Gödel

What is Cantor's continuum problem?
(*1947*)

1. The concept of cardinal number

Cantor's continuum problem is simply the question: How many points are there on a straight line in Euclidean space? In other terms, the question is: How many different sets of integers do there exist?

This question, of course, could arise only after the concept of "number" had been extended to infinite sets; hence it might be doubted if this extension can be effected in a uniquely determined manner and if, therefore, the statement of the problem in the simple terms used above is justified. Closer examination, however, shows that Cantor's definition of infinite numbers really has this character of uniqueness, and that in a very striking manner. For whatever "number" as applied to infinite sets may mean, we certainly want it to have the property that the number of objects belonging to some class does not change if, leaving the objects the same, one changes in any way whatsoever their properties or mutual relations (e.g., their colors or their distribution in space). From this, however, it follows at once that two sets (at least two sets of changeable objects of the space-time world) will have the same cardinal number if their elements can be brought into a one-to-one correspondence, which is Cantor's definition of equality between numbers. For if there exists such a correspondence for two sets A and B it is possible (at least theoretically) to change the properties and relations of each element of A into those of the corresponding element of B, whereby A is transformed into a set completely indistinguishable from B, hence of the same cardinal number. For example, assuming a square and a line segment both completely filled with mass points (so that at each point of them exactly one mass point is situated), it follows, owing to the demonstrable fact that there exists a one-to-one correspondence between the points of a square and of a line segment, and, therefore, also between the corresponding mass points, that the mass points of the square can be so rearranged as exactly to fill out the line segment, and vice versa. Such considerations, it is true, apply directly only to physical objects, but a definition of the concept of "number" which would depend on the kind of objects that are numbered could hardly be considered as satisfactory.

So there is hardly any choice left but to accept Cantor's definition of equality between numbers, which can easily be extended to a definition of "greater" and "less" for infinite numbers by stipulating that the cardinal number M of a set A is to be called less than the cardinal number N of a set B if M is different from N but equal to the cardinal number of some subset of B. On the basis of these definitions it becomes possible to

Feferman, Solomon et al. Kurt Gödel: Collected Works. Vol II N.Y. Oxford U.P. 1995

prove that there exist infinitely many different infinite cardinal numbers or "powers", and that, in particular, the number of subsets of a set is always greater than the number of its elements; furthermore, it becomes possible to extend (again without any arbitrariness) the arithmetical operations to infinite numbers (including sums and products with any infinite | number of 516
terms or factors) and to prove practically all ordinary rules of computation.

But, even after that, the problem to determine the cardinal number of an individual set, such as the linear continuum, would not be well-defined if there did not exist some "natural" representation of the infinite cardinal numbers, comparable to the decimal or some other systematic denotation of the integers. This systematic representation, however, does exist, owing to the theorem that for each cardinal number and each set of cardinal numbers[1] there exists exactly one cardinal number immediately succeeding in magnitude and that the cardinal number of every set occurs in the series thus obtained.[2] This theorem makes it possible to denote the cardinal number immediately succeeding the set of finite numbers by $\aleph_0$ (which is the power of the "denumerably infinite" sets), the next one by $\aleph_1$, etc.; the one immediately succeeding all $\aleph_i$ (where i is an integer), by $\aleph_\omega$, the next one by $\aleph_{\omega+1}$, etc., and the theory of ordinal numbers furnishes the means to extend this series farther and farther.

2. The continuum problem, the continuum hypothesis and the partial results concerning its truth obtained so far

So the analysis of the phrase "how many" leads unambiguously to quite a definite meaning for the question stated in the second line of this paper, namely, to find out which one of the $\aleph$'s is the number of points on a straight line or (which is the same) on any other continuum in Euclidean space. Cantor, after having proved that this number is certainly greater than $\aleph_0$, conjectured that it is $\aleph_1$, or (which is an equivalent proposition) that every infinite subset of the continuum has either the power of the set of integers or of the whole continuum. This is Cantor's continuum hypothesis.

[1] As to the question why there does not exist a set of all cardinal numbers, see footnote 14.

