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7 The Greek Rationalization of Nature

Mathematics is the gate and key of the sciences.
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1. *The Inspiration for Greek Mathematics*

Unfortunately, except for occasional hints, the Greek classics, such as Euclid's *Elements*, Apollonius' *Conic Sections*, and the geometrical works of Archimedes, give no indication of why these authors investigated their subjects. They give only the formal, polished deductive mathematics. In this respect, the Greek texts are no different from modern mathematics textbooks and treatises. Such books seek only to organize and present the mathematical results that have been attained and so omit the motivations for the mathematics, the clues and suggestions for the theorems, and the uses to which the mathematical knowledge is put.

To understand why the Greeks created so much vital mathematics, one must investigate their objectives. It was the urgent and irrepressible desire of the Greeks to understand the physical world that impelled them to create and value mathematics. Mathematics was part and parcel of the investigation of nature and the key to comprehension of the universe, for mathematical laws are the essence of its design.

What evidence do we have that this was the role of mathematics? It is difficult to demonstrate that any one theorem or body of theorems was created for a specific purpose because we do not have enough information about the Greek mathematicians. Ptolemy's direct statement that he created trigonometry for astronomy is an exception. However when one finds that Eudoxus was primarily an astronomer and that Euclid wrote not just the *Elements* but the *Phaenomena* (a work on the geometry of the sphere as applied to the motion of the sphere of stars), the *Optics* and *Catoptrica*, the *Elements of Music*, and small works on mechanics, all of which were mathematical, one cannot escape the conclusion that mathematics was more than an isolated discipline. Knowing how the human mind works and knowing in great detail how men such as Euler and Gauss worked, we may be fairly certain that the investigations in astronomy, optics, and music must have suggested mathematical problems, and it is most likely that the motivation for the mathematics was its application to these other areas. It is also relevant that the geometry of the sphere, known in Greek times as "sphaeric," was studied just as soon as astronomy became mathematical, which happened even before Eudoxus' time. The word "sphaeric" meant "astronomy" to the Pythagoreans.

Fortunately the inferences we may draw from the works of the mathematicians, though reasonable enough, are established beyond doubt by the overwhelming evidence in the writings of the Greek philosophers, many of whom were also prominent mathematicians, and of the Greek scientists. The bounds of mathematics were not mathematics proper. In the classical period mathematics comprised arithmetic, geometry, astronomy, and music; and in the Alexandrian period, as we have already noted in Chapter 5, the divisions of the mathematical sciences were arithmetic (theory of numbers), geometry, mechanics, astronomy, optics, geodesy, canonic (musical harmony), and logistics (applied arithmetic).

2. *The Beginnings of a Rational View of Nature*

The civilizations that preceded the Greek or were contemporary with it regarded nature as chaotic, mysterious, capricious, and terrifying. The happenings in nature were manipulated by gods. Prayers and magic might induce the gods to be kind and even to perform miracles but the life and fate of man were entirely subject to their will.

From the time our knowledge of Greek civilization and culture begins to be reasonably definite and specific, that is, from about 600 B.C., we find among the intellectuals a totally new attitude toward nature: rational, critical, and secular. Mythology was discarded, as was the belief that the gods manipulate man and the physical world according to their whims. The new doctrine holds that nature is orderly and functions invariably according to a plan. Moreover, the conviction is manifest that the human mind is powerful and even supreme; not only can the ways of nature be learned by man, but he can even predict the occurrences.

It is true that the rational approach was entertained only by the intellectuals, a small group in both the classical and Alexandrian periods. Whereas these men opposed the attribution of events to gods and demons and defied the mysteries and terrors of nature, people in general were deeply religious and believed that the gods controlled all events. They accepted mystical doctrines and superstitions as credulously as did the Egyptians and the Babylonians. In fact Greek mythology was vast and highly developed.

The Ionians began the task of determining the nature of reality. We shall not describe the qualitative theories of Thales, Anaxagoras, and their colleagues, each of whom fixed on a single substance persisting through all apparent change. The underlying identity of this prime substance is conserved but all forms of matter can be explained in terms of it. This natural philosophy of the Ionians was a series of bold speculations, shrewd guesses, and brilliant intuitions rather than the outcome of extensive and careful scientific investigations. They were perhaps a little too eager to see the whole picture and so naïvely jumped to broad conclusions. But they did substitute material and objective explanations of the structure and design of the universe for the older mythical stories. They offered a reasoned approach in place of the fanciful and uncritical accounts of the poets and they defended their contentions by reason. At least these men dared to tackle the universe with their minds and refused to rely upon gods, spirits, ghosts, devils, angels, and other mythical agents.

