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Class 21: Mathematical Truth

I. 2500 years of epistemology in fifteen minutes

The standard definition of knowledge is that it is justified true belief (JTB).
This definition traces back to Plato's characterization of knowledge as true belief with a logos, or an account, attached.
(See the dialogue *Theaetetus* 201d-201a, especially.)
For the most part, the standard definition was taken as settled until 1962, when Gettier published his counter-examples.
I mentioned the 'man who will get the job has three coins in his pocket' case.
Smith believes that the man who will get the job has three coins in his pocket, since he believes that Jones will get the job, and he saw Jones put three coins in his empty pocket.
Further, Smith has good evidence that Jones will get the job; he is justified in believing it.
But, in fact, Smith himself will get the job, and has three coins in his pocket.
So, Smith has a JTB, but because his belief is about Jones, rather than Smith, we can not say that Smith knows the proposition.
The Gettier counter-examples led to an explosion of interest in epistemology, which had traditionally focused on other matters, like skepticism and foundationalism.
Suddenly, philosophers did not know what they were chasing, any more.

One attempt to define knowledge, in the wake of Gettier's demolition of JTB, used a causal theory (CTK).
CTK can be understood as adding a fourth condition on JTB: that the justification has to include appropriate causal connections between the knower and the proposition known.
So, in the case of Smith and Jones, Smith does not have an appropriate causal connection to the object of his knowledge, which in this case is Smith himself, rather than Jones.
So, CTK gets the answer right: Smith does not know that the man who will get the job has three coins in his pocket.

Benacerraf, writing after Gettier, and the development of CTK, defends and uses CTK in 'Mathematical Truth'.
Unfortunately, by the mid-1970s, it became clear that CTK was itself flawed.
Part of the problem was the obscurity of the notion of causation on which it depended.
But, a more serious objection came from Alvin Goldman, who had himself contributed to the development of CTK.
The objection is seen in the fake barn country example I mentioned in class.
If you are, unknowingly, driving through fake barn country, and happen to see one of the rare real barns, you might believe that you have seen a barn.
You would have a JTB that you have seen a barn, and in fact you would be appropriately causally connected to a barn, so you would fulfil the extra condition arising from CTK.
But, since you would have been in the same belief state had you seen one of the fake barns, we can not really say that you know that you have seen a barn.

Here is a link to a decent account of the history that I have sketched:
http://plato.stanford.edu/entries/knowledge-analysis/#GET

II. Gödel redux

We finished last class discussing how one chooses the axioms of a mathematical theory.
Gödel argued that considerations such as simplicity and fruitfulness and success are integral to our choices of theorems.
Sarah, in particular, was worried that these kinds of considerations make our choices of axioms too arbitrary, or pragmatic.
Shapiro quotes an earlier Gödel paper, on Russell's mathematical logic, concerning the loss of certainty in mathematics.
But, isn't this the appropriate lesson to learn from the paradoxes?
We can maintain necessity, even if we lose certainty.
So, the position would be that mathematical truths are necessarily true, if true, and necessarily false, if false.
But, we aren't always certain of the claims we make, since the axioms might be wrong.
So, how do we choose the axioms.
Frege thought they were logical truths, and obvious, but his axioms led to antinomy, so they must not have been obviously true.
Hilbert made the choices of axioms arbitrary, which seemed wrong.
Brouwer made the choices of axioms dependent on our psychology, and so restricted mathematics unnecessarily.
What's left?
Gödel points to a specific mathematical intuition, analogous to sense perception, p 268.

III. Tarski's theory of truth

Benacerraf relies on Tarski's theory of truth, and on the demands of semantic theory.
Semantic theories can have three parts: a theory of truth, a theory of reference, and a theory of meaning.
A term, like 'cat', has some meaning, and refers to certain objects.
A sentence, like 'the cat is on the mat', has some meaning, and some truth conditions. It may also have a reference.
We won't spend time on meaning theories; I will do so next spring.
We are mainly concerned here with the theory of reference, since we want to know what mathematical terms refer to.
Of course, the meaning of a term and its reference may be closely connected.
Frege, in fact, defined the meaning of a term as that which determines its reference.
Reference is also closely connected to truth, since a sentence, like 'the cat is on the mat' seems to require, for its truth, that the term 'cat' refer to some specific cat, and the predicate 'is on the mat' refer to some sort of property or relation of being on the mat.

