scope of a modal operator. Instead of assertions that various systems exist, Hellman has assertions that the systems might exist.

Probably the central issue with the modal option is the nature of the invoked modality. What are we to make of the "possibilities" and "necessities" used to render mathematical statements? I presume that thinking of the possibility as *physical* possibility is a nonstarter, for reasons already given. Perhaps it is physically possible for there to be a system that exemplifies the natural-number structure, the real-number structure, or Euclidean space, but it is stretching this modal notion beyond recognition to claim that a system that exemplifies the set-theoretic hierarchy is physically possible (Maddy [1990, chapter 5] notwithstanding; see note 12 above). The relevant

modal operator is not to be understood as *metaphysical* possibility either. Intuitively, if mathematical objects–like numbers, points, and sets–exist at all, then their existence is metaphysically necessarily. According to this intuition, "the natural numbers exist" is equivalent to both "possibly, the natural numbers exist" and "necessarily, the natural numbers exist" (assuming that the modal logic system S5 is sound for metaphysical necessity). Now, recall that on the first, ontological option there must be a sufficiently large realm of objects. Presumably, the items in the ontology are not metaphysically different from natural numbers. Thus, the existence and the possible existence of the items in the background ontology are equivalent. Thus, the use of metaphysical modality does not really weaken the ontological burden of eliminative structuralism (for an elaboration of a similar point, see Resnik [1992]).

For this reason, Hellman mobilizes the *logical* modalities for his eliminative structuralism. Our arithmetic sentence Φ becomes

for any *logically possible* system S, if S exemplifies
the natural-number structure, then $\Phi[S]$.

This maneuver gives the modal option its best shot. The modal structuralist needs to assume only that it is logically possible that there is a system that exemplifies the natural-number structure, the real-number structure, and so on.

Recall that in contemporary logic textbooks and classes, the logical modalities are understood in terms of sets. To say that a sentence is logically possible is to say that *there is* a certain set that satisfies it. According to the modal option of eliminative structuralism, however, to say that there is a certain set is to say something about every logically possible system that exemplifies the structure of the set-theoretic hierarchy. This is an unacceptable circularity. It does no good to render mathematical "existence" in terms of logical possibility if the latter is to be rendered in terms of existence in the set-theoretic hierarchy. Putting the views together, the statement that a sentence is logically possible is really a statement about all set-theoretic models of set theory. Who says there are such models? Once again, we have a menacing threat of vacuity. Hellman accepts this straightforward point, and so he demurs from the standard, model-theoretic accounts of the logical modalities. Instead, he takes the logical notions as *primitive*, not to be reduced to set theory. I return to this exchange of ontology for modality in chapter 7.

The third option avoids the eliminative program altogether and adopts an *ante rem* realism toward structures. Structures exist whether they are exemplified in a nonstructural realm or not. On this option, statements in the places-are-objects mode are taken literally, at face value. In mathematics, anyway, the places of mathematical structures are as bona fide as any objects are. So, in a sense, each structure exemplifies itself. Its places, construed as objects, exemplify the structure.

First, a disclaimer: In the history of philosophy, *ante rem* universals are sometimes given an explanatory primacy. It might be said, for example, that the reason the White House is white is that it participates in the Form of Whiteness. Or what *makes* a basketball round is that it participates in the Form of Roundness. No such explanatory claim is contemplated here on behalf of *ante rem* structures. I do not hold, for example, that a given system is a model of the natural numbers because it exemplifies the natural-number structure. If anything, it is the other way around. What makes the system exemplify the natural-number structure is that it has a one-to-one successor function with an initial object, and the system satisfies the induction principle. That is, what makes a system exemplify the natural-number structure is that it is a model of arithmetic. In much of the current literature, types do not carry this sort of explanatory burden, either. Thus, in this respect, *ante rem* structures are like types.

Michael Hand [1993, 188] states that an *ante rem* structuralist is indeed committed to structures that bear the explanatory burden: "Admittedly, . . . [t]he motivation behind structuralism has nothing to do with the possible explanatory function of abstract patterns. . . . Nonetheless, the structuralist is committed to more than [the] limited motivation suggests. After all, abstract patterns, structures, are not entities newly posited by the structuralist. . . . Instead, the structuralist is making use of things we already know something about, and that we already put to use metaphysically in various ways. Since this is so, she is responsible . . . to the metaphysical uses to which we already put them." The idea behind this pronouncement seems to be that since we *ante rem* structuralists are using a notion like the traditional one-over-many, we are committed to all of the features *and uses* of *ante rem* universals *as traditionally conceived*. Hand goes on to argue, quite insightfully, that nothing can bear the explanatory burden, and he concludes that structures do not exist. According to his pronouncement, it seems, once a (philosophical) notion has been debunked, no one is allowed to use a variation on that notion—even a variation that survives the debunking. Readers sympathetic with this pronouncement are invited to construe structures as a *new* sort of notion, one that is similar in some ways to traditional *ante rem* universals but does not bear their explanatory burden. To paraphrase Kripke, call structures "shmuniversals." Hand suggests that this maneuver leaves *ante rem* structuralism unmotivated. I take this book to provide some motivation, and I leave it to the reader to judge the matter.

To sum up, the three options are ontological eliminative structuralism, modal eliminative structuralism, and *ante rem* realism. I believe that the *ante rem* option is the most perspicuous and least artificial of the three. It comes closest to capturing how mathematical theories are conceived. Nevertheless, I do not mean to rule out the other options. Indeed, it follows from the thesis of structuralism that, in a sense, all three options are equivalent. As will be shown, each delivers the same "structure of structures." The next section provides a brief account of each option and a defense of their equivalence (see also chapter 7).

4 Theories of Structure

No matter how it is to be articulated, structuralism depends on a notion of two systems that exemplify the "same" structure. That is its point. Even if one eschews structures themselves, we still need to articulate a relation among systems that amounts to "have the same structure."

There are several relations that will do for this. I mention two, both of which are equivalence relations. The first is *isomorphism*, a common (and respectable) mathematical notion. Two systems are isomorphic if there is a one-to-one correspondence from the objects and relations of one to the objects and relations of the other that preserves the relations. Suppose, for example, that the first system has a binary relation R. If f is the correspondence, then $f(R)$ is a binary relation of the second system and, for any objects m, n, of the first system, R holds between m and n in the first system if and only if $f(R)$ holds between $f(m)$ and $f(n)$ in the second system. Informally, it is sometimes said that isomorphism "preserves structure."

Isomorphism is too fine-grained for present purposes. Intuitively, one would like to say that the natural numbers with addition and multiplication exemplify the same structure as the natural numbers with addition, multiplication, and less-than. However, the systems are not isomorphic, for the trivial reason that they have different sets of relations. The first has no binary relation to correspond with the less-than relation, even though that relation is definable in terms of addition: $x < y$ if and only if $\exists z(z \neq 0 \;\&\; x + z = y)$. There is, in a sense, nothing new in the "richer" system. Similarly, we would like to say that the various formulations of Euclidean plane geometry with different primitives all exemplify the same structure.

