KLINE, Mathematical Thought from Ancient to Modern Times Oxford 1972

3 The Creation of Classical Greek Mathematics

> This, therefore, is mathematics: she reminds you of the invisible form of the soul; she gives life to her own discoveries; she awakens the mind and purifies the intellect; she brings light to our intrinsic ideas; she abolishes oblivion and ignorance which are ours by birth.
>
> PROCLUS

1. *Background*

In the history of civilization the Greeks are preeminent, and in the history of mathematics the Greeks are the supreme event. Though they did borrow from the surrounding civilizations, the Greeks built a civilization and culture of their own which is the most impressive of all civilizations, the most influential in the development of modern Western culture, and decisive in founding mathematics as we understand the subject today. One of the great problems of the history of civilization is how to account for the brilliance and creativity of the ancient Greeks.

Though our knowledge of their early history is subject to correction and amplification as more archeological research is carried on, we now have reason to believe, on the basis of the *Iliad* and the *Odyssey* of Homer, the decipherment of ancient languages and scripts, and archeological investigations, that the Greek civilization dates back to 2800 B.C. The Greeks settled in Asia Minor, which may have been their original home, on the mainland of Europe in the area of modern Greece, and in southern Italy, Sicily, Crete, Rhodes, Delos, and North Africa. About 775 B.C. the Greeks replaced various hieroglyphic systems of writing with the Phoenician alphabet (which was also used by the Hebrews). With the adoption of an alphabet the Greeks became more literate, more capable of recording their history and ideas.

As the Greeks became established they visited and traded with the Egyptians and Babylonians. There are many references in classical Greek writings to the knowledge of the Egyptians, whom some Greeks erroneously considered the founders of science, particularly surveying, astronomy, and

arithmetic. Many Greeks went to Egypt to travel and study. Others visited Babylonia and learned mathematics and science there.

The influence of the Egyptians and Babylonians was almost surely felt in Miletus, a city of Ionia in Asia Minor and the birthplace of Greek philosophy, mathematics, and science. Miletus was a great and wealthy trading city on the Mediterranean. Ships from the Greek mainland, Phoenicia, and Egypt came to its harbors; Babylonia was connected by caravan routes leading eastward. Ionia fell to Persia about 540 B.C., though Miletus was allowed some independence. After an Ionian revolt against Persia in 494 B.C. was crushed, Ionia declined in importance. It became Greek again in 479 B.C. when Greece defeated Persia, but by then cultural activity had shifted to the mainland of Greece with Athens as its center.

Though the ancient Greek civilization lasted until about A.D. 600, from the standpoint of the history of mathematics it is desirable to distinguish two periods, the classical, which lasted from 600 to 300 B.C., and the Alexandrian or Hellenistic, from 300 B.C. to A.D. 600. The adoption of the alphabet, already mentioned, and the fact that papyrus became available in Greece during the seventh century B.C. may account for the blossoming of cultural activity about 600 B.C. The availability of this writing paper undoubtedly helped the spread of ideas.

2. *The General Sources*

The sources of our knowledge of Greek mathematics are, peculiarly, less authentic and less reliable than our sources for the much older Babylonian and Egyptian mathematics, because no original manuscripts of the important Greek mathematicians are extant. One reason is that papyrus is perishable; though the Egyptians also used papyrus, by luck a few of their mathematical documents did survive. Some of the voluminous Greek writings might still be available to us if their great libraries had not been destroyed.

Our chief sources for the Greek mathematical works are Byzantine Greek codices (manuscript books) written from 500 to 1500 years after the Greek works were originally composed. These codices are not literal reproductions but critical editions, so that we cannot be sure what changes may have been made by the editors. We also have Arabic translations of the Greek works and Latin versions derived from Arabic works. Here again we do not know what changes the translators may have made or how well they understood the original texts. Moreover, even the Greek texts used by the Arabic and Byzantine authors were questionable. For example, though we do not have the Alexandrian Greek Heron's manuscript, we know that he made a number of changes in Euclid's *Elements*. He gave different proofs and added new cases of the theorems and converses. Likewise Theon of Alexandria (end of 4th cent. A.D.) tells us that he altered sections of the *Elements* in his edition.