[2] In order to prove this theorem the axiom of choice (see *Fraenkel 1928*, p. 288 ff.) is necessary, but it may be said that this axiom is, in the present state of knowledge, exactly as well-founded as the system of the other axioms. It has been proved consistent, provided the other axioms are so (see *Gödel 1940*). It is exactly as evident as the other axioms for sets in the sense of arbitrary multitudes and, as for sets in the sense of extensions of definable properties; it also is demonstrable for those concepts of definability for which, in the present state of knowledge, it is possible to prove the other axioms, namely, those explained in footnotes 20 and 26.

But, although Cantor's set theory has now had a development of more than sixty years and the problem is evidently of great importance for it, nothing has been proved so far relative to the question what the power of the continuum is or whether its subsets satisfy the condition just stated, except (1) that the power of the continuum is not a cardinal number of a certain very special kind, namely, not a limit of denumerably many smaller cardinal numbers,[3] and (2) that the proposition just mentioned about the
17 subsets of the continuum is | true for a certain infinitesimal fraction of these subsets, the analytical[4] sets.[5] Not even an upper bound, however high, can be assigned for the power of the continuum. Nor is there any more known about the quality than about the quantity of the cardinal number of the continuum. It is undecided whether this number is regular or singular, accessible or inaccessible, and (except for König's negative result) what its character of cofinality[4] is. The only thing one knows, in addition to the results just mentioned, is a great number of consequences of, and some propositions equivalent to, Cantor's conjecture.[6]

This pronounced failure becomes still more striking if the problem is considered in its connection with general questions of cardinal arithmetic. It is easily proved that the power of the continuum is equal to $2^{\aleph_0}$. So the continuum problem turns out to be a question from the "multiplication table" of cardinal numbers, namely, the problem to evaluate a certain infinite product (in fact the simplest non-trivial one that can be formed). There is, however, not one infinite product (of factors > 1) for which only as much as an upper bound for its value can be assigned. All one knows about the evaluation of infinite products are two lower bounds due to Cantor and König (the latter of which implies a generalization of the aforementioned negative theorem on the power of the continuum), and some theorems concerning the reduction of products with different factors to exponentiations and of exponentiations to exponentiations with smaller bases or exponents. These theorems reduce[7] the whole problem of computing infinite products to the evaluation of $\aleph_\alpha^{\mathrm{cf}(\aleph_\alpha)}$ and the performance of certain fundamental operations on ordinal numbers, such as determining the limit of a series of them. $\aleph_\alpha^{\mathrm{cf}(\aleph_\alpha)}$, and therewith all products and powers, can easily be

[3]See *Hausdorff 1914*, p. 68. The discoverer of this theorem, J. König, asserted more than he had actually proved (see his *1905*).

[4]See the list of definitions at the end of this paper.

[5]See *Hausdorff 1935*, p. 32. Even for complements of analytical sets the question is undecided at present, and it can be proved only that they have (if they are infinite) either the power $\aleph_0$ or $\aleph_1$ or that of the continuum (see *Kuratowski 1933*, p. 246).

[6]See *Sierpiński 1934*.

[7]This reduction can be effected owing to the results and methods of *Tarski 1925*.

computed[8] if the "generalized continuum hypothesis" is assumed, i.e., if it is assumed that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for every α, or, in other terms, that the number of subsets of a set of power $\aleph_\alpha$ is $\aleph_{\alpha+1}$. But, without making any undemonstrated assumption, it is not even known whether or not $m < n$ implies $2^m < 2^n$ (although it is trivial that it implies $2^m \leq 2^n$), nor even whether $2^{\aleph_0} < 2^{\aleph_1}$.

3. Restatement of the problem on the basis of an analysis of the foundations of set theory and results obtained along these lines

This scarcity of results, even as to the most fundamental questions in this field, may be due to some extent to purely mathematical difficulties; it seems, however (see Section 4 below), that there are also deeper reasons
behind it and that a complete solution of | these problems can be obtained 518
only by a more profound analysis (than mathematics is accustomed to give) of the meanings of the terms occurring in them (such as "set", "one-to-one correspondence", etc.) and of the axioms underlying their use. Several such analyses have been proposed already. Let us see then what they give for our problem.