Resnik [1981] has formulated a more coarse-grained equivalence relation among systems (and structures) for this purpose. First, let R be a system and P a subsystem. Define P to be a *full subsystem* of R if they have the same objects (i.e., every object of R is an object of P) and if every relation of R can be defined in terms of the relations of P. The idea is that the only difference between P and R is that some definable relations are omitted in P. So the natural numbers with addition and multiplication are a full subsystem of the natural numbers under addition, multiplication, and less-than. Let M and N be systems. Define M and N to be *structure-equivalent*, or simply *equivalent*, if there is a system R such that M and N are each isomorphic to full subsystems of R. Equivalence is a good candidate for "sameness of structure" among systems.[14]

Notice that structure equivalence is characterized in terms of *definability*, a blatant linguistic notion. One consequence is that equivalence is dependent on the resources available in the background metalanguage (or the U-language). For example, in a standard first-order background, the natural numbers with successor alone are not equivalent to the natural numbers with addition and multiplication, because addition cannot be defined from successor in the first-order theory. However, the theories are equivalent in a second-order background (see Shapiro [1991, chapter 5]). The dependence on the background theory and, in particular, on its language should not be surprising. A recurring theme in this book is that a number of ontological matters are tied to linguistic resources.

Let us briefly consider what would be involved in rigorously developing each of the three options for structuralism: the ontological in re route, the modal in re route, and the *ante rem* route. Recall that the ontological option presupposes an ultimate (nonstructural) background ontology for all of mathematics. The first item on the agenda would thus be a detailed account of this background ontology. As above, the set-theoretic hierarchy V is a natural choice for the background, in which case there

14. Structure equivalence is analogous to definitional equivalence among theories (see chapter 7).

already is a developed theory—Zermelo-Fraenkel set theory. Next, we need an account of *systems* in this background ontology. This, too, has been done already, via standard model theory. An n-ary "relation" is a set of n-tuples and an n-place function is a many-one set of $(n + 1)$-tuples. A system is an ordered pair that consists of a domain and a set of relations and functions on it. Model theorists sometimes use words like "structure," "model," and "interpretation" for what I call a "system." In set theory, isomorphism and structure equivalence are also easily defined, thus completing the requisite eliminative theory of structuralism. In other words, using common model-theoretic techniques, set theorists can speak of systems that share a common structure. Notice that we do not find what I am calling "structures" in the ontology. All we have is isomorphism and structure-equivalence *among systems*. Recall that the slogan of eliminative structuralism is "structuralism without structures."

Because isomorphism and structure-equivalence are equivalence relations, one can informally take a structure to be an isomorphism type or a structure-equivalence type. So construed, a structure is an equivalence class in the set-theoretic hierarchy. Notice, however, that each nonempty "structure" is a proper class, and so it is not in the set-theoretic hierarchy. The relevant notions could be expanded to include proper class systems, but then we could not take a structure to be an equivalence class of systems unless we moved to a third-order background.

With admirable rigor and attention to detail, Hellman [1989] develops the modal option. Modal operators are added to a standard formal language, and the aforementioned notions of "system" and isomorphism are invoked. A sentence of arithmetic, say, is rendered as a statement about all *possible* systems that satisfy the (second-order) Peano axioms.[15] Although the program is correctly characterized as "structuralist," there is no notion of *structure* in the official modal language.

Finally, the *ante rem* option requires a *theory* of structures. The plan is to stop the regress of system and structure at a universe of structures. Because structures themselves are in the ontology, we need an identity relation on structures. Resnik [1981] seems to hold that there is no such identity relation, arguing that there is no "fact of the matter" as to whether two structures are the same or different, or even whether two systems exemplify the same structure (but see Resnik [1988, 411 note 16]). Notice that this goes against the Quinean dictum "no entity without identity." Quine's thesis is that within a given theory, language, or framework, there should be definite criteria for identity among its objects. There is no reason for structuralism to be the single exception to this. If we are to have a theory of structures, we need an identity relation on them. Perhaps Resnik demurs at the development of such a theory (see Resnik [1996]). It seems to me, however, that if one is to speak coherently about structures and avoid the ontological and modal options, then such a theory is needed, at least at some stage of analysis. In Quinean terms, the need to regiment one's informal language applies to its philosophical parts as well as the more respectable scientific neighborhoods.

15. Hellman's account avoids the use of the notion of "possible system," because he does not countenance an ontology of possibilia. The program also does not directly use semantic notions like "satisfaction."

When Resnik states that there is no "fact of the matter" concerning the identity of structures, he may just mean that the ordinary use of the relevant terms does not determine a unique identity relation. This much is quite correct. To regiment our language, we would need to *define* the requisite identity relations, but there is no uniquely best candidate for this. Like the identification of places from different structures (see section 2), the identity relation we need is more a matter of decision or invention, based on convenience, rather than a matter of discovery. But we do need to decide.

We take identity among structures to be primitive, and isomorphism is a congruence among structures. That is, we stipulate that two structures are identical if they are isomorphic. There is little need to keep multiple isomorphic copies of the same structure in our structure ontology, even if we have lots of systems that exemplify each one.[16] We could also "identify" structures that are structure-equivalent, but it is technically inconvenient to do so.

With the ontological option just delimited, systems are constructed from sets in the fashion of model theory, and structures are certain equivalence types on systems. For the *ante rem* option, we axiomatize the notion of structure directly. The envisioned theory has variables that range over structures, and thus a quantifier "all structures." Each structure has a collection of "places" and relations on those places. Once again, the places-are-objects perspective is taken seriously. The theory thus has a second sort of variable that ranges over places in structures.

The category theorist characterizes a structure or a type of structure in terms of the structure-preserving functions, called "morphisms," between systems that exemplify the structures. For many purposes, this is a perspicuous approach (see McLarty [1993]), but here I provide an outline of a more traditional axiomatic treatment. In effect, structure theory is an axiomatization of the central framework of model theory.

Because it appears to be necessary to speak of relations and functions on places, I adopt a second-order background language (see Shapiro [1991]). An alternative to this would be to include a rudimentary theory of collections as part of the theory.

First, a *structure* has a collection of *places* and a finite collection of functions and relations on those places. The isomorphism relation among structures and the satisfaction relation between structures and formulas of an appropriate formal language are defined in the standard way. We could stipulate that the places of different structures are disjoint, but there is no reason to do so. Our first axiom, concerning the existence of structures is simpleminded but ontologically nontrivial:

Infinity: There is at least one structure that has an infinite number of places.

Because structures, places, relations, and functions are the only items in the ontology, everything else must be constructed from those items. Thus, a *system* is defined to be a collection of places from one or more structures, together with some

16. The sequence of natural numbers contains many isomorphic copies of itself, but there is only one natural-number structure. In structure theory, the copies are systems.

relations and functions on those places. For example, the even-number places of the natural-number structure constitute a system, and on this system, a "successor" function could be defined that would make the system exemplify the natural-number structure. The "successor" of n would be $n + 2$. Similarly, the finite von Neumann ordinals are a system that consists of places in the set-theoretic hierarchy structure, and this system also exemplifies the natural-number structure, once the requisite relations and functions are added. Other systems consist of the places of several structures, with relations defined on their "objects." For example, a nonstandard model of simple first-order arithmetic (with successor alone) consists of the natural numbers "followed by" the integers.