3. *The Development of the Belief in Mathematical Design*

The decisive step in removing the mystery, mysticism, and arbitrariness from the workings of nature and in reducing the seeming chaos to an understandable ordered pattern was the application of mathematics. The first major group to offer a rational and mathematical philosophy of nature were the Pythagoreans. They did draw some inspiration from the mystical side of Greek religion; their religious doctrines centered about the purification of the soul and its redemption from the taint and prison of the body. The members lived simply and devoted themselves to the study of philosophy, science, and mathematics. New members were pledged to secrecy at least as to religious beliefs and required to join up for life. Membership in the community was open to men and women.

The Pythagoreans' religious thinking was undoubtedly mystical, but their natural philosophy was decidedly rational. They were struck by the fact that phenomena that are most diverse from a qualitative point of view exhibit identical mathematical properties. Hence, mathematical properties must be the essence of these phenomena. More specifically, the Pythagoreans found this essence in number and in numerical relationships. Number was their first principle in the explanation of nature. All objects were made up of points or "units of existence" in combinations corresponding to the various geometrical figures. Since they thought of numbers both as points and as elementary particles of matter, number was the matter and form of the universe and the cause of every phenomenon. Hence the Pythagorean doctrine "All things are numbers." Says Philolaus, a famous fifth-century Pythagorean, "Were it not for number and its nature, nothing that exists would be clear to anybody either in itself or in its relation to other things. . . . You can observe the power of number exercising itself not only in the affairs of demons and gods but in all the acts and the thoughts of men, in all handicrafts and music."

The reduction of music, for example, to simple relationships among numbers became possible for the Pythagoreans when they discovered two facts: first, that the sound caused by a plucked string depends upon the length of the string; and second, that harmonious sounds are given off by equally taut strings whose lengths are to each other as the ratios of whole numbers. For example, a harmonious sound is produced by plucking two equally taut strings, one twice as long as the other. In our language, the interval between the two notes is an octave. Another harmonious combination is formed by two strings whose lengths are in the ratio 3 to 2; in this case the shorter one gives forth a note called the fifth above that given off by the first string. In fact, the relative lengths in every harmonious combination of plucked strings can be expressed as ratios of whole numbers. The Pythagoreans also developed a famous Greek musical scale. Though we shall not devote space to the music of the Greek period, we note that many Greek mathematicians, including Euclid and Ptolemy, wrote on the subject, especially on harmonious combinations of sounds and the construction of scales.

The Pythagoreans reduced the motions of the planets to number relations. They believed that bodies moving in space produce sounds; perhaps this was suggested by the swishing of an object whirled on the end of a string. They believed, further, that a rapidly moving body gives forth a higher note than one that moves slowly. Now according to their astronomy the greater the distance of a planet from the earth the more rapidly it moved. Hence the sounds produced by the planets, which we do not hear because we are accustomed to them from birth, varied with their distances from the earth and all harmonized. But since this "music of the spheres," like all harmony, reduced to no more than number relationships, so did the motions of the planets.

The Pythagoreans and probably Pythagoras himself wanted not just to observe and describe the heavenly motions but to find regularity in them. The idea of uniform circular motion, seemingly obvious in the case of the moon and sun, suggested that all the planetary motions were explainable in terms of uniform circular motions. The later Pythagoreans made a more striking break with tradition; they were the first to believe that the earth was spherical. Moreover, because 10 was their ideal number, they decided that the moving bodies in the heavens must be 10 in number. First, there was a central fire around which the heavenly bodies, *including the earth*, moved. They knew five planets in addition to the earth. These six bodies, the sun, the moon, and the sphere to which the stars were attached made only 9 moving bodies. Hence they asserted the existence of a tenth one, called the counter-earth, which also revolved around the central fire. We cannot see this tenth one because it moves at exactly the same speed as the earth on the

opposite side of the central fire and also because the inhabited part of earth faces away from the central fire. Here we have the first theory to put the earth in motion. However, the Pythagoreans did not assert the rotation of the earth; rather, the sphere of fixed stars revolves about the center of the universe.

The belief that the celestial bodies are eternal, divine, perfect, and unchangeable and that the sublunar bodies, that is the earth and (according to the Greeks) the comets, are subject to change, decomposition, decay, and death may also have come from the Pythagoreans. The doctrine of uniform circular motion and the distinction between celestial and sublunar bodies became embedded in Greek thought.