The Greek and Arabic versions we have may come from such versions of the originals. However, in one or another of these forms we do have the works of Euclid, Apollonius, Archimedes, Ptolemy, Diophantus, and other Greek authors. Many Greek texts written during the classical and Alexandrian periods did not come down to us because even in Greek times they were superseded by the writings of these men.

The Greeks wrote some histories of mathematics and science. Eudemus (4th cent. B.C.), a member of Aristotle's school, wrote a history of arithmetic, a history of geometry, and a history of astronomy. Except for fragments quoted by later writers, these histories are lost. The history of geometry dealt with the period preceding Euclid's and would be invaluable were it available. Theophrastus (*c.* 372–*c.* 287 B.C.), another disciple of Aristotle, wrote a history of physics, and this, too, except for a few fragments, is lost.

In addition to the above, we have two important commentaries. Pappus (end of 3rd cent. A.D.) wrote the *Synagoge* or *Mathematical Collection*; almost the whole of it is extant in a twelfth-century copy. This is an account of much of the work of the classical and Alexandrian Greeks from Euclid to Ptolemy, supplemented by a number of lemmas and theorems that Pappus added as an aid to understanding. Pappus had also written the *Treasury of Analysis*, a collection of the Greek works themselves. This book is lost, but in Book VII of his *Mathematical Collection* he tells us what his *Treasury* contained.

The second important commentator is Proclus (A.D. 410–485), a prolific writer. Proclus drew material from the texts of the Greek mathematicians and from prior commentaries. Of his surviving works, the *Commentary*, which treats Book I of Euclid's *Elements*, is the most valuable. Proclus apparently intended to discuss more of the *Elements*, but there is no evidence that he ever did so. The *Commentary* contains one of the three quotations traditionally credited to Eudemus' history of geometry (see sec. 10) but probably taken from a later modification. This particular extract, the longest of the three, is referred to as the Eudemian summary. Proclus also tells us something about Pappus' work. Thus, besides the later editions and versions of some of the Greek classics themselves, Pappus' *Mathematical Collection* and Proclus' *Commentary* are the two main sources of the history of Greek mathematics.

Of original wordings (though not the manuscripts) we have only a fragment concerning the lunes of Hippocrates, quoted by Simplicius (first half of 6th cent. A.D.) and taken from Eudemus' lost *History of Geometry*, and a fragment of Archytas on the duplication of the cube. And of original manuscripts we have some papyri written in Alexandrian Greek times. Related sources on Greek mathematics are also immensely valuable. For example, the Greek philosophers, especially Plato and Aristotle, had much to say about mathematics and their writings have survived somewhat in the same way as have the mathematical works.

The reconstruction of the history of Greek mathematics, based on sources

such as we have described, has been an enormous and complicated task. Despite the extensive efforts of scholars, there are gaps in our knowledge and some conclusions are arguable. Nevertheless the basic facts are clear.

3. *The Major Schools of the Classical Period*

The cream of the classical period's contributions are Euclid's *Elements* and Apollonius' *Conic Sections*. Appreciation of these works requires some knowledge of the great changes made in the very nature of mathematics and of the problems the Greeks faced and solved. Moreover, these polished works give little indication of the three hundred years of creative activity preceding them or of the issues which became vital in the subsequent history.

Classical Greek mathematics developed in several centers that succeeded one another, each building on the work of its predecessors. At each center an informal group of scholars carried on its activities under one or more great leaders. This kind of organization is common in modern times also and its reason for being is understandable. Today, when one great man locates at a particular place—generally a university—other scholars follow, to learn from the master.