The places of a given structure—considered from the places-are-objects perspective—are objects. As characterized here, then, each structure is also a system.

Our next axioms concern what may be called "substructures":

Subtraction: If S is a structure and R is a relation of S, then there is a structure S' isomorphic to the system that consists of the places, functions, and relations of S except R. If S is a structure and f is a function of S, then there is a structure S'' isomorphic to the system consisting of the places, functions, and relations of S except f.

Subclass: If S is a structure and c is a subclass of the places of S, then there is a structure isomorphic to the system that consists of c but with no relations and functions.

Addition: If S is a structure and R is any relation on the places of S, then there is a structure S' isomorphic to the system that consists of the places, functions, and relations of S together with R. If S is a structure and f is any function from the places of S to places of S, then there is a structure S'' isomorphic to the system that consists of the places, functions, and relations of S together with f.

That is, one can remove places, functions, and relations at will; and one can add functions and relations.

The remaining objective for my theory is to assure the existence of large structures. The next axiom is an analogue of the powerset axiom of set theory:

Powerstructure: Let S be a structure and s its collection of places. Then there is a structure T and a binary relation R such that for each subset $s' \subseteq s$ there is a place x of T such that $\forall z(z \in s' \equiv Rxz)$.

Each subset of the places of S is related to a place of T, and so there are at least as many places in T as there are subsets of the places of S. Thus, the collection of places of T is at least as large as the powerset of the places of S. The powerstructure axiom can be formulated in the second-order background language.

So far, structure theory resembles what is called Zermelo set theory. We have the existence of the natural-number structure, the real-number structure, a structure whose size is the powerset of that, and so on. The smallest standard model of the theory has the size of $V_{2\omega}$, the smallest standard model of Zermelo set theory.

To get beyond the analogue of Zermelo set theory, my next item is the analogue of the replacement principle:

Replacement: Let S be a structure and f a function such that for each place x of S, fx is a place of a structure, which we may call S_x. Then there is a structure T that is (at least) the size of the union of the places in the structures S_x. That is, there is a function g such that for every place z in each S_x there is a place y in T such that $gy = z$.

The idea is the same as in set theory. There is a structure at least as large as the result of "replacing" each place x of S with the collection of places of a structure S_x. With this axiom, every standard model of structure theory is the size of an inaccessible cardinal. In effect, structure theory is a reworking of second-order Zermelo-Fraenkel set theory.

The main principle behind structuralism is that any coherent theory characterizes a structure, or a class of structures. For what it is worth, I state this much:

Coherence: If Φ is a coherent formula in a second-order language, then there is a structure that satisfies Φ.

The problem, of course, is that it is far from clear what "coherent" comes to here. The question of when a theory is coherent, and thus describes a structure (or class of structures), will occupy us later several times (e.g., section 5 of this chapter, and chapter 4).[17] Notice, for now, that because we are using a second-order language, simple (proof-theoretic) consistency is not sufficient to guarantee that a theory describes a structure or class of structures. Because the completeness theorem fails, there are consistent second-order theories that are not satisfiable (see Shapiro [1991, chapter 4]). Consider, for example, the conjunction P of the axioms of Peano arithmetic together with the statement that P is not consistent. Contra Hilbert, consistency does not imply existence even for a structuralist. We need something more like satisfiability, but the latter is usually formulated in terms of the set-theoretic hierarchy (or some other ontology): a theory is satisfiable if *there is a model* for it. There is no getting away from this problem, but perhaps the circle is not vicious.

We can, of course, add an axiom that, say, second-order ZFC is coherent, and thus conclude that there is a structure the size of an inaccessible cardinal. Another, less ad hoc route to large structures is to assume that structure theory itself is coherent, and so is any theory consisting of structure theory plus any truth of structure theory. This suggests a reflection scheme. Let Φ be any (first- or second-order) sentence in the language of structure theory. Then the following is an axiom:

Reflection: If Φ, then there is a structure S that satisfies the (other) axioms of structure theory and Φ.

Letting Φ be a tautology, the principle entails the existence of a structure the size of an inaccessible cardinal. Letting Φ be the conjunction of the other axioms of structure theory (or ZFC) plus the existence of a structure the size of an inaccessible cardinal, the reflection principle entails the existence of a structure the size of the second inaccessible cardinal, and it goes on from there.[18]

One might think that I am inviting a version of Russell's paradox. Is there a structure of all structures? The answer is that there is not, just as there is no set of all sets. Because a "system" is a collection of places in structures (together with relations),

17. A more general principle is that every coherent *collection* Γ of formulas is satisfied by a structure, but to be picky, one should add a proviso that Γ is not the size of a proper class.

18. A variation of the reflection principle, along the lines of Bernays [1961], entails the existence of structures the size of a Mahlo cardinal, a hyper-Mahlo cardinal, up to an indescribable cardinal. A suitable reflection also entails the powerstructure and replacement axioms. See Shapiro [1987] and Shapiro [1991, chapter 6] for a study of higher-order reflection principles. See also Levy [1960].

some systems are "too big" to exemplify a structure. This defect could be avoided, as it is in set theory, by stipulating that there are no systems the size of a proper class. The relevant axioms would be that for every system *S*, there is no function from the places of *S* onto the class of all places in all structures.

The point is that structuralism is no more (and no less) susceptible to paradox than set theory, modal structuralism, or category theory. Some care is required in regimenting the informal discourse, but it is a familiar sort of care. One can ascend to another level and interpret the objects of the domain of the structure language as the places in a superstructure. But, as with set theory, we cannot take this structure to be *in* the range of the (structure) variables of the original theory. The ontology changes as we go to a metalanguage. This, however, is rarified analysis. Normally, there is little need to ascend beyond the original structuralist language, at least not in this way, just as there is little need to ascend to some sort of superset theory.

Enough on the details of structure theory. For someone familiar with axiomatic set theory, everything is straightforward. The reason the development goes smoothly is that structure theory, as I conceive it, is about as rich as set theory. It has to be if set theory itself is to be accommodated as a branch of mathematics. In a sense, set theory and the envisioned structure theory are notational variants of each other. In particular, structure theory without the reflection principle is a variant of second-order ZFC, and structure theory with the reflection principle is a notational variant of set theory with a corresponding reflection principle.

Nevertheless, for present purposes, structure theory is a more perspicuous and less artificial framework than set theory. If nothing else, structure theory regards set theory (and perhaps even structure theory itself) as one branch of mathematics among many, whereas the ontological option makes set theory (or another designated theory) the special foundation. However, even this is not a major advantage, because the equivalence and mutual interpretability of the frameworks are straightforward. Anything that can be said in either framework can be rendered in the other. Talk of structures, as primitive, is easily "translated" as talk of isomorphism or equivalence types over a universe of (primitive) sets. In the final analysis, it does not really matter where we start.

