Consciousness, philosophy, and mathematics

L. E. J. BROUWER

The... point of view that there are no non-experienced truths and that logic is not an absolutely reliable instrument to discover truths has found acceptance with regard to mathematics much later than with regard to practical life and to science. Mathematics rigorously treated from this point of view, including deducing theorems exclusively by means of introspective construction, is called intuitionistic mathematics. In many respects it deviates from classical mathematics. In the first place because classical mathematics uses logic to generate theorems, believes in the existence of unknown truths, and in particular applies the *principle of the excluded third* expressing that every mathematical assertion (i.e. every assignment of a mathematical property to a mathematical entity) either is a truth or cannot be a truth. In the second place because classical mathematics confines itself to *predeterminate* infinite sequences for which from the beginning the nth element is fixed for each n. Owing to this confinement classical mathematics, to define real numbers, has only predeterminate convergent infinite sequences of rational numbers at its disposal. Out of real numbers defined in this way, only subspecies of "ever unfinished denumerable" species of real numbers can be composed by means of introspective construction. Such ever unfinished denumerable species all being of measure zero, classical mathematics, to create the continuum out of points, needs some logical process starting from one or more axioms. Consequently we may say that classical analysis, however appropriate it be for technique and science, has less mathematical truth than intuitionistic analysis performing the said composition of the continuum by considering the species of freely proceeding convergent infinite sequences of rational numbers, without having recourse to language or logic.

As a matter of course also the languages of the two mathematical schools diverge. And even in those mathematical theories which are covered by a neutral language, i.e. by a language understandable on both sides, either school operates with mathematical entities not recognized

Excerpted by kind permission of the publisher from 10th International Congress of Philosophy, Amsterdam, 1948, *Proceedings I*, Fascicule II (Amsterdam: North-Holland Publishing Company, 1949), pp. 1243–9.

by the other one: there are intuitionist structures which cannot be fitted into any classical logical frame, and there are classical arguments not applying to any introspective image. Likewise, in the theories mentioned, mathematical entities recognized by both parties on each side are found satisfying theorems which for the other school are either false, or senseless, or even in a way contradictory. In particular, theorems holding in intuitionism, but not in classical mathematics, often originate from the circumstance that for mathematical entities belonging to a certain species, the possession of a certain property imposes a special character on their way of development from the basic intuition, and that from this special character of their way of development from the basic intuition, properties ensue which for classical mathematics are false. A striking example is the intuitionist theorem that a full function of the unity continuum, i.e. a function assigning a real number to every non-negative real number not exceeding unity, is necessarily uniformly continuous.

To elucidate the consequences of the rejection of the principle of the excluded third as an instrument to discover truths, we shall put the wording of this principle into the following slightly modified, intuitionistically more adequate form, called the *simple principle of the excluded third:*

> *Every assignment τ of a property to a mathematical entity can be judged, i.e. either proved or reduced to absurdity.*

Then for a single such assertion τ the enunciation of this principle is non-contradictory in intuitionistic as well as in classical mathematics. For, if it were contradictory, then the absurdity of τ would be true and absurd at the same time, which is impossible. Moreover, as can easily be proved, for a *finite* number of such assertions τ the simultaneous enunciation of the principle is non-contradictory likewise. However, for the simultaneous enunciation of the principle for all elements of an *arbitrary* species of such assertions τ this non-contradictority cannot be maintained.

E.g. from the supposition, for a definite real number c_1, that the assertion: c_1 *is rational,* has been proved to be either true or contradictory, no contradiction can be deduced. Furthermore, $c_1, c_2, \dots c_m$ being real numbers, neither the simultaneous supposition, for each of the values $1, 2, \dots m$ of ν, that the assertion: c_ν *is rational,* has been proved to be either true or contradictory, can lead to a contradiction. However, the simultaneous supposition for *all* real numbers c that the assertion: c *is rational,* has been proved to be either true or contradictory, does lead to a contradiction.