Other features of nature also "reduced" to number. The numbers 1, 2, 3, and 4, the *tetractys*, were especially valued because they added up to 10. In fact the Pythagorean oath is reported to have been: "I swear in the name of the Tetractys which has been bestowed on our soul. The source and roots of the everflowing nature are contained in it." The Pythagoreans asserted that nature was composed of fournesses; for example, point, line, surface, and solid, and the four elements, earth, air, fire, and water. The four elements were also central in Plato's natural philosophy. Because 10 was ideal, 10 represented the universe. The ideality of 10 required that the whole universe be describable in terms of 10 categories of opposites: odd and even, bounded and unbounded, good and evil, right and left, one and many, male and female, straight and curved, square and oblong, light and darkness, and rest and motion.

Clearly, Pythagorean philosophy mingled serious thoughts with what we would consider fanciful, useless, and unscientific doctrines. Their obsession with the importance of numbers resulted in a natural philosophy that certainly had little correspondence with nature. But they did stress the understanding of nature, not, like the Ionians, through a single substance, but through the formal structure of number relationships. Moreover they and the Ionians both saw that underlying mere sense data there must be a harmonious account of nature.

We can now see why the discovery of incommensurable lengths was so disastrous to Pythagorean philosophy: a ratio of incommensurable lengths could not be expressed as a ratio of whole numbers. In addition, they had believed that a line is made up of a finite number of points (which they identified with physical particles); but this could not be the case for a length such as $\sqrt{2}$. Their philosophy, based on the primariness of the whole numbers, would have been shattered if they had accepted irrationals as numbers.

Because the Pythagoreans "reduced" astronomy and music to number, these subjects came to be linked to arithmetic and geometry; these four were regarded as the mathematical subjects. They became and remained part of the school curriculum even into medieval times, when they were called, collectively, "the quadrivium." As we have noted, the Pythagorean interest in arithmetic (i.e. the theory of numbers) was due not to the purely aesthetic value of that subject but to a search for the meaning of natural phenomena in numerical terms; and this value caused the emphasis on special proportions and on triangular, square, pentagonal, and higher forms into which numbers could be arranged. Further, it was the Pythagorean natural philosophy centering about number that gave the subject importance with such men as Nichomachus. In fact, modern science adheres to the Pythagorean emphasis on number—though, as we shall see, in a much more sophisticated form—while the purely aesthetic modern theory of numbers derives from Pythagorean arithmetic per se.

The philosophers who came chronologically between the Pythagoreans and Plato were equally concerned with the nature of reality but did not involve mathematics directly. The arguments and views of men such as Parmenides (5th cent. B.C.), Zeno (5th cent. B.C.), Empedocles (*c.* 484–*c.* 424 B.C.), Leucippus (*c.* 440 B.C.), and Democritus (*c.* 460–*c.* 370 B.C.) were, like those of their Ionian predecessors, qualitative. They made broad assertions about reality that were, at best, barely suggested by observation. Nevertheless, each affirmed that nature is intelligible and that reality can be grasped by thought. Each was a link in the chain that led to the mathematical investigation of nature. Leucippus and Democritus are notable because they were the most explicit in affirming the doctrine of atomism. Their common philosophy was that the world is composed of an infinite number of simple, eternal atoms. These differ in shape, size, order, and position, but every object is some combination of these atoms. Though geometrical magnitudes are infinitely divisible, the atoms are ultimate indivisible particles. (The word *atom* in Greek means indivisible.) Hardness, shape, and size are physically real properties of the atoms. All other properties, such as taste, heat, and color are not in the atoms but in the perceiver; thus sensuous knowledge is unreliable because it varies with the perceiver. Like the Pythagoreans, the atomists asserted that the reality underlying the constantly changing diversity of the physical world was expressible in terms of mathematics and, moreover, that the happenings in this world were strictly determined by mathematical laws.

Plato, the foremost Pythagorean next to Pythagoras, was the most influential propagator of the doctrine that the reality and intelligibility of the physical world can be comprehended only through mathematics. For him there was no question that the world was mathematically designed, for "God eternally geometrizes." The world perceived by the senses is confused and deceptive and in any case imperfect and impermanent. Physical knowledge is unimportant, because material objects change and decay; thus the direct study of nature and purely physical investigations are worthless. The

physical world is but an imperfect copy of the ideal world, the one that mathematicians and philosophers should study. Mathematical laws, eternal and unchanging, are the essence of reality.

