KANT AND THE FOUNDATIONS OF MATHEMATICS[1]

THE heart of Kant's views on the nature of mathematics is his thesis that the judgments of pure mathematics are synthetic a priori. Kant usually offers this as one thesis, but it is fruitful to regard it as consisting of two separate claims, a metaphysical subthesis and an epistemological subthesis.

(*KM*) The truths of pure mathematics are necessary, although they do not owe their truth to the nature of our concepts.

(*KE*) The truths of pure mathematics can be known independently of particular bits of experience, although one cannot come to know them through conceptual analysis alone.

For Kant, pure mathematics includes geòmetry, arithmetic, algebra, kinematics, "pure mechanics," and, I think, analysis. On the basis of the above subtheses Kant proposes to establish a general theory which will provide particular theories for each of the pure disciplines. His aim is to reveal the nature of the propositions of the disciplines and the nature of our knowledge of those propositions.

To understand Kant's theory we shall need to untangle parts of the *Aesthetic* and reconstruct some of Kant's arguments and theses. Only then shall we be in a position to see how Kant's view of mathematics errs.

I. Kant's Basic Notions

I propose to explicate Kant's conception of necessity by appeal to the device of possible worlds. Bizarre as this approach may seem, it will, nonetheless, prove its worth in understanding Kant's

[1] I would like to thank Paul Benacerraf and Patricia Kitcher for all the patient advice and suggestions they have given me. The criticisms and encouragement of Michael Mahoney, P. F. Strawson, and Margaret Wilson have also been very helpful.

views on mathematics. Let us use the term "proposition" to convey the sense in which Kant uses "judgment" when he is interested in the object of judgment rather than the act of judging. Propositions may be regarded as what are expressed by suitable declarative sentences and we shall also take them to be truth-bearers. For Kant, every proposition ascribes a property to a subject. If that subject has that property in a world w then the proposition is true in world w. Conversely, if the subject lacks that property in w, or if it does not exist in w, then the proposition is false in w. Necessary truths are propositions true in all possible worlds.

If a proposition is true in world w, it is true because of a particular feature of w—namely, the subject's having the appropriate property in w. Thus a proposition which is necessarily true is necessarily true because all possible worlds have a particular feature in common. The connection between true propositions and features of worlds is crucial for Kant's explanations of versions of (*KM*).

To fix the concept of necessity is to specify the domain of possible worlds. For Kant a possible world is a totality of possible appearances—that is, experiences which could be experiences for us, constituted as we are. (It is assumed that we all have the same constitution). We shall follow Kant in taking the concept of a possible experience to be primitive.[2] Kant has a broad notion of necessity in that some propositions which are logically possible fail to hold in any Kantian possible world.

What features a world may have are limited by the structure of our concepts. Some propositions are true in each world in virtue of this limitation. In a derivative sense, these propositions can be said to be true in virtue of the structure of our concepts because they owe their truth to particular features of that structure. Kant calls these truths "analytic."

Kant's second subthesis uses the notion of apriority. Kant

[2] Were we to analyze the concept of a possible experience as that of a logically possible experience we should be able to draw conclusions contrary to Kant's general view. It would be analytic that all human experiences are experiences of a Euclidean world and we should be able to know mathematical truths without intuition, merely by analyzing the concept of human experience.

believes that we can know some propositions independently of experience. By this he does not mean that we can know these propositions before we have any experience at all but that, no matter what experiences we have had, provided that those experiences suffice for our acquisition of the concepts involved in an a priori proposition, then we can still know the proposition.[3] The particular stream of perceptions of the world which I have in fact had is quite irrelevant to my a priori knowledge except insofar as it plays a role in acquainting me with the appropriate concepts. The grounds of that knowledge lie elsewhere.

We can use Kant's contention that all propositions are of subject-predicate form to sharpen the question of how we know propositions and so to clarify the notion of a priori knowledge. In coming to know a proposition we recognize a connection between subject and predicate. This can happen in various ways. Analytic propositions can be known by uncovering the constitution of the subject and predicate concepts. Kant is confident of our ability to do this. Although he adds the clause that the predicate may have been "thought confusedly" in the subject of an analytic proposition (*A*7; *B*11), he assumes that we can normally analyze our concepts quite easily. He expects, for example, that his reader will quickly agree with his diagnosis of what is and is not contained in our mathematical concepts. Conceptual analysis as an avenue to a priori knowledge is taken for granted and left unexplained. Similarly, Kant feels that the fact that we have knowledge a posteriori of propositions is uncontroversial. In this case an intuition of the world tells us that the predicate applies to the subject.

Kant would not deny that most propositions which we know are known on the basis of inference. The two modes of knowledge just discussed concern how we can reach the starting points for our inferences. But if there is no way to generate synthetic conclusions from analytic premises, then some further factor must

[3] See Ch. I of my dissertation (*Mathematics and Certainty* [Princeton, 1973], unpublished) for a more detailed account of the general form of theories of a priori knowledge. It is assumed that we are all on a par as possible knowers—i.e., that anything that can be known a priori by one person can be known a priori by any other.

be brought in to account for a priori knowledge of synthetic propositions. Kant believes that there are only two routes to knowledge. Either our recognition of the connection between subject and predicate is brought about by unveiling the structure of our concepts, or our recognition requires the aid of intuition. Kant introduces the notion of *pure* intuition as that intuition which is involved in a priori knowledge of synthetic propositions. We shall examine the various guises of this notion in later sections.

Let us now look at a class of propositions whose importance Kant tended to stress. Assume that the proposition that all *A*'s are *B*'s is synthetic a priori. Since Kant believes that the epistemological notion of apriority is coextensive with the metaphysical concept of necessity, he holds the proposition to be necessary.[4] Because the proposition is synthetic it is logically possible that there be an *A* which is not a *B*. But since it is necessary we cannot experience such *A*'s. Synthetic a priori propositions thus state nonlogical constraints on what we can experience.

Kant's theses (*KM*) and (*KE*) can now be stated more clearly. (*KM*) tells us that mathematical truths state nonlogical conditions on our experience. (*KE*) contends that the principles stating these conditions can be known a priori—and Kant would also contend that we *do* know some of them a priori. Since they are not analytic, they cannot be known just by conceptual analysis, but must be known by means of pure intuition.