Consequently if we formulate the *complete principle of the excluded third* as follows:

If a, b, and c are species of mathematical entities, if further both a and b form part of c, and if b consists of those elements of c which cannot belong to a, then c is identical with the union of a and b,

the latter principle is contradictory.

A corollary of the *simple* principle of the excluded third says that:

If for an assignment τ of a property to a mathematical entity the non-contradictority, i.e. the absurdity of the absurdity, has been established, the truth of τ can be demonstrated likewise.

The analogous corollary of the *complete* principle of the excluded third is the *principle of reciprocity of complementarity,* running as follows:

If a, b, and c are species of mathematical entities, if further a and b form part of c, and if b consists of the elements of c which cannot belong to a, then a consists of the elements of c which cannot belong to b.

Another corollary of the *simple* principle of the excluded third is the *simple principle of testability* saying that

every assignment τ of a property to a mathematical entity can be tested, i.e. proved to be either non-contradictory or absurd.

The analogous corollary of the *complete* principle of the excluded third is the following *complete principle of testability:*

If a, b, d, and c are species of mathematical entities, if each of the species a, b, and d forms part of c, if b consists of the elements of c which cannot belong to a, and d of the elements of c which cannot belong to b, then c is identical with the union of b and d.

For intuitionism the principle of the excluded third and its corollaries are assertions σ about assertions τ, and these assertions σ only then are "realized", i.e. only then convey truths, if these truths have been experienced.

Each assertion τ of the possibility of a construction of bounded finite character in a finite mathematical system furnishes a case of realization of the principle of the excluded third. For every such construction can be attempted only in a finite number of particular ways, and each attempt proves successful or abortive in a finite number of steps.

If the assertion of an absurdity is called a *negative assertion,* then each negative assertion furnishes a case of realization of the principle of reciprocity of complementarity. For, let α be a negative assertion, indicating

the absurdity of the assertion β. As, on the one hand, the implication of the truth of an assertion a by the truth of an assertion b implies the implication of the absurdity of b by the absurdity of a, whilst, on the other hand, the truth of β implies the absurdity of the absurdity of β, we conclude that the absurdity of the absurdity of the absurdity of β, i.e. the non-contradictority of α, implies the absurdity of β, i.e. implies α.

In consequence of this realization of the principle of reciprocity of complementarity the principles of testability and of the excluded third are equivalent in the domain of negative assertions. For, if for α the principle of testability holds, this means that either the absurdity of the absurdity of β or the non-contradictority of the absurdity of β, i.e. by the preceding paragraph, that either the absurdity of the absurdity of β or the absurdity of β, i.e. either the absurdity of α or α can be proved, so that α satisfies the principle of the excluded third.

To give some examples refuting the principle of the excluded third and its corollaries, we introduce the notion of a *drift*. By a drift we understand the union γ of a convergent fundamental sequence of real numbers $c_1(\gamma), c_2(\gamma), \ldots$, called the *counting-numbers* of the drift, and the limiting-number $c(\gamma)$ of this sequence, called the *kernel* of the drift, all counting-numbers lying apart[1] from each other and from the kernel. If $c_\nu(\gamma) <\circ c(\gamma)$ for each ν, the drift will be called *left-winged.* If $c_\nu(\gamma) \circ> c(\gamma)$ for each ν, the drift will be called *right-winged.* If the fundamental sequence $c_1(\gamma), c_2(\gamma), \ldots$ is the union of a fundamental sequence of *left counting-numbers* $l_1(\gamma), l_2(\gamma), \ldots$ such that $l_\nu(\gamma) <\circ c(\gamma)$ for each ν, and a fundamental sequence of *right counting-numbers* $d_1(\gamma), d_2(\gamma), \ldots$ such that $d_\nu(\gamma) \circ> c(\gamma)$ for each ν, the drift will be called *two-winged.*