First of all there is Brouwer's intuitionism, which is utterly destructive in its results. The whole theory of the $\aleph$'s greater than $\aleph_1$ is rejected as meaningless.[9] Cantor's conjecture itself receives several different meanings, all of which, though very interesting in themselves, are quite different from the original problem, and which lead partly to affirmative, partly to negative answers;[10] not everything in this field, however, has been clarified sufficiently. The "half-intuitionistic" standpoint along the lines of H. Poincaré and H. Weyl[11] would hardly preserve substantially more of set theory.

This negative attitude towards Cantor's set theory, however, is by no means a necessary outcome of a closer examination of its foundations, but only the result of certain philosophical conceptions of the nature of mathematics, which admit mathematical objects only to the extent in which they

[8]For regular numbers $\aleph_\alpha$ one obtains immediately:
$$\aleph_\alpha^{\mathrm{cf}(\aleph_\alpha)} = \aleph_\alpha^{\aleph_\alpha} = 2^{\aleph_\alpha} = \aleph_{\alpha+1}.$$

[9]See *Brouwer 1909*.

[10]See *Brouwer 1907*, I, 9; III, 2.

[11]See *Weyl 1932*. If the procedure of construction of sets described there (p. 20) is iterated a sufficiently large (transfinite) number of times, one gets exactly the real numbers of the model for set theory spoken of below in Section 4, in which the continuum hypothesis is true. But this iteration would hardly be possible within the limits of the half-intuitionistic standpoint.

are (or are believed to be) interpretable as acts and constructions of our own mind, or at least completely penetrable by our intuition. For someone who does not share these views there exists a satisfactory foundation of Cantor's set theory in its whole original extent, namely, axiomatics of set theory, under which the logical system of *Principia mathematica* (in a suitable interpretation) may be subsumed.

It might at first seem that the set-theoretical paradoxes would stand in the way of such an undertaking, but closer examination shows that they cause no trouble at all. They are a very serious problem, but not for Cantor's set theory. As far as sets occur and are necessary in mathematics (at least in the mathematics of today, including all of Cantor's set theory), they are sets of integers, or of rational numbers (i.e., of pairs of integers), or of real numbers (i.e., of sets of rational numbers), or of functions of real numbers (i.e., of sets of pairs of real numbers), etc.; when theorems about all sets (or the existence of sets) in general are asserted, they can always be interpreted without any difficulty to mean that they hold for sets of integers as well as for sets of real numbers, etc. (respectively, that there exist either sets of integers, or sets of real numbers, or ... etc., which have the asserted property). This concept of set, however, according to which a set is anything obtainable from the integers (or some other well-defined
519 | objects) by iterated application[12] of the operation "set of",[13] and not something obtained by dividing the totality of all existing things into two categories, has never led to any antinomy whatsoever; that is, the perfectly "naïve" and uncritical working with this concept of set has so far proved completely self-consistent.[14]

But, furthermore, the axioms underlying the unrestricted use of this concept of set, or, at least, a portion of them which suffices for all mathematical proofs ever produced up to now, have been so precisely formulated in axiomatic set theory[15] that the question whether some given proposition follows from them can be transformed, by means of logistic symbolism, into

[12]This phrase is to be understood so as to include also transfinite iteration, the totality of sets obtained by finite iteration forming again a set and a basis for a further application of the operation "set of".

[13]The operation "set of x's" cannot be defined satisfactorily (at least in the present state of knowledge), but only be paraphrased by other expressions involving again the concept of set, such as: "multitude of x's", "combination of any number of x's", "part of the totality of x's"; but as opposed to the concept of set in general (if considered as primitive) we have a clear notion of this operation.

[14]It follows at once from this explanation of the term "set" that a set of all sets or other sets of a similar extension cannot exist, since every set obtained in this way immediately gives rise to further application of the operation "set of" and, therefore, to the existence of larger sets.