Kant argues for his theses in Section V of the introduction to the *Critique*. He takes it for granted that the truths of mathematics are necessary.

> First of all, it has to be noted that mathematical propositions, strictly so called, are always judgments *a priori*, not empirical; because they carry with them necessity, which cannot be derived from experience. If this be demurred to, I am willing to limit my statement to *pure* mathematics, the very concept of which implies that it does not contain empirical, but only pure *a priori* knowledge [*B*15].

The second sentence adds nothing to Kant's argument. If he takes the apriority of pure mathematics as following from the

[4] See *Critique of Pure Reason* (*B*4).

concept of pure mathematics, he should not assume (as he does) that geometry, arithmetic, algebra, and so forth fall under this concept. To make that assumption is to pull a substantive thesis out of thin air by a method akin to that of double definition.[5] Either we can take pure mathematics to be (by definition) that part of mathematics consisting of necessary truths; *or* we can take it to be (by definition) geometry, arithmetic, algebra, and so forth. Depending on how we choose our definition we should argue for and justify the other proposition. Kant tries to have it both ways at once and because of this, while the first sentence of the quoted passage merely states the thesis, the second fails to prove it. Clearly Kant did not take the idea that mathematical propositions might be falsified by experience at all seriously, and he expected his readers to agree with him.

Kant did anticipate opposition to his thesis that truths of mathematics are nonanalytic and offered an argument to meet it. If we take a true proposition of arithmetic or geometry, subjecting the subject concept and the predicate concept to close scrutiny, we shall not be able to find the latter contained in the former. In "thinking" the sum of seven and five, for example, we do not, according to Kant, already "think" the number twelve. But since Kant claims that we can always eventually find the constituents of a concept, he concludes that our inability to uncover the predicate concept in the subject concept shows that it is not contained therein. Kant also appeals to the nature of our mathematical knowledge to show that the truths of mathematics are not analytic. Only if mathematical truths were nonanalytic would we need the aid of intuition to convince ourselves of them. Yet if we reflect on the way in which we do recognize propositions of mathematics as true, we shall find that we always require intuition. This fact supports the thesis that intuition is necessary for mathematical knowledge and hence the thesis that mathematical truths are synthetic.

These arguments which Kant offers in support of (*KM*) and (*KE*) are not equal to the task. Aside from the fact that he has

[5] See P. Geach and M. Black, *Translations from the Philosophical Writings of Gottlob Frege*, (Oxford, 1952), pp. 159-170.

not faced squarely the problem of establishing the necessity of mathematics, Kant's optimism about our powers of conceptual analysis must also be questioned. Frege, for example, would reply that the exhibition of the analyticity of the truths of arithmetic is a long and difficult affair. Where Kant has erred is in supposing that our ability to uncover the constituents is such that we can decide what concepts contain by casual reflection. Hence Kant's argument would fail to show that the propositions of, for example, arithmetic are nonanalytic and would merely indicate a case where naïve reflection is an untrustworthy guide to conceptual structure. Alternatively, if Kant were to stick with the idea that analytic truth can be revealed so easily, he would be trivializing his sense of analyticity in playing down deep and exciting conceptual relations. In this trivial sense of analyticity his conclusion might follow, but it would be uninteresting and would not eliminate the possibility of our learning arithmetic truths by probing our concepts more deeply.

But exhibiting the shakiness of Kant's argument does not present the real difficulty with (*KM*) and (*KE*). That is revealed in Kant's efforts to explain these theses.

II. The Explanation of Geometry

Since Kant's theory of geometry is much clearer than his view of other parts of pure mathematics, it is advisable to start with it. The core of Kant's geometrical doctrine is the restrictions of (*KM*) and (*KE*) to the case of geometry, to wit:

> (*GM*) The truths of geometry are necessary, although they do not owe their truth just to the nature of our concepts.
>
> (*GE*) The truths of geometry can be known independently of particular bits of experience, although we cannot know them through conceptual analysis alone.

We can understand these theses by reference to the discussion in section I.

(*GM*) and (*GE*) are to be explained by the thesis of the transcendental ideality of space. This thesis asserts that space is an a priori form of intuition. It consists of the following two claims:

(*SM*) All possible intuitions of what we normally take to be the external world are subject to conditions imposed by space, which can therefore be said to be the form of outer intuition.

(*SE*) We can know the principles which state these conditions and which thus describe space. We can know them a priori by means of a pure intuition of space.

(*SM*) is supposed to be the only explanation for (*GM*) and (*SE*) the only explanation for (*GE*). Since Kant regards the truth of (*GM*) and (*GE*) as established, by showing that (*SM*) and (*SE*) are the only explanations for these truths, he takes himself to have demonstrated the truth of (*SM*) and (*SE*).[6]

We begin with the argument for (*SM*). (*GM*) is our premise. It asserts that the truths of geometry are synthetic and necessary—that is, that they state nonlogical conditions on what we can experience. Further, by an assumption ascribed to Kant above, geometrical truths must either be about some particular feature of the world—that feature in virtue of which they are true—or they must state some particular property of our concepts. Since they are not analytic, the latter cannot be the case. So geometric truths are true because of some facet of the world. But Euclidean geometry is true in virtue of the fact that space is Euclidean. Geometry thus describes the structure of space.[7] Geometrical truths are true in every possible world. Hence every possible world has the same spatial structure—namely, that described by geometry. Thus there are laws which describe the spatial structure of any possible world—of any world, that is, of which we can have experience. We can explain this conclusion only by supposing

[6] Kant does not present the argument in a way which makes it clear that (*GM*) and (*GE*) are distinct. Despite this, we shall see below that certain passages do seem to indicate his awareness of the distinction between them.

[7] As an anachronistic argument on Kant's behalf we might point out that different geometries ascribe different spatial structures to the world. Replying to that argument by insisting that one can talk only about a geometry and a physics *together* applying to the world would, however, undermine Kant's assumption of an intimate tie between propositions and features of worlds. Without that assumption, Kant's whole line of reasoning would collapse and, worse still, (*SM*), (*GM*), and (*KM*) would all need to be refurbished. Yet perhaps one might try, as Poincaré did, to maintain the special status of Euclidean geometry.

that space imposes conditions on what we are able to experience. These conditions are not logical conditions. Hence they are conditions not on what we can understand but on what we can intuit. Space may therefore properly be called the form of intuition. This establishes (*SM*).