The same goes for the modal option, but the articulation and details of that equivalence will be postponed (chapter 7). The upshot is the same as with set theory and structure theory. Anything that can be said in the modal structural system (of Hellman [1989]) can be rendered in either the set language or the structure language.

In short, on any structuralist program, *some* background theory is needed. The present options are set theory, modal model theory, and *ante rem* structure theory. The fact that any of a number of background theories will do is a reason to adopt the program of *ante rem* structuralism. *Ante rem* structuralism is more perspicuous in that the background is, in a sense, minimal. On this option, we need not assume any more about the background ontology of mathematics than is required by structuralism itself.[19] But when all is said and done, the different accounts are equivalent.

19. McLarty [1993] makes the same claim on behalf of a category-theoretic foundation of mathematics.

The smooth translations between the various theories also suggest that none of them can claim a major epistemological advantage over the others. The sticky epistemic problems get "translated" as well. Probably the deepest epistemic problem with standard set theory is how we can know anything about the abstract, acausal universe of sets. Which sets exist? How do we know? Formidable problems indeed. In the case of structure theory, the corresponding problems concern how we can know anything about the realm of structures. Which structures exist? How can we tell? The very same problem, in the case of modal structuralism, is how we know anything about the various possibilities. Which structures are possible? How can we tell?

The upshot of this section, then, is that there are several ways to render structuralism in a rigorous, carefully developed background theory, and there is very little to choose among the options. In a sense, they all say the same thing, using different primitives. The situation with structuralism is analogous to that of geometry. Points can be primitive, or lines can be primitive. It does not matter because, in either case, the same structure is delivered. The same goes for structuralism itself. Set theory and structure theory are equivalent, in the sense defined above. To speak loosely, the same "structure of structures" is delivered. Modal structuralism also fits, once the notion of "equivalence" is modified for the modal language.

5 Mathematics: Structures, All the Way Down

I articulate the picture of *ante rem* structuralism here, to demonstrate why this account is more perspicuous than the others, and to continue the dialectic of articulating the notions of structure, theory, and object.

Thus far, I have spoken freely of ordinary, nonmathematical structures, such as baseball defenses, governments, and chess configurations, along with mathematical structures like the natural numbers and the set-theoretic hierarchy. One might wonder whether the word "structure" is univocal across these uses. What if anything distinguishes *mathematical* structures from the others?

One possible answer is that in principle, there is no difference in kind between mathematical and nonmathematical structures. This has a clean, holistic ring to it—at least on the ontological front. A cocky holist might go on to claim that the only difference between the "mathematical" structures and the others is that the former are the ones studied by mathematicians qua mathematicians. If enough mathematicians took a professional interest in baseball defenses, then baseball defenses would be mathematical structures. If mathematicians took a professional interest in chess, then chess configurations would be mathematical structures. Typically, the structures studied by mathematicians are complex and interesting, but this does not mark a difference in kind.[20]

A slightly more cautious claim would be that the difference between mathematical and ordinary structures is not so much in the structures themselves but in the way

20. Mea culpa. In the past, when responding to questions, I would usually take this cocky holistic line. This would be greeted with frowns and incredulous stares from my audiences—with the possible exception of ontic holists.

they are studied. Mathematics is the *deductive* study of structures. The mathematician gives a description of the structure in question, independently of any systems this structure may be the structure of. Anything the mathematician, qua mathematician, goes on to say about the structure must follow from this description. Ordinary structures are not usually studied this way, or not studied this way exclusively. Recall the passage from Resnik [1982] in section 1. When the imaginary linguists discover that Tenglish is not the structure of spoken English, presumably by comparing the structure as defined to the spoken language, they lose interest in the structure. Their methodology is focused on what Tenglish is supposed to be a structure of. In contrast, if Tenglish is internally coherent, a mathematician can go on to study the structure, independently of whether it is exemplified in any real or even possible linguistic community. On this orientation, ontic holism is maintained, but mathematics is distinguished by its deductive epistemology.

This account is not cautious enough. Although there are interesting borderline cases between mathematical and ordinary structures, which will further occupy us when we get to applications (chapter 8), there are important differences between the two types of structures. A vague border is still a border.

One difference between the types of structures concerns the nature of the *relations* between the officeholders of exemplifying systems. Consider our standby, the baseball-defense structure. Imagine a system that consists of nine people placed in the configuration of a baseball defense but hundreds of miles apart—the "right fielder" in New York, the "center fielder" in Detroit, and so on. This system does not exemplify the structure of baseball defense, although one might say that it simulates or models the structure. There is an implicit requirement that the player at first base be within a certain distance of first base, the pitcher, and so forth. If not, then it is no baseball defense. In mathematical structures, on the other hand, the relations are all *formal*, or *structural*. The only requirements on the successor relation, for example, are that it be a one-to-one function, that the item in the zero place not be in its range, and that the induction principle hold. No spatiotemporal, mental, personal, or spiritual properties of any exemplification of the successor function are relevant to its being the successor function.

Although these examples may point in a certain direction, there is a problem of precisely formulating this notion of a "formal" relation. There are clear cases of formal relations and there are clear cases of nonformal relations. Surely, if a relation involves a *physical* magnitude like distance or a personal property like intelligence or age, then it is not formal. Being thirty-five years of age or older is not a formal property. One can leave things at this intuitive level, letting borderline cases take care of themselves. Accordingly, the border between mathematical and nonmathematical structures may not be sharp. Perhaps we can do better. If each relation of a structure can be completely defined using only logical terminology and the other objects and relations of the system, then they are all formal in the requisite sense. A slogan might be that formal languages capture formal relations. This is still not an adequate definition of "formal" or "structural" relation, however, because it is not clear how to formulate the logical/nonlogical boundary without begging any questions (see Shapiro [1997]).

Tarski [1986] proposed a criterion for the logical/nonlogical boundary that seems particularly apt here—whatever its fate in the philosophy of logic (see Sher [1991] for an insightful elaboration). His idea is that a notion is logical if its extension is unchanged under every permutation of the domain. Thus, for example, the property among sets of being nonempty is logical, because any permutation of the domain takes nonempty sets (of objects in the domain) to nonempty sets, and any such permutation takes the empty set to itself. The property of being thirty-five years of age or older is not logical, because there are permutations of the domain (of people) that take someone older than thirty-five to someone younger.

Because, in a permutation, any object can be replaced by any other, a notion that is invariant under all permutations ignores any nonstructural or intrinsic features of the individual objects. In these terms, the present proposal is that a relation is *formal* if it can be completely defined in a higher-order language, using only terminology that denotes Tarski-logical notions and the other objects and relations of the system, with the other objects and relations completely defined at the same time. All relations in a mathematical structure are formal in this sense.