[15]See, e.g., *Bernays 1937, 1941, 1942, 1942a, 1943, von Neumann 1925*; cf. also *von Neumann 1928a* and *1929, Gödel 1940*.

a purely combinatorial problem concerning the manipulation of symbols which even the most radical intuitionist must acknowledge as meaningful. So Cantor's continuum problem, no matter what philosophical standpoint one takes, undeniably retains at least this meaning: to ascertain whether an answer, and if so what answer, can be derived from the axioms of set theory as formulated in the systems quoted.

Of course, if it is interpreted in this way, there are (assuming the consistency of the axioms) a priori three possibilities for Cantor's conjecture: It may be either demonstrable or disprovable or undecidable.[16] The third alternative (which is only a precise formulation of the conjecture stated above that the difficulties of the problem are perhaps not purely mathematical) is the most likely, and to seek a proof for it is at present one of the most promising ways of attacking the problem. One result along these lines has been obtained already, namely, that Cantor's conjecture is not disprovable from the axioms of set theory, provided that these axioms are consistent (see Section 4).

It is to be noted, however, that, even if one should succeed in proving its undemonstrability as well, this would (in contradistinction, for example, to the proof for the transcendency of π) by no means settle the question
definitively. | Only someone who (like the intuitionist) denies that the 520
concepts and axioms of classical set theory have any meaning (or any well-defined meaning) could be satisfied with such a solution, not someone who believes them to describe some well-determined reality. For in this reality Cantor's conjecture must be either true or false, and its undecidability from the axioms as known today can only mean that these axioms do not contain a complete description of this reality; and such a belief is by no means chimerical, since it is possible to point out ways in which a decision of the question, even if it is undecidable from the axioms in their present form, might nevertheless be obtained.

For first of all the axioms of set theory by no means form a system closed in itself, but, quite on the contrary, the very concept of set[17] on which they are based suggests their extension by new axioms which assert the existence of still further iterations of the operation "set of". These axioms can also be formulated as propositions asserting the existence of very great cardinal numbers or (which is the same) of sets having these cardinal numbers.

[16]In case of the inconsistency of the axioms the last one of the four a priori possible alternatives for Cantor's conjecture would occur, namely, it would then be both demonstrable and disprovable by the axioms of set theory.

[17]Similarly also the concept "property of set" (the second of the primitive terms of set theory) can constantly be enlarged and, furthermore, concepts of "property of property of set" etc. be introduced whereby new axioms are obtained, which, however, as to their consequences for propositions referring to limited domains of sets (such as the continuum hypothesis) are contained in the axioms depending on the concept of set.

The simplest of these strong "axioms of infinity" assert the existence of inaccessible numbers (and of numbers inaccessible in the stronger sense) $> \aleph_0$. The latter axiom, roughly speaking, means nothing else but that the totality of sets obtainable by exclusive use of the processes of formation of sets expressed in the other axioms forms again a set (and, therefore, a new basis for a further application of these processes).[18] Other axioms of infinity have been formulated by P. Mahlo.[19] Very little is known about this section of set theory; but at any rate these axioms show clearly, not only that the axiomatic system of set theory as known today is incomplete, but also that it can be supplemented without arbitrariness by new axioms which are only the natural continuation of the series of those set up so far.

That these axioms have consequences also far outside the domain of very great transfinite numbers, which are their immediate object, can be proved; each of them (as far as they are known) can, under the assumption of consistency, be shown to increase the number of decidable propositions even in the field of Diophantine equations. As for the continuum problem, there is little hope of solving it by means of those axioms of infinity which can be set up on the basis of principles known today (the above-mentioned proof for the undisprovability of the continuum hypothesis, e.g., goes through for all of them without any change). But probably there exist others based on hitherto unknown principles; also there may exist, besides the ordinary
521 axioms, the axioms of infinity and | the axioms mentioned in footnote 17, other (hitherto unknown) axioms of set theory which a more profound understanding of the concepts underlying logic and mathematics would enable us to recognize as implied by these concepts.