To construct the argument for (*SE*), we shall have to tackle the notion of pure intuition. To do so we must begin with the particular case of geometry. For, despite the fact that Kant does introduce the key concept of pure intuition in a quite general theory, his use of it is quite hard to understand except by reference to the geometrical version of it.

The notion of pure intuition is obscured through the treatment of (*KM*) and (*KE*) together and (*SM*) and (*SE*) together. At the beginning of the *Aesthetic* Kant tells us that intuition is a mode of knowledge "in immediate relation" to objects and that pure intuitions are intuitions in which everything belonging to experience is subtracted (*A*20-21; *B*24-25). But he goes on at once to equate pure intuition with the form, or faculty, of intuition. The situation is all the more complicated in that we are supposed to know the features of pure intuition (the faculty or form) through pure intuitions (representations without empirical content).

Intuitions must have an object. Kant tries to provide an object for the intuitions which yield our geometrical knowledge by showing how space can be the object of intuitions. I shall refer to the form of intuition as form-space and the object of appropriate pure intuitions as object-space.[8] Kant's idea is that by having an intuition of object-space we come to know the properties of form-space. What needs explaining is how object-space can be intuited, how intuitions of object-space can be pure, and how they can give knowledge of the properties of form-space.

Kant's explanation centers on the notion that we can construct object-space a priori with the help of our geometrical concepts. He claims that we can exhibit such concepts as "line," "point," "circle," and so forth, to ourselves a priori. Without recourse to

[8] My reason for using this distinction is that it may be a conceptual error, for Kant, to identify space (the form of our perceptions) with space (the object of our perceptions when we do geometry). Even if no such error is involved, the gain in clarity which the distinction brings is obvious.

experience, we can construct geometrical figures in thought. In doing so, we bring object-space into being as the object of a pure intuition. Only by means of our construction do we have an object at all. For form-space is only the form of intuition, providing for intuition once an object meets its conditions. By drawing geometrical diagrams "in the mind's eye" we are able to construct determinate object-space with its metric and projective properties. Kant sums this up as follows:

To know anything in space (for instance, a line) I must *draw* it, and thus synthetically bring into being a determinate combination of the given manifold, so that the unity of this act is at the same time the unity of consciousness (as in the concept of a line); and it is through this unity of consciousness that an object (a determinate space) is first known [*B*138].

Kant's idea may be clarified by an analogy. Let us imagine that we are condemned to look toward a surface, normally unlit, onto which pictures are periodically flashed. We can discern some order and pattern in the pictures by learning the geometrical properties of the surface. There is a way to do this without attending carefully to the pictures. We are able to draw luminous figures on the surface. We do so by following rules. To draw a triangle I follow the rule for triangles. The appearance of the resulting figure is determined partly by the rule, partly by the surface, and is, perhaps, partly due to free choices which I have made. Because of the determining role of the surface, the drawn figure can reveal properties of the surface. In Kant's terms, constructing the figure makes the surface an object of intuition. We shall discuss the nature of the rules below.

Similarly, Kant would contend that the drawing of geometrical figures reveals the properties of space. The constructed triangle yields a representation of object-space. Form-space partly determines the appearance of the triangle and so the representation discloses properties of form-space. These properties are to be learned from inspection of the constructed figure for whose appearance they are partly responsible. Kant claims that the construction of object-space can be carried out without recourse to experience. For we can follow the rules for representing mathematical concepts no matter what our experiences of the world have been.

We are now ready for Kant's argument for (*SE*). By (*GE*), truths of geometry are known a priori. Although we know most of them by following proofs, some of them must be known immediately a priori. The basic truths of geometry cannot be known by deriving them from other truths. Nor are they knowable by analyzing our concepts; that would make them analytic. Hence we must know them through intuition. Because we can, and do, know them a priori, they must be known through a nonempirical kind of intuition. Kant now offers his construction of object-space and the notion of pure intuition as described above, as the only explanation for our ability to know the properties of space in nonempirical intuition. In accepting (*GE*) we are forced to accept (*SE*) and the account of the construction of object-space, for, it is claimed, (*SE*) explains (*GE*) and there just is no alternative.

(*SM*) and (*SE*) are related. For without presupposing (*SM*), we could not set up the account of the construction of object-space as we did. The story of our mental picturings is saved from immediate collapse into the empiricist description of conceptual analysis by the determining role that form-space is supposed to play in it. Despite this connection, Kant's tendency to conflate the theses (*GM*) and (*GE*) and the theses (*SM*) and (*SE*) in the *Aesthetic* and his ambiguous use of the term "pure intuition" render his theory of space all the more obscure.[9] For example, in the section on the "Transcendental Exposition of the Concept of Space," given in the second edition, Kant argues from (*GE*) to (*SE*) and, without a break and without disambiguating his term,

[9] I think that Kant's obscurity here misleads Jaakko Hintikka. In his paper "On Kant's Notion of Intuition" (printed in T. Penelhum and J. J. MacIntosh [eds.], *The First Critique* [Belmont, Calif., 1969],), Hintikka glosses Kant's task as proving "that the ideas of space and time are inseparably tied up to human sensibility" (*ibid.*, p. 45). But, on the view advanced in the present paper, there are *two* tasks which can be characterized by this ambiguous phrase. Hintikka's main discussion focuses on the metaphysical task—the move from (*GM*) to (*SM*)—without dealing with Kant's attempts to explain mathematical *knowledge*. It is thus not surprising that Hintikka should conclude by divorcing "intuition" from its epistemological role.

slides into the argument from (*GM*) to (*SM*) (compare *B*40-41). In this and similar passages he fails to make it clear that there are two distinct parts to his ideality thesis.

At times, however, Kant insists that his theory of space solves *two* problems which baffled his predecessors. He points out that Leibniz' relational theory of space "can neither account for the possibility of a priori mathematical knowledge, nor bring the propositions of experience into necessary agreement with it" (*A*40-41, *B*57-58). Kant's attack on Leibniz in the *Inaugural Dissertation* also prods this double weakness.[10] Leibniz is supposedly neither able to explain why geometrical truths have the necessity they do have nor equipped to show adequately how we have a priori knowledge of these propositions. Kant regards his theory of space as clearing up both difficulties.

III. Other Parts of Pure Mathematics

The reconstruction of the *Aesthetic* enables us to bring some clarity to Kant's views on other parts of mathematics.