If this definition of "formal" is adopted, then it is immediate that any relation that is logical in Tarski's sense is formal. However, it does not follow that all formal relations are logical. For example, neither 0 nor the successor function is Tarski-logical, because there are permutations of the natural numbers that take 0 to something else and there are permutations that do not preserve the successor function. Suppose, however, that we go up a level. Notice that any permutation of the natural numbers takes the successor function to the successor function of a (possibly different) natural-number system on the same domain. Likewise for 0. That is, if f is a permutation of the natural numbers, then $f(0)$ occupies the 0 place in a new system S, and m is the successor of n in S if $f^{-1}(m)$ is the successor of $f^{-1}(n)$ in the original natural numbers. The new system S exemplifies the natural-number structure. Thus, the notion of *natural-number system* $\langle N, 0, s\rangle$ is logical in Tarski's sense: any permutation of the domain takes a natural-number system to a natural-number system. In general, for any mathematical structure S, the notion of "exemplifies structure S" is logical in Tarski's sense. This is a pleasing feature of the combination of structuralism and the given account of logical notions. It manifests the two slogans that mathematics is the science of structure, and that logic is topic-neutral.

Another important difference between mathematical and ordinary structures concerns the sorts of items that can occupy the places in the structures. Imagine a system that consists of a ballpark with nine piles of rocks, or nine infants, placed where the fielders usually stand. Imagine also a system of chalk marks on a diagram of a field, on which a baseball manager makes assignments and discusses strategy. Intuitively, neither of these systems exemplifies the defense structure. A system is not a baseball defense unless its positions are filled by people prepared to play ball. Piles of rocks, infants, and chalk marks are excluded. Prima facie, these requirements on the officeholders in potential defense systems are not "structural." For example, the requirement that the officeholders be people prepared to play is not described solely in terms of relations among the offices and their occupants. The system of rock piles and the system of chalk marks can perhaps be said to *model* or *simulate* the baseball-defense structure, but they do not exemplify it. Similarly, there is no possible system that exemplifies the U.S. government (before the year 2017) in which my eldest daughter is president. The president must be thirty-five years of age, chosen by the electoral college, and be a native-born citizen. The age, birth, and election requirements are not structural, in that those requirements are not described in terms of relations of the officeholders to each other. There are, again, systems that model or simulate the government, and my daughter has the office of president in some of these, but simulating and exemplifying are not the same thing.

In contrast, *mathematical* structures are *freestanding*. Every office is characterized completely in terms of how its occupant relates to the occupants of the other offices of the structure, and any object can occupy any of its places. In the natural-number structure, for example, there is no more to holding the 6 office than being the successor of the item in the 5 office, which in turn is the successor of the item in the 4 office. Anything at all can play the role of 6 in a natural-number system. Any *thing*. There are no requirements on the individual items that occupy the places; the requirements are solely on the relations between the items. A consequence of this feature is that in mathematics there is no difference between simulating a structure and exemplifying it.[21]

The freestanding nature of mathematical structures and the "formal" or "structural" nature of their relations are connected to each other. Suppose that a structure *S* has a nonformal relation, say, one that involves a physical magnitude, such as distance. For example, let it be required that the occupants of two particular places be ninety feet apart. Then *S* cannot be free-standing. The places of *S* that bear the distance relations cannot be filled with abstract objects, for example, because such objects do not have distance relations with each other. Similarly, if some relations of *S* require the objects to be movable, then objects that cannot be easily moved, like stars, cannot fill those places. If, on the other hand, all of the relations in a structure are formal, then any objects at all can fill the places. Insofar as the relations are formal, the structure is freestanding.

As we have seen, the places in the natural-number structure can be occupied by places in other structures (like finite von Neumann ordinals). Even more, the places in the natural-number structure can be occupied by the same or other natural numbers. The even numbers and the natural numbers greater than 4 both exemplify the natural-number structure. In the former, 6 plays the 3 role, and in the latter 8 plays the 3 role. In the series of primes, 7 plays the 3 role. The *ante rem* account of structures easily accommodates this freestanding feature of mathematical structures. Places of structures, considered from the places-are-objects perspective, can occupy places in the same or in different structures.

As noted earlier, there is one trivial example. In the system of natural numbers, 3 itself plays the 3 role. That is, the number 3, in the places-are-objects perspective, occupies the 3 office. The natural-number structure itself exemplifies the natural-

21. I am indebted to Diana Raffman and Michael Tye for several insightful conversations on these matters.

number structure. Hand [1993] argues that the freestanding feature of structures, construed *ante rem*, invites a Third Man regress. It is a Third ω. Both the system of finite von Neumann ordinals and the system of Zermelo numerals exemplify the natural-number structure. So do the natural numbers themselves, qua places-are-objects. Thus, the argument goes, we need a new structure, a super natural-number structure, which the original natural-number structure shares with the finite von Neumann ordinals and the Zermelo numerals. Actually, we need no such thing. The best reading of "the natural-number structure itself exemplifies the natural-number structure" is something like "the places of the natural-number structure, considered from the places-are-objects perspective, can be organized into a system, and this system exemplifies the natural-number structure (whose places are now viewed from the places-are-offices perspective)." In each case, there is no need for a Third.[22] The natural-number structure, as a system of places, exemplifies itself. The Third ω is the first ω.

Eliminative in re structuralist programs do not fully accommodate the freestanding nature of mathematical structures. As we have seen, on both eliminative options, there is no places-are-objects perspective. On this view, numbers are not objects, and so cannot be organized into systems. Strictly speaking, on either eliminative program, neither the natural-number structure nor numbers exist (as objects), and so such items cannot fill the places of structures.

On the other side of the ledger, there is not even a prima facie Third Man concern with eliminative in re structuralism. In general, if a structure is not freestanding, then there is no problem with a Third. No one would say, for example, that the baseball-defense structure is itself a baseball defense. You cannot play ball with the places of a structure; people are needed. Thus, if one is still bothered by the possibility of a Third ω, it might be best to eschew freestanding structures and adopt an eliminative program.

Parsons [1990] delimits an important distinction between different levels of *abstracta*: "Pure mathematical objects are to be contrasted not only with concrete objects, but also with certain abstract objects, that I call quasi-concrete, because they are directly 'represented' or 'instantiated' in the concrete. Examples might be geometric figures (as traditionally conceived), symbols whose tokens are physical utterances or inscriptions, and perhaps sets or sequences of concrete objects" (p. 304). Parsons's contrast is aligned with the matters under discussion here. His quasi-concrete objects are naturally organized into systems; his point is that the structures of such systems are not freestanding. Prima facie, only inscriptions of some sort can exemplify linguistic types, and, at least traditionally, only points in space can exemplify geometric points.

Parsons argues that a "purely structuralist account does not seem appropriate for quasi-concrete objects, because the representation relation is something additional

22. See Dieterle [1994, chapter 1] for a further discussion of the Third Man argument in the context of structuralism and a more detailed reply to Hand [1993]. Dieterle relates the present issue to some contemporary treatments of the traditional Third Man problem.

to intra structural relations." Because quasi-concrete objects "have a claim to be the most elementary mathematical objects," structuralism is not the whole story about mathematics. There is more to mathematics than what is indicated by the slogan "the science of structure."