Furthermore, however, even disregarding the intrinsic necessity of some new axiom, and even in case it had no intrinsic necessity at all, a decision about its truth is possible also in another way, namely, inductively by studying its "success", that is, its fruitfulness in consequences and in particular in "verifiable" consequences, i.e., consequences demonstrable without the new axiom, whose proofs by means of the new axiom, however, are considerably simpler and easier to discover, and make it possible to condense into one proof many different proofs. The axioms for the system of real numbers, rejected by the intuitionists, have in this sense been verified to some extent owing to the fact that analytical number theory frequently allows us to prove number-theoretical theorems which can subsequently be verified by elementary methods. A much higher degree of verification than that, however, is conceivable. There might exist axioms so abundant in

[18]See *Zermelo 1930*.

[19]See his *1911*, pp. 190–200, *1913*, pp. 269–276. From Mahlo's presentation of the subject, however, it does not appear that the numbers he defines actually exist.

their verifiable consequences, shedding so much light upon a whole discipline, and furnishing such powerful methods for solving given problems (and even solving them, as far as that is possible, in a constructivistic way) that quite irrespective of their intrinsic necessity they would have to be assumed at least in the same sense as any well-established physical theory.

4. Some observations about the question: In what sense and in which direction may a solution of the continuum problem be expected?

But are such considerations appropriate for the continuum problem? Are there really any strong indications for its unsolubility by the known axioms? I think there are at least two.

The first one is furnished by the fact that there are two quite differently defined classes of objects which both satisfy all axioms of set theory written down so far. One class consists of the sets definable in a certain manner by properties of their elements,[20] the other of the sets in the sense of arbitrary multitudes irrespective of if, or how, they can be defined. Now, before it is settled what objects are to be numbered, and on the basis of what one-to-one correspondences, one could hardly expect to be able to determine their number (except perhaps in case of some fortunate coincidence). If, however, someone believes that it is meaningless to speak of sets except in the sense of extensions of definable properties, or, at least, that no other sets exist, then, too, he can hardly expect more than a small fraction of the problems of set theory to be solvable without making use of this, in his opinion essential, characteristic of sets, namely, that they are | all derived from 522
(or in a sense even identical with) definable properties. This characteristic of sets, however, is neither formulated explicitly nor contained implicitly in the accepted axioms of set theory. So from either point of view, if in addition one has regard to what was said above in Section 2, it is plausible that the continuum problem will not be solvable by the axioms set up so far, but, on the other hand, may be solvable by means of a new axiom

[20]Namely, definable "in terms of ordinal numbers" (i.e., roughly speaking, under the assumption that for each ordinal number a symbol denoting it is given) by means of transfinite recursions, the primitive terms of logic, and the ϵ-relation, admitting, however, as elements of sets and of ranges of quantifiers only previously defined sets. See my papers *1939a* and *1940*, where an exactly equivalent, although in its definition slightly different, concept of definability (under the name of "constructibility") is used. The paradox of Richard, of course, does not apply to this kind of definability, since the totality of ordinals is certainly not denumerable.

which would state or at least imply something about the definability of sets.[21]

The latter half of this conjecture has already been verified; namely, the concept of definability just mentioned (which is itself definable in terms of the primitive notions of set theory) makes it possible to derive the generalized continuum hypothesis from the axiom that every set is definable in this sense.[22] Since this axiom (let us call it "A") turns out to be demonstrably consistent with the other axioms, under the assumption of the consistency of these axioms, this result (irrespective of any philosophical opinion) shows the consistency of the continuum hypothesis with the axioms of set theory, provided that these axioms themselves are consistent.[23] This proof in its structure is analogous to the consistency proof for non-Euclidean geometry by means of a model within Euclidean geometry, insofar as it follows from the axioms of set theory that the sets definable in the above sense form a model for set theory in which furthermore the proposition A and, therefore, the generalized continuum hypothesis is true. But the definition of "definability" can also be so formulated that it becomes a definition of a concept of "set" and a relation of "element of" (satisfying the axioms of set theory) in terms of entirely different concepts, namely, the concept of "ordinal numbers", in the sense of elements ordered by some relation of "greater" and "less", this ordering relation itself, and the notion of "recursively defined function of ordinals", which can be taken as primitive and be described axiomatically by way of an extension of Peano's axioms.[24] (Note that this does not apply to my original formulation presented in the papers quoted above, because there the general concept of "set" with its element relation occurs in the definition of "definable set", although the definable sets remain the same if, afterwards, in the definition of "definability" the term "set" is replaced by "definable set".)