The first of the schools, the Ionian, was founded by Thales (*c.* 640–*c.* 546 B.C.) in Miletus. We do not know the full extent to which Thales may have educated others, but we do know that the philosophers Anaximander (*c.* 610–*c.* 547 B.C.) and Anaximenes (*c.* 550–480 B.C.) were his pupils. Anaxagoras (*c.* 500–*c.* 428 B.C.) belonged to this school, and Pythagoras (*c.* 585–*c.* 500 B.C.) is supposed to have learned mathematics from Thales. Pythagoras then formed his own large school in southern Italy. Toward the end of the sixth century, Xenophanes of Colophon in Ionia migrated to Sicily and founded a center to which the philosophers Parmenides (5th cent. B.C.) and Zeno (5th cent. B.C.) belonged. The latter two resided in Elea in southern Italy, to which the school had moved, and so the group became known as the Eleatic school. The Sophists, active from the latter half of the fifth century onward, were concentrated mainly in Athens. The most celebrated school is the Academy of Plato in Athens, where Aristotle was a student. The Academy had unparalleled importance for Greek thought. Its pupils and associates were the greatest philosophers, mathematicians, and astronomers of their age; the school retained its pre-eminence in philosophy even after the leadership in mathematics passed to Alexandria. Eudoxus, who learned mathematics chiefly from Archytas of Tarentum (Sicily), founded his own school in Cyzicus, a city of northern Asia Minor. When Aristotle left Plato's Academy he founded another school, the Lyceum, in Athens. The Lyceum is commonly referred to as the Peripatetic school. Not all of the great mathematicians of the classical period can be identified with a school, but for the sake of coherence we shall occasionally discuss the work of a man in connection with a particular school even though his association with it was not close.

4. *The Ionian School*

The leader and founder of this school was Thales. Though there is no sure knowledge about Thales' life and work, he probably was born and lived in Miletus. He traveled extensively and for a while resided in Egypt, where he carried on business activities and reportedly learned much about Egyptian mathematics. He is, incidentally, supposed to have been a shrewd businessman. During a good season for olive growing, he cornered all the olive presses in Miletus and Chios and rented them out at a high fee. Thales is said to have predicted an eclipse of the sun in 585 B.C., but this is disputed on the ground that astronomical knowledge was not adequate at that time.

He is reputed to have calculated the heights of pyramids by comparing their shadows with the shadow cast by a stick of known height at the same time. By some such use of similar triangles he is supposed to have calculated the distance of a ship from shore. He is also credited with having made mathematics abstract and with having given deductive proofs for some theorems. These last two claims, however, are dubious. Discovery of the attractive power of magnets and of static electricity is also attributed to Thales.

The Ionian school warrants only brief mention so far as contributions to mathematics proper are concerned, but its importance for philosophy and the philosophy of science in particular is unparalleled (see Chap. 7, sec. 2). The school declined in importance when the Persians conquered the area.

5. *The Pythagoreans*

The torch was picked up by Pythagoras who, supposedly having learned from Thales, founded his own school in Croton, a Greek settlement in southern Italy. There are no written works by the Pythagoreans; we know about them through the writings of others, including Plato and Herodotus. In particular we are hazy about the personal life of Pythagoras and his followers; nor can we be sure of what is to be credited to him personally or to his followers. Hence when one speaks of the work of Pythagoras one really refers to the work done by the group between 585 B.C., the reputed date of his birth, and roughly 400 B.C. Philolaus (5th cent. B.C.) and Archytas (428–347 B.C.) were prominent members of this school.

Pythagoras was born on the island of Samos, just off the coast of Asia Minor. After spending some time with Thales in Miletus, he traveled to other places, including Egypt and Babylon, where he may have picked up some mathematics and mystical doctrines. He then settled in Croton. There he founded a religious, scientific, and philosophical brotherhood. It was a formal school, in that membership was limited and members learned from leaders. The teachings of the group were kept secret by the members, though the secrecy as to mathematics and physics is denied by some historians. The Pythagoreans were supposed to have mixed in politics; they allied themselves with the aristocratic faction and were driven out by the popular or democratic party. Pythagoras fled to nearby Metapontum and was murdered there about 497 B.C. His followers spread to other Greek centers and continued his teachings.