Let α be a mathematical assertion so far neither tested nor recognized as testable. Then in connection with this assertion α and with a drift γ the creating subject can generate an infinitely proceeding sequence $R(\gamma, \alpha)$ of real numbers $c_1(\gamma, \alpha), c_2(\gamma, \alpha), \ldots$ according to the following direction: As long as during the choice of the $c_n(\gamma, \alpha)$ the creating subject has experienced neither the truth, nor the absurdity of α, each $c_n(\gamma, \alpha)$ is chosen equal to $c(\gamma)$. But as soon as between the choice of $c_{r-1}(\gamma, \alpha)$ and that of $c_r(\gamma, \alpha)$ the creating subject has experienced either the truth or the absurdity of α, $c_r(\gamma, \alpha)$, and likewise $c_{r+\nu}(\gamma, \alpha)$ for each natural

[1] If for two real numbers a and b defined by convergent infinite sequences of rational numbers $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ respectively, two such natural numbers m and n can be calculated that $b_\nu - a_\nu > 2^{-n}$ for $\nu \geqslant m$, we write $b \circ> a$ and $a <\circ b$, and a and b are said to lie *apart* from each other. If $a = b$ is absurd, we write $a \neq b$. If $a <\circ b$ is absurd, we write $a \geqslant b$. If both $a = b$ and $a <\circ b$ are absurd, we write $a > b$. The absurdities of $a <\circ b$ and $a < b$ prove to be mutually equivalent, and the absurdity of $a \geqslant b$ proves to be equivalent to $a < b$.

number ν, is chosen equal to $c_r(\gamma)$. This sequence $R(\gamma, \alpha)$ converges to a real number $D(\gamma, \alpha)$ which will be called a *direct checking-number of* γ *through* α.

Again, in connection with α and with a two-winged drift γ the creating subject can generate an infinitely proceeding sequence $S(\gamma, \alpha)$ of real numbers $\omega_1(\gamma, \alpha), \omega_2(\gamma, \alpha), \ldots$ according to the following direction: As long as during the choice of the $\omega_n(\gamma, \alpha)$ the creating subject has experienced neither the truth, nor the absurdity of α, each $\omega_n(\gamma, \alpha)$ is chosen equal to $c(\gamma)$. But as soon as between the choice of $\omega_{r-1}(\gamma, \alpha)$ and that of $\omega_r(\gamma, \alpha)$ the creating subject has experienced the truth of α, $\omega_r(\gamma, \alpha)$, and likewise $\omega_{r+\nu}(\gamma, \alpha)$ for each natural number ν, is chosen equal to $d_r(\gamma)$. And as soon as between the choice of $\omega_{s-1}(\gamma, \alpha)$ and that of $\omega_s(\gamma, \alpha)$ the creating subject has experienced the absurdity of α, $\omega_s(\gamma, \alpha)$, and likewise $\omega_{s+\nu}(\gamma, \alpha)$ for each natural number ν, is chosen equal to $l_s(\gamma)$. This sequence $S(\gamma, \alpha)$ converges to a real number $E(\gamma, \alpha)$ which will be called an *oscillatory checking-number of* γ *through* α.

Let γ be a right-winged drift whose counting-numbers are rational. Then the assertion of the rationality of $D(\gamma, \alpha)$ is testable, but not judgable, and its non-contradictority is not equivalent to its truth. Furthermore we have $D(\gamma, \alpha) > c(\gamma)$, but not $D(\gamma, \alpha) \circ\!> c(\gamma)$.

Let γ be a two-winged drift whose right counting-numbers are rational, and whose left counting-numbers are irrational. Then the assertion of the rationality of $E(\gamma, \alpha)$ is neither judgeable, nor is it testable, nor is its non-contradictority equivalent to its truth. Furthermore $E(\gamma, \alpha)$ is neither $\geqslant c(\gamma)$, nor $\leqslant c(\gamma)$.