523 | A second argument in favor of the unsolubility of the continuum problem on the basis of the ordinary axioms can be based on certain facts (not known or not existing at Cantor's time) which seem to indicate that Cantor's

[21]D. Hilbert's attempt at a solution of the continuum problem (see his *1926*), which, however, has never been carried through, also was based on a consideration of all possible definitions of real numbers.

[22]On the other hand, from an axiom in some sense directly opposite to this one the negation of Cantor's conjecture could perhaps be derived.

[23]See my paper *1940* and note *1939a*. For a carrying through of the proof in all details, my paper *1940* is to be consulted.

[24]For such an extension see *Tarski 1924*, where, however, the general concept of "set of ordinal numbers" is used in the axioms; this could be avoided, without any loss in demonstrable theorems, by confining oneself from the beginning to recursively definable sets of ordinals.

conjecture will turn out to be wrong;[25] for a negative decision the question is (as just explained) demonstrably impossible on the basis of the axioms as known today.

There exists a considerable number of facts of this kind which, of course, at the same time make it likely that not all sets are definable in the above sense.[26] One such fact, for example, is the existence of certain properties of point sets (asserting an extreme rareness of the sets concerned) for which one has succeeded in proving the existence of undenumerable sets having these properties, but no way is apparent by means of which one could expect to prove the existence of examples of the power of the continuum. Properties of this type (of subsets of a straight line) are: (1) being of the first category on every perfect set,[27] (2) being carried into a zero set by every continuous one-to-one mapping of the line on itself.[28] Another property of a similar nature is that of being coverable by infinitely many intervals of any given lengths. But in this latter case one has so far not even succeeded in proving the existence of undenumerable examples. From the continuum hypothesis, however, it follows that there exist in all three cases not only examples of the power of the continuum,[29] but even such as are carried into themselves (up to denumerably many points) by *every* translation of the straight line.[30]

And this is not the only paradoxical consequence of the continuum hypothesis. Others, for example, are that there exist: (1) subsets of a straight line of the power of the continuum which are covered (up to denumerably many points) by *every* dense set of intervals, or (in other terms) which contain no undenumerable subset nowhere dense on the straight line,[31] (2) subsets of a straight line of the power of the continuum which contain no undenumerable zero set,[32] (3) subsets of Hilbert

[25]Views tending in this direction have been expressed also by N. Luzin in his *1935*, p. 129 ff. See also *Sierpiński 1935*.

[26]That all sets are "definable in terms of ordinals" if *all* procedures of definition, i.e., also quantification and the operation $\hat{x}$ with respect to *all* sets, irrespective of whether they have or can be defined, are admitted could be expected with more reason, but still it would not at all be justified to assume this as an axiom. It is worth noting that the proof that the continuum hypothesis holds for the definable sets, or follows from the assumption that all sets are definable, does not go through for this kind of definability, although the assumption that these two concepts of definability are equivalent is, of course, demonstrably consistent with the axioms.

[27]See *Sierpiński 1934a* and *Kuratowski 1933*, p. 269 ff.

[28]See *Luzin and Sierpiński 1918* and *Sierpiński 1934a*.

[29]For the 3rd case see *Sierpiński 1934*, p. 39, Theorem 1.

[30]See *Sierpiński 1935a*.

[31]See *Luzin 1914*, p. 1259.