One of the great Greek contributions to the very concept of mathematics was the conscious recognition and emphasis of the fact that mathematical entities, numbers, and geometrical figures are abstractions, ideas entertained by the mind and sharply distinguished from physical objects or pictures. It is true that even some primitive civilizations and certainly the Egyptians and Babylonians had learned to think about numbers as divorced from physical objects. Yet there is some question as to how much they were consciously aware of the abstract nature of such thinking. Moreover, geometrical thinking in all pre-Greek civilizations was definitely tied to matter. To the Egyptians, for example, a line was no more than either a stretched rope or the edge of a field and a rectangle was the boundary of a field.

The recognition that mathematics deals with abstractions may with some confidence be attributed to the Pythagoreans. However, this may not have been true at the outset of their work. Aristotle declared that the Pythagoreans regarded numbers as the ultimate components of real, material objects.[1] Numbers did not have a detached existence apart from objects of sense. When the early Pythagoreans said that all objects were composed of (whole) numbers or that numbers were the essence of the universe, they meant it literally, because numbers to them were like atoms are to us. It is also believed that the sixth- and fifth-century Pythagoreans did not really distinguish numbers from geometrical dots. Geometrically, then, a number was an extended point or a very small sphere. However, Eudemus, as reported by Proclus, says that Pythagoras rose to higher principles (than had the Egyptians and Babylonians) and considered abstract problems for the pure intelligence. Eudemus adds that Pythagoras was the creator of pure mathematics, which he made into a liberal art.

The Pythagoreans usually depicted numbers as dots in sand or as pebbles. They classified the numbers according to the shapes made by the arrangements of the dots or pebbles. Thus the numbers 1, 3, 6, and 10 were called triangular because the corresponding dots could be arranged as triangles (Fig. 3.1). The fourth triangular number, 10, especially fascinated the Pythagoreans because it was a prized number for them, and had 4 dots on

1. *Metaphys.* I, v, 986a and 986a 21, Loeb Classical Library ed.

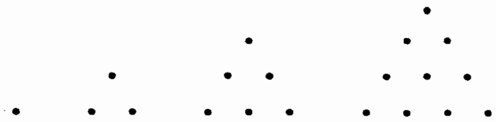


Figure 3.1

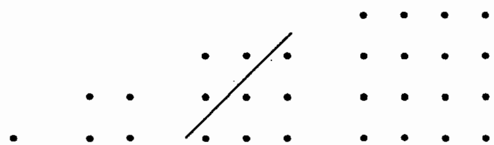


Figure 3.2

each side, 4 being another favorite number. They realized that the sums 1, $1 + 2$, $1 + 2 + 3$, and so forth gave the triangular numbers and that $1 + 2 + \cdots + n = (n/2)(n + 1)$.

The numbers 1, 4, 9, 16, ... were called square numbers because as dots they could be arranged as squares (Fig. 3.2). Composite (nonprime) numbers which were not perfect squares were called oblong.

From the geometrical arrangements certain properties of the whole numbers became evident. Introducing the slash, as in the third illustration of Figure 3.2, shows that the sum of two consecutive triangular numbers is a square number. This is true generally, for as we can see, in modern notation,

$$\frac{n}{2}(n + 1) + \frac{n + 1}{2}(n + 2) = (n + 1)^2.$$

That the Pythagoreans could prove this general conclusion, however, is doubtful.