Since propositions of arithmetic, algebra, and so forth are synthetic a priori, they must state nonlogical conditions on our experience. Appropriate restrictions of (*KM*) would be explained by theses akin to (*SM*). We shall have to connect arithmetic, algebra, and so on with features of the forms of intuition, space, and time. Furthermore, by an argument parallel to that just rehearsed for the geometrical case, we must be able to know the truths of these parts of pure mathematics with the aid of pure intuitions (representations without empirical content). So far we have made sense of the notion of pure intuition only in the context of our geometrical knowledge. Two tasks must be completed for each discipline. We should associate arithmetic, algebra, and so forth with aspects of space and time, and describe for each the nature of appropriate kinds of pure intuition.

Arithmetic is the easiest case. Kant did not believe, as is often

[10] See D. Kerferd and K. Walferd (eds.), *Selected Pre-Critical Writings*, (New York, 1968), p. 71. I shall refer to this volume as *PC*.

supposed, that arithmetic stands to time as geometry does to space. In the transcendental exposition of the concept of time, Kant does not mention arithmetic but refers, somewhat vaguely, to the "general doctrine of motion" (*B*49). Kant would not have been content with this wave of the hand if he could have suited his theory by offering the much more obvious case of arithmetic as an example. Furthermore, his use of an arithmetical example in the argument for the synthetic status of mathematical truths describes an intuition through which the cited arithmetic truth is known, and it is hard to understand this intuition as a pure intuition of time alone. Finally, we have the word of the *Inaugural Dissertation*:

Hence PURE MATHEMATICS deals with *space* in GEOMETRY, and *time* in PURE MECHANICS. In addition to these concepts there is a certain concept which in itself indeed is intellectual, but whose actuation in the concrete requires the assisting notions of time and space (by successively adding a number of things and setting them simultaneously beside one another). This is the concept of *number*, which is the concept treated in ARITHMETIC [*PC*, p. 62].

But if arithmetic does not state properties of time, what is it about?

The truths of geometry are true in virtue of particular features of space. That does not mean that geometry exhausts the properties of space. The following possibility remains open. Arithmetical propositions are true in virtue of certain structural features of space *and* of time. We can refer to these properties collectively as "combinatorial" features of space-time. The same combinatorial feature can be observed to hold of space and of time; for example, a unit length added to a two-unit length makes a three-unit length whether we think in terms of space-units or time-units. Arithmetical truths may, perhaps, portray such common combinatorial features.

This is somewhat vague, but is, nonetheless, *an* answer to the problem of associating arithmetic with the forms of intuition. Kant is more specific about what pure arithmetical intuition (the nonempirical representation) is like. Considering our knowledge that $7 + 5 = 12$, he claims that we cannot know the proposition by concepts alone. Instead, "starting with the number 7, and for the concept of 5 calling in the aid of the fingers of my hand as intuition, I now add one by one to the number 7 the units which I previously took together to form the number 5, and with the aid of that figure (the hand) see the number 12 come into being" (*B*16). If we take this as our model, we can suppose that what makes arithmetic possible is a pure intuition of space and time together, which spells out the hint given in the *Dissertation.* We know that $7 + 5 = 12$ by instantiating our concepts of 7 and 5, using stroke symbols, for example, and by successively juxtaposing a stroke to the block of seven for every stroke in the block of five. To put the example graphically: we draw strokes counting from one to seven; we then continue "one-eight" (stroke), "two-nine" (stroke), until twelve "comes into being" with "five-twelve" (stroke).

There is one obvious difficulty with this idea. In the case of geometry we could find a role for the structure of space in giving a partial determination to our representations. Using the analogy of the surface we were able to give content to the notion that the representation was a representation of space. A similar suggestive analogy is harder to find for the case of arithmetic.[11] As a result of our difficulty in finding a role for the structure of space-time in the determination of our representation the threat of collapse into empiricist-style conceptual analysis gains new vigor for this case.

Kant does not seem to favor the option that intuition of stroke symbols leads to knowledge of general propositions of arithmetic.[12] This introduces a problem when we turn to algebra which Kant understands as a generalized arithmetic dealing with "quantity

[11] Although one can imagine a discrete space-time governed by a modular arithmetic (letting the modulus be, for example, 1,000). If we then suppose ourselves to be drawing stroke symbols on an imaginary surface we shall reveal that there are only 1,000 "places" on that surface. The details resist elaboration, but I think this sketch suggests a way to adapt our gloss of pure intuition to the arithmetical case.

[12] See the "Axioms of Intuition" (*A*164-165; *B*205-206). Kant did not avail himself of an option here which was later taken by Hilbert. One can suppose that general laws of arithmetic are known by intuition of indeterminate stroke symbols.

as such."[13] That problem emerges only with the epistemological thesis. Using Kant's hint of the relation between algebra and arithmetic, we can gloss the metaphysical thesis as claiming that, while arithmetical propositions owe their truth to relatively simple and concrete features of space and time, algebraic truths are true in virtue of more abstract and, perhaps, more fundamental features of the forms of intuition. (Such features would be reflected in laws like the law of commutativity of addition.) Combinatorial features of space-time would be partitioned into two classes, the specific and the more general, the former accounting for arithmetical truth and the latter for algebraic truth.

Worse than the difficulty of rendering this distinction (or, indeed, the notion of "combinatorial property") clear and precise is the problem of explaining the algebraic version of pure intuition. Kant has two approaches to this problem. The first is an obscure doctrine which claims that algebra is intuitive because it uses "symbolic construction" (cf. *A*717, *B*745). What Kant means is that algebra uses symbols and proceeds by manipulating these symbols. (The term "symbol" is loaded, as we shall see.) His early essay on the principles of natural theology makes the point quite clearly. Mathematics has an advantage over philosophy in its ability to use symbolism and in its power of ignoring the things symbolized. Kant assumes, significantly, that there could not be a philosophical symbolism which could produce the same benefits. He remarks:

> The signs used in the philosophical way of thinking are never anything other than words, which can neither show, in their composition, the parts of the concepts out of which the whole idea, indicated by the word, consists; nor can they show in their combinations the relations of philosophical thoughts [*PC*, p. 9].