The long belief in the universal validity of the principle of the excluded third in mathematics is considered by intuitionism as a phenomenon of history of civilization of the same kind as the old-time belief in the rationality of π or in the rotation of the firmament on an axis passing through the earth. And intuitionism tries to explain the long persistence of this dogma by two facts: firstly the obvious non-contradictority of the principle for an arbitrary single assertion; secondly the practical validity of the whole of classical logic for an extensive group of *simple everyday phenomena*. The latter fact apparently made such a strong impression that the play of thought that classical logic originally was, became a deep-rooted habit of thought which was considered not only as useful but even as aprioristic.

Obviously the field of validity of the principle of the excluded third is identical with the intersection of the fields of validity of the principle of testability and the principle of reciprocity of complementarity. Furthermore the former field of validity is a *proper* subfield of each of the latter ones, as is shown by the following examples:

Let A be the species of the direct checking-numbers of drifts with rational counting-numbers, B the species of the irrational real numbers, C the union of A and B. Then all assertions of rationality of an element of C satisfy the principle of testability, whilst there are assertions of rationality of an element of C not satisfying the principle of the excluded third. Again, all assertions of equality of two real numbers satisfy the principle of reciprocity of complementarity, whereas there are assertions of equality of two real numbers not satisfying the principle of the excluded third.

In the domain of mathematical assertions the property of absurdity, just as the property of truth, is a *universally additive property*, that is to say, if it holds for each element α of a species of assertions, it also holds for the assertion which is the union of the assertions α. *This property of universal additivity does not obtain for the property of non-contradictority.* However, non-contradictority does possess the weaker property of *finite additivity*, that is to say, if the assertions ρ and σ are non-contradictory, the assertion τ which is the union of ρ and σ, is also non-contradictory. For, let us start for a moment from the supposition ω that τ is contradictory. Then the truth of ρ would entail the contradictority of σ, which would clash with the data, so that the truth of ρ is absurd, i.e. ρ is absurd. This consequence of the supposition ω clashing with the data, the supposition ω is contradictory, i.e. τ is non-contradictory.

Application of this theorem to the special non-contradictory assertions that are the enunciations of the principle of the excluded third for a single assertion, establishes the above-mentioned non-contradictority of the simultaneous enunciation of this principle for a finite number of assertions.

Within some species of mathematical entities the absurdities of two non-equivalent[2] assertions may be equivalent. E.g. each of the following three pairs of non-equivalent assertions relative to a real number a:

I 1. $a=a$;	I 2. either $a \leqslant 0$ or $a \geqslant 0$
II 1. $a \geqslant 0$;	II 2. either $a=0$ or $a \circ\!> 0$
III 1. $a > 0$;	III 2. $a \circ\!> 0$

furnishes a pair of equivalent absurdities.

It occurs that within some species of mathematical entities some absurdities of constructive properties can be given a constructive form. E.g. for a natural number a the absurdity of the existence of two natural numbers different from a and from 1 and having a as their product is equivalent to the existence, whenever a is divided by a natural number dif-

[2]By non-equivalence we understand absurdity of equivalence, just as by noncontradictority we understand absurdity of contradictority.

ferent from a and from 1, of a remainder. Likewise, for two real numbers a and b the relation $a \geqslant b$ introduced above as an absurdity of a constructive property can be formulated constructively as follows: Let $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ be convergent infinite sequences of rational numbers defining a and b respectively. Then, for any natural number n, a natural number m can be calculated such that $a_\nu - b_\nu \circ\!> -2^{-n}$ for $\nu \geqslant m$.

On the other hand there seems to be little hope for reducing irrationality of a real number a, or one of the relations $a \neq b$ and $a > b$ for real numbers a and b, to a constructive property, if we remark that a direct checking-number of a drift whose kernel is rational and whose counting-numbers are irrational, is irrational without lying apart from the species of rational numbers; further that a direct checking-number of an arbitrary drift differs from the kernel of the drift without lying apart from it, and that a direct checking-number of a right-winged drift lies to the right of the kernel of the drift without lying apart from it.