To pass from one square number to the next one, the Pythagoreans had the scheme shown in Figure 3.3. The dots to the right of and below the lines in the figure formed what they called a gnomon. Symbolically, what they saw here was that $n^2 + (2n + 1) = (n + 1)^2$. Further, if we start with 1 and

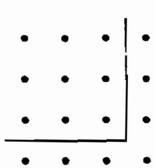


Figure 3.3

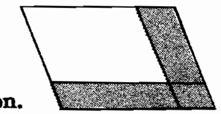


Figure 3.4. The shaded area is the gnomon.

add the gnomon 3 and then the gnomon 5, and so forth, what we have in our symbolism is

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

As to the word "gnomon," originally in Babylonia it probably meant an upright stick whose shadow was used to tell time. In Pythagoras' time it meant a carpenter's square, and this is the shape of the above gnomon. It also meant what was left over from a square when a smaller square was cut out of one corner. Later, with Euclid, it meant what was left from a parallelogram when a smaller one was cut out of one corner provided that the parallelogram in the lower right-hand corner was similar to the one cut out (Fig. 3.4).

The Pythagoreans also worked with polygonal numbers such as pentagonal, hexagonal, and higher ones. As we can see from Figure 3.5, where each dot represents a unit, the first pentagonal number is 1, the second, whose dots form the vertices of a pentagon, is 5; the third is $1 + 4 + 7$, or 12, and so forth. The nth pentagonal number, in our notation, is $(3n^2 - n)/2$. Likewise the hexagonal numbers (Fig. 3.6) are 1, 6, 15, 28, ... and generally $2n^2 - n$.

A number that equaled the sum of its divisors including 1 but not the number itself was called perfect; for example, 6, 28, and 496. Those exceeding the sum of the divisors were called excessive and those which were less were called defective. Two numbers were called amicable if each was the sum of the divisors of the other, for example, 284 and 220.

The Pythagoreans devised a rule for finding triples of integers which could be the sides of a right triangle. This rule implies knowledge of the Pythagorean theorem, about which we shall say more later. They found that when m is odd, then m, $(m^2 - 1)/2$, and $(m^2 + 1)/2$ are such a triple. However,

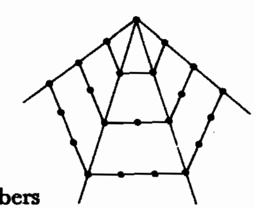


Figure 3.5. Pentagonal numbers

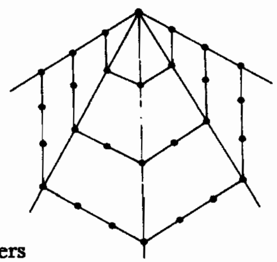


Figure 3.6. Hexagonal numbers

this rule gives only some sets of such triples. Any set of three integers which can be the sides of a right triangle is now called a Pythagorean triple.

The Pythagoreans studied prime numbers, progressions, and those ratios and proportions they regarded as beautiful. Thus if p and q are two numbers, the arithmetic mean A is $(p + q)/2$, the geometric mean G is $\sqrt{pq}$, and the harmonic mean H, which is the reciprocal of the arithmetic mean of $1/p$ and $1/q$, is $2pq/(p + q)$. Now G is seen to be the geometric mean of A and H. The proportion $A/G = G/H$ was called the perfect proportion and the proportion $p:(p + q)/2 = 2pq/(p + q):q$ was called the musical proportion.

Numbers to the Pythagoreans meant whole numbers only. A ratio of two whole numbers was not a fraction and therefore another kind of number, as it is in modern times. Actual fractions, expressing parts of a monetary unit or a measure, were employed in commerce, but such commercial uses of arithmetic were outside the pale of Greek mathematics proper. Hence the Pythagoreans were startled and disturbed by the discovery that some ratios—for example, the ratio of the hypotenuse of an isosceles right triangle to an arm or the ratio of a diagonal to a side of a square—cannot be expressed by whole numbers. Since the Pythagoreans had concerned themselves with whole-number triples that could be the sides of a right triangle, it is most likely that they discovered these new ratios in this work. They called ratios expressed by whole numbers commensurable ratios, which means that the two quantities are measured by a common unit, and they called ratios not so expressible, incommensurable ratios. Thus what we express as $\sqrt{2}/2$ is an incommensurable ratio. The ratio of incommensurable magnitudes was called αλογος (*alogos*, inexpressible). The term αρρητος (*arratos*, not having a ratio) was also used. The discovery of incommensurable ratios is attributed to Hippasus of Metapontum (5th cent. B.C.). The Pythagoreans were supposed to have been at sea at the time and to have thrown Hippasus overboard for having produced an element in the universe which denied the Pythagorean doctrine that all phenomena in the universe can be reduced to whole numbers or their ratios.