Kant's subsequent remarks indicate how he thinks that mathematical symbols can do better. Mathematical signs reveal properties of the objects symbolized which are not contained in the concepts of these objects. This is not altogether absurd. Kant clearly thinks that a particular diagram of a circle can be the

[13] See Sec. 2 of the *Enquiry on the Clarity of the Principles of Natural Theology and Ethics*, esp. *PC*, p. 8.

symbol for all circles and that the stroke symbol "|||" is the symbol for 3. Thus the cases of geometry and arithmetic demonstrate how the use of symbols (in this loaded sense) can be especially useful. We can see, by their aid, that circles intersect in at most two points and that 3 is greater than 2. But Kant cannot extend his conclusions to algebra where the symbolism is different. Perhaps because algebra deals with such colorless objects as magnitudes-in-general, its signs cannot serve as pictures in the way in which signs of geometry and arithmetic can.[14] No matter how long we stare at the sign design $\ulcorner a + b = b + a \urcorner$ we shall not discover, by means of our scrutiny of these signs, the truth of the law of commutativity of addition. Kant's stress on the mathematician's use of symbolism fails to tie algebra to geometry and arithmetic; instead, it reveals that there are important differences between the type of arithmetical symbolism which interests him and ordinary algebraic symbolism. The theory of "symbolic construction" for algebra only amounts to the weak claim that algebra is "intuitive" in being able to operate with signs. It does not divorce algebra from branches of analytic knowledge, which manipulate signs in the same way.

So we are still left with the problem of finding a way in which we can have a priori knowledge of basic algebraic truths. We can take the law of commutativity of addition as an example of a fundamental algebraic law (it is clearly a basic principle of "the general arithmetic of indeterminate magnitudes") (*PC*, p. 8). In the last paragraph we concluded that it could not be known just by presenting the signs to ourselves.

Kant does not leave the issue with the shuffle around "symbolic construction."

Even the method of algebra with its equations, from which the correct answer, together with its proof, is deduced by reduction, is not indeed geometrical in nature, but is still constructive in a way characteristic of the science. *The concepts attached to the symbols*, *especially concerning the relations of magnitudes*, *are presented in intuition*;... [*A*734, *B*762; my emphasis].

[14] When we reflect we see that there are *two* different types of sign used in geometry and arithmetic. There are the "revealing" signs which Kant talks about and also signs like "*AB*," "$\triangle$*DEF*," "3," "7 × 9," which are just as incapable of showing us mathematical truths as the signs used in algebra.

Applying this to our example, Kant would account for our knowledge that $a + b = b + a$ by showing how we construct two magnitudes—both of which can stand for *all* magnitudes—exhibit the concept of addition (the relation of magnitudes in which we are interested here) and so grasp the commutativity principle. But there is a problem with the idea that the magnitudes exhibited can stand for all.

Kant believes that mathematics proceeds to universal conclusions from intuitions of particular objects. By examining a diagram of one particular triangle we come to know properties common to all triangles.[15] Similarly, Kant could suppose us to proceed to knowledge of properties shared by *all* magnitudes by intuiting *particular* magnitudes. Since he speaks of algebra as generalized arithmetic, we might think that the appropriate magnitudes to be intuited would be *numbers*. Kant's remarks about axioms for arithmetic suggest, however, that intuition of stroke symbols is not intended to lead us to general conclusions, that knowledge of algebra is not to be founded on intuitions of indeterminate stroke symbols. In any case, he has an alternative at hand in the finite line segment. *Geometrical* picturing can be used to reveal the principles of algebra. The line segment which we intuit can stand for all the line segments or for all magnitudes. If one use of the particular figure is legitimate the other will be, too. Kant can thus find a way of accounting for our knowledge of principles of algebra. Significantly, that account explains our knowledge of these principles in their geometrical instantiations. Thus while Kant's metaphysical thesis concerning algebra construes algebra as generalized arithmetic, algebraic *knowledge* would be gleaned in a way similar to that followed in knowing geometry.

IV. Proofs and Kant's Foundational Program

We have been focusing on immediate knowledge. Kant's theory of pure mathematics distinguishes those truths of pure mathematics which are apprehended immediately from those

[15] This will be discussed in detail below.

which are inferred. Thus, in the *Methodology*, Kant characterizes the general form of mathematical disciplines. The mathematician "constructs his concepts in *a priori* intuition," and by doing so he is able to "combine the predicates of the object both *a priori* and immediately" (*A*732, *B*760), thus obtaining starting points for proofs; then "through a chain of inferences guided throughout by intuition, he arrives at a fully evident and universally valid solution of the problem" (*A*717, *B*745).

Perhaps all that Kant has in mind when he describes proofs as "guided by intuition" is the notion that mathematical inferences are intuitive because they consist of transitions from one set of symbols to another. (As if we first exhibited concepts to ourselves to teach ourselves the proper ways of manipulating signs and then inspected the signs themselves, scrutinizing them to ensure that each move accorded with the established rules.) A more substantial reading can be given for the case of geometry. Various of Kant's examples of geometrical proof indicate that he regards proofs as sequences of mental constructions on figures already constructed. Having drawn a figure in thought, embellishing it reveals successively more complex and recondite features of space. So we can be guided to recognize properties which we cannot learn "all at once." It is hard to envisage, however, how we can construe this substantive use of pure intuition in proof for the cases of arithmetic and algebra.

But whether we take proofs to be intuitive in the sense that the constructed objects are kept in view throughout, or whether we suppose that one inspects constructed objects just at the beginning of proofs, the rest being surveillance of signs, there is a theoretical challenge for Kant to face. If we grant, for the moment, that Kant can claim that some traditional parts of pure mathematics fit the pattern that he sees in *all* parts of pure mathematics, there remains the task of showing that the rest of pure mathematics can be accommodated. In particular, he would have to show that the new eighteenth-century disciplines in pure mathematics match his ideal—or else they would have to be dismissed as not belonging to pure mathematics. Kant would have to exhibit the intuitive foundations of complex algebra and of analysis if he wishes to maintain that those subjects belong to pure mathematics.