Several responses to Parsons's charge are available. First, the structuralist might argue that quasi-concrete objects are not really mathematical objects. This is surely counterintuitive, because sets, geometric figures, and strings seem to be mathematical if anything is. Second, one might argue that Parsons's distinction is ill founded. There are not really any levels of *abstracta* to accommodate. As indicated, I demur from this option. The distinction is well taken, if not precise. Third, and less radically, one might claim that structures of quasi-concrete *abstracta* lie on the border between mathematical and ordinary structures and that the structures of quasi-concrete *abstracta* can be replaced with freestanding, formal ones. As far as mathematics goes, this replacement is virtually without loss. In particular, we can concede Parsons's point and try to delimit the role of quasi-concrete objects, showing how they are perhaps restricted to motivation and epistemology. The latter strategy is consistent with Parsons's own conclusions.

A brief look at the history of mathematics shows that the structures of quasi-concrete objects have been gradually supplanted by freestanding structures whose relations are formal. Consider geometry. From antiquity through the eighteenth century, geometry was the study of physical space, perhaps idealized. The points and lines of Euclidean geometry are points and lines of space. Thus, they are concrete or quasi-concrete, and their structure is not freestanding. Moreover, relations like "betweenness" and "congruence" are not formal. For point *B* to be between *A* and *C*, it must lie on a line connecting them, with *A* (physically) on one side and *C* on the other. For two line segments to be congruent, they must be the same length. Because of various internal developments, however, geometry came to be construed more and more formally, and thus more and more structurally. Along the way, nonspatial systems were construed as exemplifying the structure of various geometries. In analytic geometry, for example, the structure of Euclidean geometry is exemplified with a system of triples of real numbers. There is, of course, a "betweenness" relation of real analysis, in which π is between 3.1 and 3.2. This relation is similar to the "betweenness" of geometry, but the similarity is just structural, or formal. Real numbers are not actually parts of locations in space—but, as we now know, the structures are the same.[23] The subsequent use of idealized "points" and the use of analogues of complex analysis in geometry provided the crucial motivation for the move to a formal, structural construal of geometry. It became ever more difficult to understand the techniques, "constructions," and even the ontology of geometry as connected essentially to physical space. In chapter 5, I take a further look at some of these developments.

Unlike geometry, string theory does not have a long and hallowed history, but one can see a similar, if abbreviated development. Intuitively, strings are linguistic

23. Dedekind [1872] effectively exploited the structural similarities between the points on a line and real numbers in his celebrated treatment of continuity.

types, the forms of *written marks*. Because only written marks can be tokens of these types, the structures are not freestanding. Moreover, the central operation of the theory is concatenation, which is not formal. Two strings are concatenated when they are placed (physically) next to each other. Thus, strings are quasi-concrete. Nevertheless, it is not much of a stretch to see a more formal look to contemporary string theory. For one thing, it is now common to consider more abstract models of string theory. With Gödel numbering, for example, logicians consider natural numbers or sets to be strings, in which case concatenation is given an interpretation as an arithmetic or set-theoretic operation. In other words, systems of numbers or sets exemplify the structure of strings. Moreover, logicians now regularly consider infinitely long strings in a variety of contexts. Surely those strings cannot be instantiated with physical or spoken inscriptions. I return to the nature of strings and their role in epistemology in chapter 4.

Set theory is a most interesting case study for *ante rem* structuralism, and Parsons himself treats it at some length (in [1990] and in much more detail in [1995]). Like geometry and string theory, the intuitive ideas that underlie and motivate current axiomatic set theory are not structural. Teachers and elementary textbooks usually define a set to be a *collection* of its elements. Although it is quickly added that a set is not to be thought of as the result of a physical or even a mental collecting, there still seems to be more to membership than a purely formal relation between officeholders. Parsons and others note that there are actually different conceptions of "set" that are invoked in the motivation of axiomatic set theory.[24] One of them "is the conception of a set as a totality 'constituted' by its elements, so that it stands in some kind of ontological dependence on its elements, but not vice versa. This would give to the membership relation some additional content, still very abstract but recognizably more than a pure structuralism would admit" (Parsons [1990, 332]). A second motivating notion is the idea of a set as the extension of a predicate, so that each set is somehow ontologically dependent on the predicate and not on its elements. Parsons argues that neither of these motivating notions quite matches the one delivered in Zermelo-Fraenkel set theory. The Zermelo-Fraenkel notion departs "from concrete intuition at least when it admits infinite sets," and it departs from the predicative notion when it "admits impredicatively defined sets" ([1990, 336]). The upshot is that it may be best to view the structure delivered in modern set theory as freestanding and formal: "The result of these extensions . . . is that the elements of the original [nonstructural] ideas that are preserved in the theory have a purely formal character. For example, the priority of the elements of a set to the set, which is usually motivated by appealing to the first of [the] two informal conceptions is reflected in the theory itself by the fact that membership is a well-founded relation" (p. 336). Well-foundedness can be characterized in a second-order language using no nonlogical terminology: a relation *E* is well-founded if and only if $\forall P[\exists x Px \rightarrow \exists x(Px \,\&\, \forall y(Eyx \rightarrow \neg Py))]$ (see

24. Parsons is a major contributor to a substantial literature on the philosophical underpinnings of axiomatic set theory. See Benacerraf and Putnam [1983, part 4].

Shapiro [1991, chapter 5]). Thus, well-foundedness is a formal property in the Zermelo-Fraenkel structure, and it replaces the quasi-concrete notion of "priority."

Parsons's careful conclusion is that one can and should overcome some intuitive, prereflective reasons for taking the domain of Zermelo-Fraenkel set theory non-structurally. The "universe of sets" is a collection of places "related in a relation called 'membership' satisfying conditions that can be stated in the language of set theory" (Parsons [1990, 332]). The set-theoretic hierarchy is a freestanding structure.[25]

This conclusion generalizes to all of mathematics, or at least to all of pure mathematics: "A structuralist view of higher set theory will then oblige us to accept the idea of a system of objects that is really no more than a structure. But then there is no convincing reason not to adopt it in other domains of mathematics, in particular in the case of the natural numbers. It would be highly paradoxical to accept Benacerraf's conclusion that numbers are not objects and yet accept as such the sets of higher set theory" (Parsons [1990, 332]). Amen. The path urged here, via *ante rem* structuralism, is to accept both numbers and sets, on a par, as objects. They are places-as-objects. Parsons comes close to the same conclusion: "The absence of notions whose non-formal properties really matter . . . makes mathematical objects on the structuralist view continue to seem elusive, and encourages the belief that there is some scandal to human reason in the idea that there are such objects. My claim is that something close to the conception of objects of this kind, already encouraged by the modern developments of arithmetic, geometry, and algebra, is forced on us by higher set theory" (p. 335).