The proof that $\sqrt{2}$ is incommensurable with 1 was given by the Pythagoreans. According to Aristotle, their method was a *reductio ad absurdum*—that is, the indirect method. The proof showed that if the hypotenuse were commensurable with an arm then the same number would be both odd and even. It runs as follows: Let the ratio of hypotenuse to arm of an isosceles right triangle be $\alpha:\beta$ and let this ratio be expressed in the smallest numbers. Then $\alpha^2 = 2\beta^2$ by the Pythagorean theorem. Since α^2 is even, α must be even, for the square of any odd number is odd.[2] Now the ratio $\alpha:\beta$ is in its lowest terms. Hence β must be odd. Since α is even, let $\alpha = 2\gamma$. Then $\alpha^2 = 4\gamma^2 = 2\beta^2$. Hence $\beta^2 = 2\gamma^2$ and so β^2 is even. Then β is even. But β is also odd and so there is a contradiction.

This proof, which is of course the same as the modern one that $\sqrt{2}$ is irrational, was included in older editions of Euclid's *Elements* as Proposition 117 of Book X. However, it was most likely not in Euclid's original text and so is omitted in modern editions.

Incommensurable ratios are expressed in modern mathematics by irrational numbers. But the Pythagoreans would not accept such numbers. The Babylonians did work with such numbers by approximating them, though they probably did not know that their sexagesimal fractional approximations could never be made exact. Nor did the Egyptians recognize the distinctive nature of irrationals. The Pythagoreans did at least recognize that incommensurable ratios are entirely different in character from commensurable ones.

This discovery posed a problem that was central in Greek mathematics. The Pythagoreans had, up to this point, identified number with geometry. But the existence of incommensurable ratios shattered this identification. They did not cease to consider all kinds of lengths, areas, and ratios in geometry, but they restricted the consideration of numerical ratios to commensurable ones. The theory of proportions for incommensurable ratios and all kinds of magnitudes was provided by Eudoxus, whose work we shall consider shortly.

Some geometrical results are also credited to the Pythagoreans. The most famous is the Pythagorean theorem itself, a key theorem of Euclidean geometry. The Pythagoreans are also supposed to have discovered what we learn as theorems about triangles, parallel lines, polygons, circles, spheres, and the regular polyhedra. They knew in particular that the sum of the angles of a triangle is 180°. A limited theory of similar figures and the fact that a plane can be filled out with equilateral triangles, squares, and regular hexagons are included among their results.

The Pythagoreans started work on a class of problems known as

2. Any odd whole number can be expressed as $2n + 1$ for some n. Then $(2n + 1)^2 = 4n^2 + 4n + 1$, and this is necessarily odd.

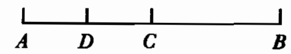
application of areas. The simplest of these was to construct a polygon equal in area to a given polygon and similar to another given one. Another was to construct a specified figure with an area exceeding or falling short of another by a given area. The most important form of the problem of application of areas is: Given a line segment, construct on part of it or on the line segment extended a parallelogram equal to a given rectilinear figure in area and falling short (in the first case) or exceeding (in the second case) by a parallelogram similar to a given parallelogram. We shall discuss application of areas when we study Euclid's work.