To justify the use of complex numbers one must demonstrate that the practice of applying the usual algebraic operations to those numbers is legitimate. Now we have seen above that Kant's problem with our basic knowledge of algebra can be solved by using geometrical picturing. Let us suppose that some way of representing the (linear) multiplication of line segments has been established and that we use direction to indicate sign. Then, when we are justifying the basic laws of algebra, our pictures will reveal to us that the multiplication of a quantity by itself always yields a positive quantity. These pictures cannot therefore justify us in extending the scope of our laws by enlarging the domain of quantities to include complex numbers. The pictures we use may even tempt us to take the nonexistence of square roots of positive quantities to be a law of algebra. And it might be hard to argue that this *is* temptation.[16]

An alternative would be to find a new, more general way to represent magnitudes to ourselves in pure intuition. The emphasis on geometrical construction defines the task further. We need a geometrical model for the complex numbers. Using this model we could exhibit to ourselves the general laws of algebra (taken now as the laws of complex algebra), and the use of the dubious imaginary numbers could be placed on a firm foundation. Mathematicians contemporary with Kant took the task of making complex numbers familiar to be significant. Kant's geometrical bias was reflected in the way men like Wechsel, Argand, and Gauss completed the task.

Kant's problem with analysis does not concern the exhibition of an unfamiliar concept but the derivation of the theorems of analysis from basic principles apprehended immediately in pure intuition. As in the case of algebra, analysis would be founded upon geometrical constructions, and, although Kant could not have developed nineteenth-century function theory on that basis, he could have exhibited the foundations of the analysis he knew.[17]

[16] See below, p. 50.

[17] That is, Kant could definitely have reconstructed the analysis developed by Newton and his successors. Whether or not Euler's algebraic analysis had already left the province of geometrical intuition is a matter for historical speculation. Of course, Eulerian analysis does not *look* much like geometry—

For, in accordance with the geometrical spirit of the seventeenth century, the originators of the calculus often tended to regard it as an offshoot of geometry. Newton even attempted to show, using his method of first and last ratios, that his calculus could be grounded in "the geometry of the Ancients."[18] That attempt could be adapted as a Kantian answer to the problem of founding analysis. Newton's assumptions could be justified by appeal to pure intuition, and the kinematic conception of geometry which Newton used (an approach which regards figures as generated by the motion of points) is re-echoed in Kant's constructive geometrical acts (the "drawing of the line in thought"). The Newtonian idea that "continuity" is an unproblematic notion could also be defended in Kant's terms. The *Anticipations of Perception* even suggest the line of defense; and it is noteworthy that, in this passage, Kant uses Newton's own favored terminology and speaks of "flowing magnitudes" (*A*170).

Insofar as Kant could contend that *any* knowledge of mathematics can be gleaned from pure intuitions, he could carry that contention through for *all* parts of pure mathematics that he knew. The strength of his position lies in its seeming ability to account for everything. When mathematics attempted to go beyond concepts which were intuitively accessible (as in the development of the function concept), the currency of ideas similar to Kant's can be seen in the strength of the protests. And when mathematics *did* forsake intuitive geometry for good, the mathematical community abandoned Kant's ideal. The death blow was not struck by Bolyai, Lobatschewsky, and Klein but by the men in the tradition which led to Weierstrass's function, continuous everywhere but differentiable nowhere.

But Kant's theory was wrong from the beginning. His attempt at explaining mathematical knowledge gives no explanation at all.

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but then neither does much of Cauchy's treatment. Yet Kant's geometrical approach might even be adequate to the intuitive concept of the continuum which Cauchy employed.

[18] See my "Fluxions, Limits and Infinite Littlenesse," (*Isis*, March 1973) for the details of Newton's program.

V. Circles of Mathematical Knowledge

Kant contends that he has explained how we can know the basic propositions of geometry a priori. Pure intuitions are supposed to teach us *general* truths which describe the structure of space *exactly*; such are the axioms from which geometry begins. I shall show that they cannot do this. Kant's account of our geometrical knowledge is circular in two different ways.

My objections will be developed against the case of geometry, but they apply equally to the cases of arithmetic and algebra; I have chosen to advance them against Kant's account of geometry because that is where his explanation of mathematical knowledge appears most cogent. I think it is easy to see that the criticism I raise could be applied with equal force in the cases of other disciplines.

The first circle arises as follows. We begin with the criticism which Berkeley leveled against Locke's theory of abstract ideas. One cannot draw a triangle which has only the properties common to all triangles. So, if Kant supposes that we learn general propositions about triangles by drawing particular figures to ourselves in thought, he will either have to show that Berkeley's point is wrong or else explain how we carefully refrain from generalizing over the peculiarities of the figure.

Let us take a simple example. Suppose that I construct a scalene triangle. From my figure I can generalize that all triangles have the side-sum property (the property that the sum of the lengths of any two sides is greater than the length of the third); but I must not infer that all triangles are scalene. Why is the one inference legitimate and not the other?

Kant answers this question in a cryptic passage in the *Methodology*:

> mathematics can achieve nothing by concepts alone but hastens at once to intuition, in which it considers the concept *in concreto*, though not empirically, but only in an intuition which it presents *a priori*, that is, which it has constructed, and in which whatever follows from the universal conditions of the construction must be universally valid of the object of the concept constructed [*A*715-716, *B*743-744].

This is best understood by means of our surface analogy. We imagined ourselves revealing the contours of a surface by drawing figures on it, and envisaged the drawing as rule-governed. We did not make it clear at that stage what the significance of the rules was to be. Kant calls the set of rules which we follow to produce the object of a concept the *schema* of that concept, noting, apropos of a discussion of Berkeley's attack on Locke that "it is schemata, not images of objects, which underlie our pure sensible concepts. No image could ever be adequate to the concept of a triangle in general" (*A*140-141, *B*180). Kant's solution to the problem is thus to claim that we can draw general conclusions using only those features of the image on which the rule has pronounced. In the above example, my production of a *scalene* triangle was brought about by a free decision of mine over and above my application of the rule. It is therefore illegitimate to use the scalene peculiarity to draw the conclusion that all triangles are scalene.

The trouble with this reply is that it seems to make the exhibition of a particular triangle in intuition quite unnecessary. For if all that we are allowed to do is to draw out features of triangles prescribed by the schema of the concept "triangle," then we can do this by conceptual analysis alone. We shall arrive only at analytic propositions in this way but, given Kant's above reply, it is not clear that we are entitled to learn more anyway. By resisting generalization over accidental features of the drawn figure, we seem to restrict our ability to generalize to properties which the schema demands be exhibited in *all* triangles. So we need only look to the schema and not to the constructed object.