Theories of truth seem pretty easy.
Aristotle said, "To say of what is that it is not, or of what is not that it is, is false, while to say of what is that it is, and of what is not that it is not, is true" (Metaphysics 1011b25).
This statement is often taken to indicate a correspondence theory of truth, truth is the correspondence between words and worlds.
The central problem with the theory of truth is the liar:

L: L is false

The liar sentence, like Russell's paradox, is self-referential.
Russell developed the theory of types in such a way as to prevent impredicative definitions, definitions which refer to themselves.
He relied on the vicious circle principle to eliminate such definitions.
Tarski's theory of truth similarly proscribes self-reference.
Tarski rehabilitates the old Aristotelian view of truth by segregating object language from metalanguage.
Imagine a language that does not contain the word 'true'.
Call that language the object language.
We can talk about the object language in a meta-language, as long as we have names for all of the sentences of the object language.
Furthermore, we can add a predicate, 'True', which applies to some sentences of the object language.
We can partition, thus, the sentences of the object language into two classes: the true and the false.
We'll need names of each of the object-language sentences in the metalanguage; we can just take their ordinary names if we like.
The minimal condition for the truth predicate, for getting truth right, is what we call the T-schema:

p is true iff x
where 'p' is the name of any object-language sentence, and x are the truth conditions of that sentence

Here are some instances of the T-schema:

'the cat is on the mat' is true iff the cat is on the mat.
'2+2=4' iff 2+2=4
'George Bush is president' is true iff the eldest son of GHW Bush is the head of the executive branch of the United States of America.

Note that the truth conditions do not have to be expressed using the same terms as the sentence on the left.
Further, we could just name the sentence 'George Bush is president' X, in the metalanguage, so that we get:

X is true iff the eldest son of GHW Bush is the head of the executive branch of the United States of America.

Since 'true' does not appear in the object language, we can not set up L: there is no such sentence in the object language.

To determine which sentences are true and which are false, we have to examine the truth conditions as given on the right hand side of instances of the T-schema.
To determine truth conditions, we use standard model theory.
Here, for those of you who did not get a chance to read it, is a little piece of mine on Model Theory.

Model Theory

All communication involves the production and interpretation of statements. Semantics studies the interpretations of statements. The study of the statements of a formal system is called metatheory. Metatheory may be divided into proof theory and model theory. Proof theory studies the rules guiding inferences within the system. Model theory is a mathematical approach to semantics, in particular to the assignment of truth values to the statements of a theory.

A formal system consists of a language (vocabulary and formation rules) as well as axioms and rules for generating theorems. Given a formal system, the first step in model theory is to specify an interpretation of each particle of the system. Then, we provide rules governing the assignments of truth values to complex expressions on the basis of assignments of truth values to their component parts. A model is an interpretation of a system on which its theorems are true.

The semantics for the propositional calculus are easily given without model theory. Truth tables suffice to interpret the connectives, and propositional variables can be replaced by propositions or sentences.

The semantics for predicate logic normally proceeds using set theory. We specify a domain of interpretation for the variables of the system. For example, it is natural to use the domain of natural numbers to model the Peano Axioms, and to use sets to model the axioms of set theory. Models of physical theories naturally take the physical world as their domains. Non-standard models, using unintended domains of quantification, are available.

The next step in constructing a model is to assign elements of the domain to particles of the system. We assign particular objects to the constants. Predicates are normally interpreted as sets of objects in the domain, and n-place relations are taken as ordered n-tuples within the domain. An existentially quantified sentence is true in a model if there is an object in the domain of interpretation with the properties mentioned in the sentence. A universally quantified expression is true if the properties mentioned hold of every object in the domain.

For modal logics, Kripke models provide possible-worlds semantics. In a Kripke model, we start with a set of ordinary models, one for each possible world, and an accessibility relation among them. A statement is taken to be possible if there is an accessible possible world in which the statement is true. A statement is taken to be necessary if it is true in all possible worlds.

Model theory, developed in large part by Alfred Tarski and Abraham Robinson in the mid-twentieth century, has become a standard tool for studying set theory and algebraic structures. Major results of model theory include Paul Cohen's proof that the continuum hypothesis is independent of the axioms of Zermelo-Fraenkel (ZF) set theory including the axiom of choice, and that the axiom of choice itself is independent of the other axioms of ZF. Model theory is responsible for the so-called Skolem paradox, one of its earliest results.

IV. On Field's project

Field rewrote NGT replacing reference to mathematics with reference to the geometry of space-time.
The indispensability argument, which we will discuss next week, alleges that whether we should believe that mathematical objects exist depends on whether we need mathematics in science, especially in physics.
So, Field was attempting to show that we need mathematics in physics.

V. Benacerraf questions and comments

1. What is a semantic theory? What is a homogeneous semantic theory? How would a homogeneous (or standard) semantic theory treat mathematical sentences?

Comments:

See above for some comments about semantic theories. A standard semantic theory says that two sentences with the same grammatical structure are to be analyzed in the same, formal way. Their content might differ, but the structure should not depend on the content.

Work through the details of 'There are at least three FGs that bear R to a.

$(\exists x)(\exists y)(\exists z)(Lx \bullet Ly \bullet Lz \bullet Oxn \bullet Oyn \bullet Ozn)$

$(\exists x)(\exists y)(\exists z)(Px \bullet Py \bullet Pz \bullet Gxn \bullet Gyn \bullet Gzn)$

Here, Lx: x is a large city, and Px: x is a perfect number.

These claims are themselves liable to analysis, since the be a large city is not just to be large, and to be a city; similarly for perfect numbers. Benacerraf's point is that however we resolve that question, the resolution will be parallel for the two cases.