The most vital contribution of the Greeks to mathematics is the insistence that all mathematical results be established deductively on the basis of explicit axioms. Hence the question arises as to whether the Pythagoreans proved their geometric results. No unequivocal answer can be given, but it is very doubtful that deductive proof on any kind of axiomatic basis, explicit or implicit, was a requirement in the early or middle period of Pythagorean mathematics. Proclus does affirm that they proved the angle sum theorem; this may have been done by the late Pythagoreans. The question of whether they proved the Pythagorean theorem has been extensively pursued, and the answer is that they probably did not. It is relatively easy to prove it by using facts about similar triangles, but the Pythagoreans did not have a complete theory of similar figures. The proof given in Proposition 47 of Book I of Euclid's *Elements* (Chap. 4, sec. 4) is a difficult one because it does not use the theory of similar figures, and this proof was credited by Proclus to Euclid himself. The most likely conclusion about proof in Pythagorean geometry is that during most of the life of the school the members affirmed results on the basis of special cases, much as they did in their arithmetic. However, by the time of the late Pythagoreans, that is, about 400 B.C., the status of proof had changed because of other developments; so these latter-day members of the brotherhood may have given legitimate proofs.

6. *The Eleatic School*

The Pythagorean discovery of incommensurable ratios brought to the fore a difficulty that preoccupied all the Greeks, namely, the relation of the discrete to the continuous. Whole numbers represent discrete objects, and a commensurable ratio represents a relation between two collections of discrete objects, or two lengths that have a common unit measure so that each length is a discrete collection of units. However, lengths in general are not discrete collections of units; this is why ratios of incommensurable lengths appear. Lengths, areas, volumes, time, and other quantities are, in other words, continuous. We would say that line segments, for example, can have irrational as well as rational lengths in terms of some unit. But the Greeks had not attained this view.

Figure 3.7



The problem of the relation of the discrete to the continuous was brought into the limelight by Zeno, who lived in the southern Italian city of Elea. Born some time between 495 and 480 B.C., Zeno was a philosopher rather than a mathematician, and like his master Parmenides was said to have been a Pythagorean originally. He proposed a number of paradoxes, of which four deal with motion. His purpose in posing these paradoxes is not clear because not enough of the history of Greek philosophy is known. He was said to be defending Parmenides, who had argued that motion or change is impossible. He was also attacking the Pythagoreans, who believed in extended but indivisible units, the points of geometry. We do not know precisely what Zeno said but must rely upon quotations from Aristotle, who cites Zeno in order to criticize him, and from Simplicius, who lived in the sixth century A.D. and based his statements on Aristotle's writings.

The four paradoxes on motion are distinct, but the import of all four taken together was probably intended to be the significant argument. Two opposing views of space and time were held in Zeno's day: one, that space and time are infinitely divisible, in which case motion is continuous and smooth; and the other, that space and time are made up of indivisible small intervals (like a movie), in which case motion is a succession of minute jerks. Zeno's arguments are directed against both theories, the first two paradoxes being against the first theory and the latter two against the second theory. The first paradox of each pair considers the motion of a single body and the second considers the relative motion of bodies.

Aristotle in his *Physics* states the first paradox, called the Dichotomy, as follows: "The first asserts the nonexistence of motion on the ground that that which is in motion must arrive at the half-way stage before it arrives at the goal." This means that to traverse AB (Fig. 3.7), one must first arrive at C; to arrive at C one must first arrive at D; and so forth. In other words, on the assumption that space is infinitely divisible and therefore that a finite length contains an infinite number of points, it is impossible to cover even a finite length in a finite time.

Aristotle, refuting Zeno, says there are two senses in which a thing may be infinite: in divisibility or in extent. In a finite time one can come into contact with things infinite in respect to divisibility, for in this sense time is also infinite; and so a finite extent of time can suffice to cover a finite length. Zeno's argument has been construed by others to mean that to go a finite length one must cover an infinite number of points and so must get to the end of something that has no end.

The second paradox is called Achilles and the Tortoise. According to

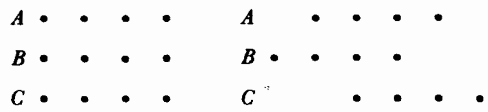


Figure 3.8

Aristotle: "It says that the slowest moving object cannot be overtaken by the fastest since the pursuer must first arrive at the point from which the pursued started so that necessarily the slower one is always ahead. The argument is similar to that of the Dichotomy, but the difference is that we are not dividing in halves the distances which have to be passed over." Aristotle then says that if the slowly moving object covers a finite distance, it can be overtaken for the same reason he gives in answering the first paradox.