Plato went further than the Pythagoreans in wishing not merely to understand nature through mathematics but to substitute mathematics for nature itself. He believed that a few penetrating glances at the physical world would supply some basic truths with which reason could then carry on unaided. From that point there would be no nature, just mathematics, which would substitute for physical investigations as it does in geometry.

Plato's attitude toward astronomy illustrates his position on the knowledge to be sought. This science is not concerned with the movements of the visible heavenly bodies. The arrangement of the stars in the heavens and their apparent movements are indeed wonderful and beautiful to behold, but mere observation and explanation of the motions fall far short of true astronomy. Before we can attain to the latter we "must leave the heavens alone," for true astronomy deals with the laws of motion of true stars in a mathematical heaven of which the visible heaven is but an imperfect expression. Plato encourages devotion to a theoretical astronomy, whose problems please the mind, not the eye, and whose objects are apprehended by the mind, not visually. The varied figures that the sky presents to the eye are to be used only as diagrams to assist the search for the higher truths. The uses of astronomy in navigation, calendar-reckoning, and the measurement of time were alien to Plato.

Plato's views on the role of mathematics in astronomy are an integral part of his philosophy, which held that there is an objective, universally valid reality consisting of forms or ideas. These realities were independent of human beings and were immutable, eternal, and timeless. We become aware of these ideas through recollection or anamnesis; although they are present in the soul, it must be stimulated to recall them or fetch them up from its depths. These ideas are the only reality. Included among them but occupying a lesser rank are mathematical ideas, which are regarded as intermediate between the sensible world and such higher ideas as goodness, truth, justice, and beauty. In this comprehensive philosophy, mathematical ideas played a double role; not only were they part of reality themselves but, as we have already pointed out in Chapter 3, they helped train the mind to view eternal ideas. As Plato put it in Book VII of *The Republic*, the study of geometry made easier the vision of the idea of goodness: "Geometry will draw the soul toward truth, and create the spirit of philosophy. . . ."

Aristotle, while deriving many ideas from his teacher Plato, had a quite different concept of the study of the real world and of the relation of mathematics to reality. He criticized Plato's otherworldliness and his reduction of science to mathematics. Aristotle was a physicist; he believed in material things as the primary substance and source of reality. Physics and science generally must study the physical world to obtain truths; genuine knowledge is obtained from sense experience by intuition and abstraction. Then reason can be applied to the knowledge so obtained.

Matter alone is not significant. As such it is indeterminate, simply the potentiality of form; matter becomes significant when it is organized into various forms. Form and the changes in matter that give rise to new forms are the interesting features of reality and the real concern of science.

According to Aristotle matter is not, as some earlier Greeks believed, composed of one primitive substance. The matter we see and touch is composed of four basic elements: earth, water, fire, and air. Also, each element has its own characteristic qualities. Earth is cold and dry; water is cold and moist; air is hot and moist; and fire is hot and dry. Hence the qualities of any given object depend upon the proportions of the elements that enter into it; and thereby solidity, hardness, coarseness, and other qualities are determined.

The four elements have other qualities. Earth and water have gravity; air and fire have levity. Gravity causes an element to seek to be at rest at the center of the earth; levity causes it to seek the heavens. Thus by knowing the proportions of the elements that enter into a given object, one can also determine its motion.

Aristotle regarded solids, fluids, and gases as three different types of matter, distinguished by the possession of different substantial qualities. The transition from solid to fluid, for example, meant the loss of one quality and the substitution of another. Thus changing mercury into rigid gold involved taking from mercury the substance that possessed fluidity and substituting some other substance.

Science also had to consider the causes of change. For Aristotle there were four types of causes. The first was the material or immanent cause; for a statue made of bronze, bronze is the immanent cause. The second was the formal cause; for the statue it is the design or shape. The formal cause of harmony is the pattern of 2 to 1 in the octave. The third cause was the effective cause, the agent or doer; the artist and his chisel are the effective cause of the statue. The fourth was the final cause, or the purpose that the phenomenon served; the statue serves to please people, to offer beauty. Final cause was the most important of the four because it gave the ultimate reasons for events or phenomena. Everything had a final cause.