The way to answer this is to hearken back to our surface analogy. We can divide into three types the properties which a figure drawn on the surface possesses. Some properties belong to the figure just because it has been drawn in accordance with a particular rule; we shall call these *R*-properties. Others belong to it in virtue of the application of the rule on the structure of the surface; these will be termed *S*-properties. Finally, there will just be the accidental properties of the figure which result from free choice and are not determined at all by the application of the rule; let us refer to these as *A*-properties. If we now revert to

Kant's theory of the construction of object-space and to the example of the triangle which we draw to represent object-space to ourselves, we can set up parallel categories. *R*-properties are just the properties which the schema *alone* determines; for the triangle an example of an *R*-property would be the property of having three sides. *S*-properties are those properties which the schema and the structure of space *together* determine; the side-sum property and the property of having the internal angle-sum equal to 180 degrees are both supposed to be *S*-properties. Finally, there are the *A*-properties, peculiarities of the particular figure drawn, such as the scaleneness of the triangle.

Now we can know that all triangles have the *R*-properties which they do have merely by analyzing our concepts. Again, since none of the *A*-properties of the particular triangle we construct is shared by all triangles we must not conclude that all triangles have an *A*-property just because we notice that our particular triangle has that property. Where pure intuition is supposed to help is in leading us to the *S*-properties which are shared by all triangles. By this means we arrive at propositions which are synthetic a priori and are basic to geometry.

Kant is, however, still in difficulties. Let us consider three geometrical propositions. (*a*) All triangles have three sides. (*b*) All triangles have the side-sum property. (*c*) All triangles are scalene. (*a*) is analytic and knowable by conceptual analysis. (*b*) is assumed to be synthetic a priori and is just the kind of proposition which we are supposed to know a priori through pure intuition. We now imagine ourselves coming to know (*b*) in the way Kant suggests. We draw a scalene triangle and see that this triangle has the side-sum property. We also see that it is scalene. If we are now to conclude that all triangles have the side-sum property but recognize that we cannot conclude that all triangles are scalene, then we must be able to distinguish *S*-properties from *A*-properties. It is not enough for Kant to provide the distinction between these types of properties. He has to show that we can use the distinction to make the right moves and avoid the wrong ones.

Unfortunately, it is difficult to see how we can distinguish *S*-properties from *A*-properties without already knowing the properties of space. For there is nothing intrinsic to a property which makes it an *S*-property rather than an *A*-property. Suppose that we do not take (*b*) to be analytic (that is, we do not take it to provide a partial explication of the concept of distance). Then there can be spaces and metric relations on them such that (*b*) is false. More straightforward is the case of the angle-sum property of a triangle. Kant assumes that we can recognize that this is an *S*-property of a triangle. In a universe like that described by Reichenbach, however, where a cross-section of the space is a plane with a protruding hemisphere,[19] having the sum of its angles equal to 180 degrees would be an *A*-property of a triangle. Conversely, it does not seem impossible that what Kant takes to be an *A*-property might turn out to be an *S*-property in some spaces. Perhaps there are spaces in which all triangles are scalene.

The upshot of this is that, to recognize something as an *S*-property we already have to know what the properties of space are. Without knowing that we were not confronting the Reichenbachian space we could not take the angle-sum property to be an *S*-property. The intuition is supposed, however, to show us that we are experiencing Euclidean space. But we cannot draw this conclusion until we have distinguished the *S*-properties. Clearly the account is turning in circles.[20]

In fact we can make the point without the appeal to bizarre spaces and we can make it for any basic general proposition of geometry. Let *G* be a basic geometrical truth. *G* is supposed to be synthetic a priori. Its synthetic status arises because its truth value is, partially, determined by the structure of space. It is logically possible that space have a structure such that *G* be false. (Otherwise, *G* would be analytic.) Further, it is logically

[19] See Hans Reichenbach, *The Philosophy of Space and Time*, (New York, 1958), p. 11.

[20] Kant might, perhaps, just deny that there is any need for explanation of how we recognize the *S*-properties. To do so would be to turn his theory of constructions into an irrelevant piece of window dressing. For the issue with which he is grappling is the issue of how we know geometrical truths and, unless we explain how we know the *S*-properties, to answer that we know geometry through knowing which properties are *S*-properties is like saying that we know geometrical propositions because we know them. If the original demand for explanation needs to be taken seriously, so does the new question. But, as I have shown, the theory of constructions is quite helpless here.

possible that *G* might have been false in such a way that many figures actually had the property ascribed to them by *G*. How would we have determined from inspection of such a figure that the property was only an *A*-property and that we should not therefore generalize over it? We can answer this question only if we can decide what counts as the application of a rule on the structure of space and what was our free decision in drawing the figure. Yet to distinguish *S*-properties from *A*-properties *is* just to recognize the structure of space. We could not therefore come to know *G* in the way which Kant describes.[21]

Kant's explanation of our geometrical knowledge is also trapped in a second circle. So far we have been supposing that it is only with general propositions that problems arise. We have focused on the difficulty of deciding how to get nonanalytic general conclusions from particular figures. We could also have asked for the justification of the basis for generalization. Has Kant really explained how we know that a *particular* figure has a *particular* property? We shall use two examples to answer the question.

The first is a very clear case due to Charles Parsons.[22] We consider the proposition that all line segments are infinitely divisible. One thing is obvious. We cannot come to this knowledge by observing a line segment infinitely divided. So how do we describe a pure intuition—or sequence of such intuitions which will lead us to knowledge? We may follow Parsons in supposing that there is one appropriate form for the description. We can represent to ourselves the line segment bisected. From this representation we can proceed to another in which we represent (say) the left-hand half of the bisected segment in all the detail in which we formerly represented the whole segment. Now we can bisect this segment and again represent the left-hand part of the

[21] Whether or not Kant has any other theory as to how the properties of space are known, he certainly has no other *clear* theory. It may be that one can dredge up from the *Aesthetic* hints of an alternative approach to mathematical knowledge, but I have preferred to concentrate on the more interesting and detailed approach which Kant favors throughout the *Critique*.

[22] See *Infinity and Kant's Conception of the "Possibility of Experience," Philosophical Review*, LXXIII (1964), 182-197; reprinted in R. P. Wolff (ed.), *Kant: A Collection of Critical Essays* (Notre Dame, Ind., 1968).

segment in enough detail. Continuing this process, we divide the segment as many times as we wish. Given any number n we see that we can divide the segment more than n times. Hence we conclude that the segment is infinitely divisible.