The next two paradoxes are directed against "cinematographic" motion. The third paradox, called the Arrow, is given by Aristotle as follows: "The third paradox he [Zeno] spoke about, is that a moving arrow is at a standstill. This he concludes from the assumption that time is made up of instants. If it would not be for this supposition, there would be no such conclusion." According to Aristotle, Zeno means that at any instant during its motion the arrow occupies a definite position and so is at rest. Hence it cannot be in motion. Aristotle says that this paradox fails if we do not grant indivisible units of time.

The fourth paradox, called the Stadium or the Moving Rows, is put by Aristotle in these words: "The fourth is the argument about a set of bodies moving on a race-course and passing another set of bodies equal in number and moving in the opposite direction, the one starting from the end, the other from the middle and both moving at equal speed; he [Zeno] concluded that it follows that half the time is equal to double the time. The mistake is to assume that two bodies moving at equal speeds take equal times in passing, the one a body which is in motion, and the other a body of equal size which is at rest, an assumption which is false."

The probable point of Zeno's fourth paradox can be stated as follows: Suppose that there are three rows of soldiers, A, B, and C (Fig. 3.8), and that in the smallest unit of time B moves one position to the left, while in that time C moves one position to the right. Then relative to B, C has moved two positions. Hence there must have been a smaller unit of time in which C was one position to the right of B or else half the unit of time equals the unit of time.

It is possible that Zeno merely intended to point out that speed is relative. C's speed relative to B is not C's speed relative to A. Or he may have meant there is no absolute space to which to refer speeds. Aristotle says that Zeno's fallacy consists in supposing that things that move with the same speed past a moving object and past a fixed object take the same time. Neither Zeno's argument nor Aristotle's answer is clear. But if we think of

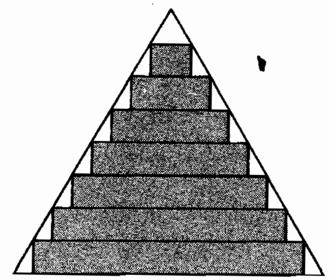


Figure 3.9

this paradox as attacking indivisible smallest intervals of time and indivisible smallest segments of space, which Zeno was attacking, then his argument makes sense.

We may include with the Eleatics Democritus (*c.* 460–*c.* 370 B.C.) of Abdera in Thrace. He is reputed to have been a man of great wisdom who worked in many fields, including astronomy. Since Democritus belonged to the school of Leucippus and the latter was a pupil of Zeno, many of the mathematical questions Democritus considered must have been suggested by Zeno's ideas. He wrote works on geometry, on number, and on continuous lines and solids. The works on geometry could very well have been significant predecessors of Euclid's *Elements*.

Archimedes says Democritus discovered that the volumes of a cone and a pyramid are 1/3 of the volumes of the cylinder and prism having the same base and height, but that the proofs were made by Eudoxus. Democritus regarded the cone as a series of thin indivisible layers (Fig. 3.9), but was troubled by the fact that if the layers were equal they should yield a cylinder and if unequal the cone could not be smooth.

7. *The Sophist School*

After the final defeat of the Persians at Mycale in 479 B.C., Athens became the major city in a league of Greek cities and a commercial center. The wealth acquired through trading, which made Athens the richest city of its time, was used by the famous leader Pericles to build up and adorn the city. Ionians, Pythagoreans, and intellectuals generally were attracted to Athens. Here emphasis was given to abstract reasoning and the goal of extending the domain of reason over the whole of nature and man was set.

The first Athenian school, the Sophist, embraced learned teachers of grammar, rhetoric, dialectics, eloquence, morals, and—what is of interest to us—geometry, astronomy, and philosophy. One of their chief pursuits was the use of mathematics to understand the functioning of the universe.