So far, so good; but where is the problem? We have spoken of the "transition . . . from dealing with domains of a more concrete nature to speaking of objects only in a purely structural way." The problem is that this transition "leaves a residue. The more concrete domains, often of quasi-concrete objects, still play an ineliminable role in the explanation and motivation of mathematical concepts and theories. . . . The explanatory and justificatory role of more concrete models implies . . . that [structuralism] is not the right legislation even for the interpretation of modern mathematics" (p. 338). So Parsons proposes a caveat to structuralism. If we kick away the ladder of the concrete or quasi-concrete objects, then we cannot motivate or even justify some mathematical theories. For example, teachers often refer to sequences of linguistic types in order to motivate the natural-number structure. Hilbert [1925] himself invoked a collection of sequences of strokes, a quasi-concrete structure, to define the objects of finitary mathematics. Parsons also notes that at least the lower portions of the set-theoretic hierarchy have quasi-concrete instantiations. The quasi-concrete seem to be a main exemplar of mathematical structures.

Maddy [1990, 174–175] makes a related point, claiming that there is an epistemological disanalogy between arithmetic and set theory. She agrees that a structuralist understanding of the natural numbers is "appealing partly because our understanding of arithmetic doesn't depend on which instantiation of the number structure

25. Hellman [1989, 53–73] also treats set theory structurally, but not as the theory of a freestanding structure.

we choose to study." Set theory is different: "Experience with any endless row might lead us to think that every number has a successor, but it is experience with sets themselves that produces the intuitive belief that any two things can be collected into a set. . . . [T]hough any instantiation of the natural number structure can give us access to information about that structure, our information about the set-theoretic hierarchy structure comes from our experience with one particular instantiation."

I can put the point in present terms. Recall the coherence principle in the development of structuralism in section 4:

If Φ is a coherent formula in a second-order language,
then there is a structure that satisfies Φ.

This is a central (albeit vague) principle of structuralism. There is no getting around the fact that systems of quasi-concrete objects play a central role in convincing us that some mathematical theories are coherent and thus characterize well-defined structures (especially because consistency is not sufficient for coherence). The best way to show that a structure exists is to find a system that exemplifies it. At some point, we have to appeal to items that are not completely structural (unless somehow everything—every thing—is completely structural; see chapter 8). And at some point, we have to appeal to items that are not completely concrete, given the size of most mathematical structures. So we appeal to the quasi-concrete. If we completely eschew quasi-concrete systems, we lose any motivation or intuitive justification that even arithmetic and geometry (and string theory) are well motivated or even coherent.

Another reason to think that the quasi-concrete cannot be eliminated is that I have appealed to quasi-concrete items in order to characterize the very notion of a *structure*. Recall that a structure is the form of a system, and a system is a collection of objects under various relations. The notion of "collection" is an intuitive one. There is something fishy about appealing to the set-theoretic hierarchy, as a freestanding *ante rem* structure, in order to explicate the notion of "collection" in the characterization of "system" and thus "structure." Where did we get on this merry-go-round, and how do we get off?

A related point concerns the practice of characterizing specific structures using a second-order language. Such languages make literal use of intuitive notions like "predication" or "collection." A crucial step in the defense of second-order languages is that we have a serviceable, intuitive grasp of notions like "all subsets" (see Shapiro [1991]). This notion is also quasi-concrete. Boolos [1984] (and [1985]) has proposed an alternate understanding of monadic, second-order logic, in terms of plural quantifiers, which many philosophers have found attractive. Parsons [1995] contains an insightful discussion of pluralities in the context of structuralism. He shows that we are dealing with yet another quasi-concrete notion.

In all cases, then, the conclusion is the same. We can try to hide the quasi-concrete, but there is no running away from it. Parsons's caveat is well taken. However, the caveat does not undermine the main ontological thesis of *ante rem* structuralism, the idea that the subject matter of a branch of pure mathematics is well construed as a class of freestanding structures with formal relations. The role of concrete and quasi-concrete systems is the motivation of structures and the justification that structures with certain properties exist. The history of mathematics shows a trend from concrete and quasi-concrete systems to more formal, freestanding structures. There is no contradiction in the idea of a system of quasi-concrete objects' exemplifying a freestanding *ante rem* structure. Nevertheless, Parsons's caveat is a reminder not to forget the roots of each theory. Without reference to the quasi-concrete, some mathematical theories are left unmotivated and unjustified. In the next chapter, I turn to the epistemology of structuralism. However, to recall the dialectical pattern of this book, I am not finished with ontological matters. First, a brief interlude to note a connection with the philosophy of mind.

6 Addendum: Function and Structure

In contemporary philosophy, several views go by the name of "functionalism." If we limit ourselves to philosophy of mind and philosophy of psychology, the framework of this chapter provides convenient terminology in which to recapitulate some common themes. Functionalism is an in re structuralism of sorts.

Ned Block [1980] describes three types of functionalism. First, *functional analysis* is a research strategy aimed at finding explanations of a certain type: "A functional explanation is one that relies on a decomposition of a system into its component parts; it explains the working of the system in terms of the capacities of the parts and the way that the parts are integrated with one another. For example, we can explain how a factory can produce refrigerators by appealing to the capacities of the various assembly lines, their workers and machines, and the organization of these components" (p. 171). Block uses the word "system" to refer to a collection of related objects or people, just as I do here. A functional explanation is an account of what a system is like and what it does. The explanation begins by noting that the system exemplifies a certain structure and then invokes features of the structure itself, ignoring properties of the system (and its constituents) that do not relate to the structure. In the sketch cited, the only relevant facts about the people on the assembly lines are their relationships to each other and to the items playing other roles in the structure. Their hair color and gender do not matter. I take it that an explanation of why a shift defense is effective against a left-handed pull hitter is also a functional analysis.

Second, Block defines *computation-representation functionalism* to be a special case of functional analysis in which "psychological explanation is seen as akin to providing a computer program for the mind. . . . [F]unctional analysis of mental processes [is taken to] the point where they are seen to be composed of [mechanical] computations. . . . The key notions . . . are representation and computation. Psychological states are seen as systematically representing the world via a language of thought, and psychological processes are seen as computations involving these representations" (p. 179). Again, the connections with structuralism are straightforward. According to computation-representation functionalism, the theorist is to find an equivalence between psychological processes and something like a natural language, a formal language, or a computer language. This equivalence is of a piece with isomorphism and structure equivalence. The plan is to establish a systematic correlation between microprocesses and something like grammatical transformation rules

or machine-language instructions. The brain is an ensemble of microprocesses and is seen to be "equivalent" to either the functioning of language as a whole or to the function of a programmed computer. Much of the work in the emerging discipline of cognitive science can be seen as an attempt to fit this mold.