There are two possibilities for the way in which the increase of detail is accomplished. The first is to suppose, as Parsons does, that we increase the acuity of our vision, bringing the leftmost parts of the segment under ever more intense scrutiny. Now there is obviously a physical limit to our ability to do this, a threshhold length beyond which we cannot increase the detail of the leftmost segment sufficiently to bisect it. Let us refer to the leftmost segment at this stage as L. The claim of the Kantian is that, although our myopia prevents us from bisecting L, this is only a physical disability; "in principle" we can bisect L. Because we see that we can bisect L and that we can bisect the descendant segments L', L'', and so on indefinitely, we see that the original segment is infinitely divisible. There are now two questions: first, what kind of possibility is being invoked here? and second, how do we know that, in the appropriate sense of possibility, we can continue to bisect L, L', and so forth?

By assumption, it is not practically possible for us to bisect L. Let us then suppose that we know only that it is logically possible for us to bisect L, L', and so forth. We cannot conclude from this that it is possible in Kant's sense that L be divided as many times as we like. What is logically possible may not be possible according to the intuitability criterion. The only way to find out if the logically possible is indeed possible is to give oneself an appropriate intuition. For the case in hand that course has already been rejected. Hence if we read the principle that ensures the bisectability of L, L', and so forth as guaranteeing only *logical* possibility of bisection, it is too weak to lead to the conclusion we want.

Kant might reply to this by describing some way in which we can show that logical possibility and his kind of possibility coincide in certain cases, or perhaps even in certain families of cases. By so doing he would be able to conclude that the logical possibility of further bisection guarantees the possibility of such division according to the stronger notion of possibility which he uses. Naturally, the way in which we could come to know results about the equivalence of the two kinds of possibility would have to be explained. It is clear that Kant cannot suppose that these results are known through analysis of the concept of human experience. That would be to undercut the significance of pure intuition altogether, by making the propositions of geometry knowable through the analysis of concepts. It is not obvious how the machinery which Kant develops can be adapted to any other means by which we could know a priori that logical possibility and Kantian possibility coincide for certain cases. Indeed there are dangers that in trying to escape the conclusion that only intuition can show us if the logically possible is really possible Kant would have to sacrifice some theses which are closest to his heart.[23]

So we must conclude that the proposition which is supposed to be known is that it is possible to bisect L, L' and so on indefinitely, where we are to understand possibility in Kant's sense. How is this to be known? The proposition cannot be known through conceptual analysis (that would render what we are taking to be a truth of geometry analytic) and it must be knowable a priori. Thus it has to be knowable through pure intuition. But now we are back with the problem from which we began and which the process of bisection was supposed to clear up for us. It is clear that no progress has been made.

Nor is there any solution in the idea that we can, successively, replace our pictures of the whole line with pictures of the half-line. This idea is simpler. We construct the original line segment and bisect it. We now replace this picture with a more detailed picture of the left-hand part of the bisected segment and bisect

[23] This claim depends on my view that Kant is engaged in an epistemological as well as a metaphysical enterprise. I have taken him to offer an account of mathematical knowledge and construed "intuition" as a *sensuous* route to such knowledge ("the science of things sensual"; cf. *PC*, p. 62). To retreat to vagueness at this point is to give up the attempt at explanation, and appeals to "intellectual intuition" are the counsel of vagueness.

I hope that the investigations of Secs. III and IV show the power of sensuous intuition to account for *everything* Kant took to be pure mathematics if it can account for anything. Given this, one may admire Kant's attempt at a thorough and detailed theory of mathematical knowledge while admitting his failure. One does not save him by blurring the central concept.

that. So we continue as long as need be. Here we are immediately in difficulties. Apart from the need for justification of our ability to continue the process indefinitely, what is required is an account of how we know that the successive pictures really do represent the leftmost segments of the previous pictures in more detail. Intuition cannot reveal this to us; for we have no way of intuiting that the fine structure of segments is preserved from picture to picture. Nor is it a conceptual truth that this procedure of "enlargement" really is just a method of presenting segments in greater detail.

What is wrong in both cases is that, to draw from the sequence of intuitions the conclusion that the original segment is infinitely divisible, we need to know something equivalent to that conclusion. We set out to investigate the properties of space which, on Kant's theory, determine what we can and cannot intuit. We try to learn these properties through a series of representations of space. The series of representations which we *can in practice* give ourselves does not suffice to show the truth of the conclusion. Hence we suppose that the series can be extended to a series of representations which *would* show the conclusion. In making this supposition we commit ourselves to the notion that such a series is possible, but this possibility in turn involves the property we were supposed to be discovering. Once again we have a circle. We can know that space has a property only by knowing that a series of intuitions is possible. But we can know that that series of intuitions is possible only if we know that space has the original property. We begin by trying to discover the limitations of experience; we end up by assuming them.

The inadequacy of pure intuition does not arise only in connection with the notion of infinity. It stems immediately from the idea that we can, on inspection, determine the exact nature of a figure, whether physical or "drawn in thought." We can, according to Kant, know by means of pure intuition that there is one and only one straight line joining two given points. At first we might think that we understand what he means by this. We construct the points and draw the line between them. If, however, we were to see that this line is unique, then we should have to be able to distinguish it from any other line which we are able to draw between the two points. Now there are cases in which we cannot distinguish the one straight line from very slightly curved ones, even on close inspection. Confronted with figures in which this is the case we "see" that only one line is straight—but that is only because background geometrical knowledge is available to us.

Kant is supposing that we are in the process of gaining this knowledge. Hence he must think that we can distinguish the one straight line from the curves which are "nearly straight." But we cannot. And although this may come about from physical limitations of ours, until we have learned our geometry we are in no position to know that our failure is a medical accident—or indeed that it *is failure*. For all we know, it could just as well be success in discerning a property of space.

The problem lies with the picture behind Kant's theory. That picture presents the mind bringing forth its own creations and the naïve eye of the mind scanning those creations and detecting their properties with absolute accuracy. Kant attempts to derive a clear theory of mathematical knowledge from that picture, the theory described above. Whether that picture has been abandoned in the less clear theories of his constructivist successors is a further question.

PHILIP KITCHER

University of Vermont