Where was mathematics in this scheme of things? The physical sciences were fundamental to the study of nature, and mathematics helped by describing formal properties such as shape and quantity. It also provided explanations of facts observed in material phenomena. Thus geometry provided the reasons for facts provided by optics and astronomy, and arithmetical proportions could give the reasons for harmony. But mathematics was definitely an abstraction from the real world, since mathematical objects are not independent of or prior to experience. They exist in human minds as a class of ideas mediating between the sensible objects themselves and the essence of objects. Because they are abstracted from the physical world, they are applicable to it; but they have no reality apart from visible and tangible things. Mathematics alone can never provide an adequate definition of substance. Qualitative differences, as among colors, cannot be reduced to differences in geometry. Hence in the study of causes, mathematics can provide at best some knowledge of the formal cause—that is, a description. It can describe what happens in the physical world, can correlate concomitant variations, but can say nothing about the efficient and final causes of movement or change. Thus Aristotle distinguished sharply between mathematics and physics and assigned a minor role to mathematics. He was not interested in prediction.

From this survey we may see that all the philosophers who forged and molded the Greek intellectual world stressed the study of nature for comprehension and appreciation of its underlying reality. From the time of the Pythagoreans, practically all asserted that nature was designed mathematically. During the classical period, the doctrine of the mathematical design of nature was established and the search for the mathematical laws instituted. Though this doctrine did not motivate all of the mathematics subsequently created, once established it was accepted and consciously pursued by most of the great mathematicians. During the time that this doctrine held sway, which was until the latter part of the nineteenth century, the search for the mathematical design was identified with the search for truth. Though a few Greeks—for example, Ptolemy—realized that mathematical theories were merely human attempts to provide a coherent account, the belief that mathematical laws were the truth about nature attracted some of the deepest and noblest thinkers to mathematics.

We should also note, in order to appreciate more readily what happened in the seventeenth century, the Greek emphasis on the power of the mind. Because the Greek philosophers believed that the mind was the most powerful agent in comprehending nature, they adopted first principles that appealed to the mind. Thus the belief that circular motion was the basic type, defended by Aristotle on the ground that the circle is complete whereas a rectilinear figure, because it is bounded by many curves (line segments), is incomplete and therefore secondary in importance, appealed to the mind on aesthetic grounds. That the heavenly bodies should move with only constant or uniform velocity, was a conception which appealed to the mind perhaps because it was simpler than nonuniform motion. The combination of uniform and circular motion seemed to befit heavenly bodies. That the sublunar bodies should be different from the planets, sun, and stars seemed reasonable also, because the heavenly bodies preserved a constant appearance whereas change on earth was evident. Even Aristotle, who stressed abstractions only insofar as they helped to understand the observable world, said that we must start from principles that are known and manifest to the mind and then proceed to analyze things found in nature. We proceed, he said, from universals to particulars, from man to men, just as children call all men father and then learn to distinguish. Thus even the abstractions made from concrete objects presuppose some general principles emanating from the mind. This doctrine, the power of the mind to yield first principles, was overthrown in the seventeenth century.

4. *Greek Mathematical Astronomy*

Let us examine now what the Greeks produced in the mathematical description of natural phenomena. It is from the time of Plato that the several sciences created by the Greeks take on significant content and direction. Though we intend to review Greek astronomy, let us note in passing one aspect of Euclidean geometry. We have already observed that spherical geometry was developed for astronomy. Geometry was in fact part of the larger study of cosmology. Geometric principles were, to the Greeks, embodied in the entire structure of the universe, of which space was the primary component. Hence the study of space itself and of figures in space was of importance to the larger goal. Geometry, in other words, was in itself a science, the science of physical space.

It was Plato who, though fully aware of the impressive number of astronomical observations made by the Babylonians and Egyptians, emphasized the lack of an underlying or unifying theory or explanation of the seemingly irregular motions of the planets. Eudoxus, who was for a while a student at the Academy, took up Plato's problem of "saving the appearances." His answer is the first reasonably complete astronomical theory. He wrote four books on astronomy, *Mirror*, *Phenomena*, *Eight-Year Period*, and *On Speeds*, only fragments of which are known. From these fragments and accounts by other writers, we know the essence of Eudoxus' theory.

The motions of the sun and moon, as viewed from the earth, can be crudely described as circular with constant speed. However, their deviations from circular orbits are great enough to have been observed and to require explanation. The motions of the planets as seen from the earth are even more complex, for during any one revolution they reverse their course, go backward for a while, and then go forward again. Moreover, their speeds on these paths are variable.

To show that the actual, rather complicated, and apparently lawless motions could be understood in terms of simple circular geometrical motions, Eudoxus proposed the following scheme: For any one heavenly body there was a set of three or four spheres, all concentric with the earth as the center, each rotating about an axis. The innermost sphere carried the body, which