Block's third theme, *metaphysical functionalism*, has the most interesting connections with structuralism. This functionalism is not, or is not merely, a theory of psychological explanation but is rather a theory of the nature of the mind and mental states like pain, belief, and desire. The metaphysical functionalist is "concerned not with how mental states account for behavior, but with what they are" (p. 172). According to the metaphysical functionalist, mental states are functional states. In present terms, the metaphysical functionalist characterizes a structure, and *identifies* mental states with places in this structure. In other words, a functional state just is a place in a structure. As Block puts it, "Metaphysical functionalists characterize mental states in terms of their causal roles, particularly, in terms of their causal relations to sensory stimulations, behavioral outputs, and other mental states. Thus, for example, a metaphysical functionalist theory of pain might characterize pain in part in terms of its tendency to be caused by tissue damage, by its tendency to cause the desire to be rid of it, and by its tendency to produce action designed to separate the damaged part of the body from what is thought to cause the damage" (p. 172). According to metaphysical functionalism, then, pain is to be characterized in terms of its relation to other mental states and to certain inputs and outputs. This is not much different from characterizing a natural number in terms of its relations to other numbers. Of course, the characterization of the natural numbers is rigorous and precise, whereas the above characterization of pain is admittedly inadequate. The metaphysical functionalist envisions a program for filling it in, much as the Peano postulates fill in the details of the natural-number structure.

Block describes this functionalist program in terms much like those of the present chapter. He envisions that we start with a psychological theory T that describes the relations among pain, other mental states, sensory inputs, and behavioral outputs. Reformulate T as a single sentence, with mental-state terms all as singular terms. So T has the form

$$T(s_1, \ldots, s_n),$$

where $s_1, \ldots, s_n$ are the aforementioned singular terms for mental states. Now, if s_i is the term for "pain," then we can define an organism y to *be in pain* as follows, adapting the technique of Ramsey sentences:

$$y \text{ has pain if and only if } \exists x_1 \ldots \exists x_n (T(x_1, \ldots, x_n) \text{ and } y \text{ has } x_i).$$

In other words, y is in pain if and only if y has states that relate to each other in various ways and tend to produce such and such outputs when confronted with thus and so inputs. Block illustrates this with a nonmental example: "Consider the 'theory' that says: 'The carburetor mixes gasoline and air and sends the mixture to the ignition chamber, which, in turn . . . ' [Block's ellipsis] Let us consider 'gasoline' and 'air' to be input terms, and let x_1 replace 'carburetor', and x_2 replace 'ignition chamber'" (p. 175). Then, according to the metaphysical functionalist, we can say that y is a carburetor if and only if "$\exists x_1 \ldots \exists x_n$[(The x_1 mixes gasoline and air and sends the mixture to the x_2, which, in turn, . . .) and y is an x_1]" (p. 175).

Block thus uses the term "functional state" for something like the present "place in a structure." To continue the automotive examples, "valve-lifter" is a functional term, because anything "that lifts valves in an engine with a certain organizational structure is a valve-lifter." Similarly, "carburetor" is a functional term, as are mental-state terms like "pain," "belief," and "desire." They all denote places in structures. Block uses "structural term" to refer to something like an officeholder. For example, "camshaft" is said to be a structural term, relative to "valve-lifter," because a "camshaft is *one* kind of device for lifting valves" (p. 174). In contemporary philosophy, "C-fiber" would be a structural term.

Presumably, the theory of pain is more sophisticated than the theory of carburation, but the form of the metaphysical functionalist analysis is the same. Notice that the structuralist definition of the natural numbers also has this form. We say that a given object z plays the 2 role in a certain system S if and only if S satisfies the Peano axioms and z is the S-successor of the S-successor of the zero object of S.

The structures delimited by metaphysical functionalism are not freestanding, and most of their places are not formal. Carburetors must *mix gasoline and air*. One cannot locate a carburetor in anything but an internal-combustion device. Computers and humans do not have systems that mix gasoline and air in preparation for combustion, and so computers and humans do not have carburetors. In the case of (physical) pain, the indicated inputs and outputs must also be held fixed. If an organism does not have something like the capacity for tissue damage, then it is not capable of pain. We can locate the exemplification of the pain structure in humans and animals, and perhaps we can locate a pain system in extraterrestrials and in future machines, but certainly not in abstract objects or planets.

If the concepts given functional definitions are made a little more formal and freestanding—along the lines of the development of geometry—then borderline cases of the concepts are produced. Eventually, the boundary with mathematics is crossed. Suppose there were a device that mixed two things other than gasoline and air, and sent the mixture to an ignition chamber. The functional definition would be something like this:

$$\exists w_1 \exists w_2 \exists x_1 \ldots \exists x_n[(\text{The } x_1 \text{ mixes } w_1 \text{ and } w_2 \text{ and sends the mixture to the } x_2, \text{ which, in turn, } \ldots) \text{ and } y \text{ is an } x_1].$$

Would the y be a carburetor? Perhaps. Suppose it did not mix the two things but did something else to them, and rather than sending the result somewhere, did something else with it:

$$\exists X \exists w_1 \exists w_2 \exists z_1 \ldots \exists z_m \exists x_1 \ldots \exists x_n[(Xx_1w_1w_2 \,\&\, \ldots) \,\&\, y = x_1].$$

Clearly, this does not define "carburetor" in any sense of the word. At the limit, we would produce a purely formal definition, which characterizes a freestanding structure. In theory, any object could play the x_1 role, including the number 2 and Julius Caesar. There would be systems of sets and numbers that exemplify the resulting structure. "Carburetor" would be an object of pure mathematics, and carburetor theory would have gone the route of geometry, dealing with an *ante rem* structure.

4

Epistemology and Reference

1 Epistemic Preamble

For a philosopher who takes the full range of contemporary mathematics seriously, the most troublesome issues lie in epistemology. The situation is especially acute for traditional realism in ontology. Almost every realist agrees that mathematical objects are *abstract*. Although there is surprisingly little discussion of the abstract/concrete dichotomy in the literature,[1] the idea seems to be that *abstracta* are not located in space-time and are (thus) outside the causal nexus. We do not bump up against abstract objects, nor do we see them or hear them. If mathematical objects are like this, then how can we know anything about them? How can we formulate warranted beliefs about mathematical objects and have any confidence that our beliefs are true? Most of us believe that every natural number has a successor, and I would hope that at least some of us are fully justified in this belief. But how?

Benacerraf's celebrated [1973] develops this difficulty into an objection to realism in ontology by invoking the so-called causal theory of knowledge. According to this epistemology, there is no knowledge of a type of object unless there is some sort of causal connection between the knower and at least samples of the objects. On this account, it seems, knowledge of *abstracta* is impossible, because, by definition, there is no causal contact with such objects. In recent decades, the causal theory of knowledge has been roundly criticized from several quarters, and not just by friends of *abstracta*. There is no consensus on any epistemology, causal or otherwise. There is no leading contender.

1. One very notable exception to the lack of discussion on the abstract/concrete dichotomy is the fine study in Hale [1987]. See also Zalta [1983]. As noted in previous chapters, mathematicians use the "abstract/concrete" label for a different distinction. For them, arithmetic is a "concrete" study, because its subject is a single structure (up to isomorphism). Group theory is more "abstract." The mathematicians' "abstract/concrete" is my "algebraic/nonalgebraic."