[32]See *Sierpiński 1924*, p. 184.

space of the power of the continuum which contain no undenumerable subset of finite dimension,[33] (4) an infinite sequence A^i of decompositions of
524 any set M of the power of the continuum into continuum | many mutually exclusive sets A^i_x such that, in whichever way a set $A^i_{x_i}$ is chosen for each i, $\prod_i (M - A^i_{x_i})$ is always denumerable.[34] Even if in (1)–(4) "power of the continuum" is replaced by "$\aleph_1$", these propositions are very implausible; the proposition obtained from (3) in this way is even equivalent with (3).

One may say that many of the results of point-set theory obtained without using the continuum hypothesis also are highly unexpected and implausible.[35] But, true as that may be, still the situation is different there, insofar as in those instances (such as, e.g., Peano's curves) the appearance to the contrary can in general be explained by a lack of agreement between our intuitive geometrical concepts and the set-theoretical ones occurring in the theorems. Also, it is very suspicious that, as against the numerous plausible propositions which imply the negation of the continuum hypothesis, not one plausible proposition is known which would imply the continuum hypothesis. Therefore one may on good reason suspect that the role of the continuum problem in set theory will be this, that it will finally lead to the discovery of new axioms which will make it possible to disprove Cantor's conjecture.

Definitions of some of the technical terms

Definitions 4–12 refer to subsets of a straight line, but can be literally transferred to subsets of Euclidean spaces of any number of dimensions; definitions 13–14 refer to subsets of Euclidean spaces.

1. I call "character of cofinality" of a cardinal number m (abbreviated by "cf(m)") the smallest number n such that m is the sum of n numbers $< m$.
2. A cardinal number m is regular if $\text{cf}(m) = m$, otherwise singular.
3. An infinite cardinal number m is inaccessible if it is regular and has no immediate predecessor (i.e., if, although it is a limit of numbers $< m$, it is not a limit of fewer than m such numbers); it is inaccessible in the stronger sense if each product (and, therefore, also each sum) of fewer than m numbers $< m$ is $< m$. (See *Sierpiński and Tarski 1930*, *Tarski 1938*. From the generalized continuum hypothesis follows the equivalence of these two notions. This equivalence, however, is a much weaker and much more plausible proposition. $\aleph_0$ evidently is inaccessible in both senses. As for finite numbers, 0 and 2 and no others are inaccessible in the stronger sense (by the above definition), which suggests that the same will hold also for the correct extension of the concept of inaccessibility to finite numbers.)
4. A set of intervals is dense if every interval has points in common with some interval of the set. (The endpoints of an interval are not considered as points of the interval.)
5. A zero set is a set which can be covered by infinite sets of intervals with arbitrarily small lengths-sum.
6. A neighborhood of a point P is an interval containing P.
7. A subset A of B is dense in B if every neighborhood of any point of B contains points of A.
8. A point P is in the exterior of A if it has a neighborhood containing no point of A.
9. A subset A of B is nowhere dense on B if those points of B which are in the exterior of A are dense in B. (Such sets A are exactly the subsets of the borders of the open sets in B, but the term "border-set" is unfortunately used in a different sense.)
10. A subset A of B is of the first category in B if it is the sum of denumerably many sets nowhere dense in B.
11. | Set A is of the first category on B if the intersection $A \cdot B$ is of the 5:
first category in B.
12. A set is perfect if it is closed and has no isolated point (i.e., no point with a neighborhood containing no other point of the set).
13. Borel sets are defined as the smallest system of sets satisfying the postulates:
   (1) The closed sets are Borel sets.
   (2) The complement of a Borel set is a Borel set.
   (3) The sum of denumerably many Borel sets is a Borel set.
14. A set is analytic if it is the orthogonal projection of some Borel set of a space of next higher dimension. (Every Borel set therefore is, of course, analytic.)
15. Quantifiers are the logistic symbols standing for the phrases: "for all objects x" and "there exist objects x". The totality of objects x to which they refer is called their range.
16. The symbol "$\hat{x}$" means "the set of those objects x for which ...".

[33] See *Hurewicz 1932*.

[34] See *Braun and Sierpiński 1932*, p. 1, proposition (Q). This proposition and the one stated under (3) in the text are equivalent with the continuum hypothesis.

[35] See, e.g., *Blumenthal 1940*.