It occurs that within some species of mathematical entities some non-contradictorities of constructive properties ζ can be given either a constructive form (possibly, but not necessarily, in consequence of reciprocity of complementarity holding for ζ) or the form of an absurdity of a constructive property. E.g. for real numbers a and b the non-contradictority of $a = b$ is equivalent to $a = b$, and the non-contradictority of: *either* $a = b$ *or* $a \circ\!> b$, is equivalent to $a \geqslant b$; further the non-contradictority of $a \circ\!> b$ is equivalent to the absurdity of $a \leqslant b$ as well as to the absurdity of: *either* $a = b$ or $a <\!\circ\, b$.

On the other hand, if we think of the property of non-contradictority of rationality existing for all direct checking-numbers of drifts whose counting-numbers are rational, there seems to be little hope for reducing non-contradictority of rationality of a real number to a constructive property or to an absurdity of a constructive property.

If we understand by the *simple absurdity* of the property η the absurdity of η, and by the $(n+1)$*-fold absurdity* of η the absurdity of the n-fold absurdity of η, then a theorem established above expresses that *threefold absurdity is equivalent to simple absurdity.* And a corollary of this theorem is that *n-fold absurdity is equivalent to simple or to double absurdity according as n is odd or even.*

I should like to terminate here. I hope I have made clear that intuitionism on the one hand subtilizes logic, on the other hand denounces logic as a source of truth. Further that intuitionistic mathematics is inner architecture, and that research in foundations of mathematics is inner inquiry with revealing and liberating consequences, also in non-mathematical domains of thought.

The philosophical basis of intuitionistic logic

MICHAEL DUMMETT

The question with which I am here concerned is: What plausible rationale can there be for repudiating, within mathematical reasoning, the canons of classical logic in favour of those of intuitionistic logic? I am, thus, not concerned with justifications of intuitionistic mathematics from an eclectic point of view, that is, from one which would admit intuitionistic mathematics as a legitimate and interesting form of mathematics alongside classical mathematics: I am concerned only with the standpoint of the intuitionists themselves, namely that classical mathematics employs forms of reasoning which are not valid on any legitimate way of construing mathematical statements (save, occasionally, by accident, as it were, under a quite unintended reinterpretation). Nor am I concerned with exegesis of the writings of Brouwer or of Heyting: the question is what forms of justification of intuitionistic mathematics will stand up, not what particular writers, however eminent, had in mind. And, finally, I am concerned only with the most fundamental feature of intuitionistic mathematics, its underlying logic, and not with the other respects (such as the theory of free choice sequences) in which it differs from classical mathematics. It will therefore be possible to conduct the discussion wholly at the level of elementary number theory. Since we are, in effect, solely concerned with the logical constants – with the sentential operators and the first-order quantifiers – our interest lies only with the most general features of the notion of a mathematical construction, although it will be seen that we need to consider these in a somewhat delicate way.

Any justification for adopting one logic rather than another as the logic for mathematics must turn on questions of *meaning.* It would be impossible to contrive such a justification which took meaning for granted, and represented the question as turning on knowledge or certainty. We are certain of the truth of a statement when we have conclusive grounds for it and are certain that the grounds which we have *are* valid grounds for it and *are* conclusive. If classical arguments for mathematical statements are called in question, this cannot possibly be because

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Philosophy of mathematics : selected readings / edited by Paul Benacerraf,...

Title: Philosophy of mathematics : selected readings /
Publisher: Cambridge [Cambridgeshire] ; New York : Cambridge University Press, 1983.
Description: viii, 600 p. ; 23 cm.
ISBN: 0521227968
052129648X (pbk.)
Subjects: Mathematics --Philosophy.

Database: Hamilton College Library
Location: BURKE
Call Number: QA8.4 .P48 1983
Status: c.1 Charged out - Due on 05-19-08

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