Hamilton College
Russell Marcus
rmarcus1@hamilton.edu

Class 20: Gödel

I. A bit of background

For the discussion of Gödel's work, you might want to skim the technical details. Feferman distinguishes three versions of the continuum hypothesis. The weak continuum hypothesis says that every uncountable set of reals is the same cardinality of the set of all reals. The continuum hypothesis says that $2^{\aleph_0} = \aleph_1$. And, the generalized continuum hypothesis says that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$, for all cardinals α.

Even this late version of Gödel's paper came out before the size of the continuum was shown, by Paul Cohen in the mid-60s, to be independent of, meaning that it is undecidable by, the other axioms of set theory. That the size of the continuum is independent means that one can add an axiom to the standard axioms asserting that the continuum is any size greater than the size of the natural numbers, and not derive a contradiction. So $2^{\aleph_0}=\aleph_1$ and $2^{\aleph_0}=\aleph_2$ and $2^{\aleph_0}=\aleph_3$... are all consistent with the other axioms. (Gödel discusses the axioms of von Neumann-Bernays set theory, which is now usually called NBG after von Neumann, Bernays and Gödel himself, but the same point holds of ZF.) Gödel could smell the results, as you can tell by reading the paper, but he had seen proofs of only smaller solutions. See the final postscript, from Gödel himself, on pp 269-70. (Page numbers here refer to the numbers at the top of each page, not to the marginal page numbers.)

The philosophical interest of the paper comes mainly from §3 and, more importantly, Note 4 of Gödel's Postscript.

II. Questions and comments

1. Describe Gödel's pure concept of set. See fn 14. How does the concept of sets as things, "Dividing the totality of all existing things into two categories" (259) lead to contradiction? How does Gödel's concept of set avoid the paradoxes?

Comments:

Gödel is building up sets using numbers as ur-elements. That is, he is not presenting pure set theory. But, the use of ur-elements is not the central issue. The key point here is to use a bottom-up definition of set, like that in ZF, rather than a top-down one, like Cantor's, or Frege's. In a bottom-up definition, we start with well-formed sets, and form new sets, applying constructive axioms, those which form new sets, like the pair set axiom, to the sets we already have. The result is often called the constructive set-theoretic universe.

2. What does the undecidability of the continuum hypothesis by the standard axioms of set theory show, for Gödel? (See 260) What strategy does Gödel suggest involving axioms of infinity?

Comments:

Gödel argues that we need better axioms, more constructive axioms, in order to settle the matter. He believes that the continuum hypothesis has a truth value, and if the current axioms do not settle the truth value, then we should strengthen them. But, we have to figure out which ones to take!

3. How are success and fruitfulness criteria for mathematical truth? How does Gödel link success to verifiability? (See 261.) Does using these criteria make our mathematical commitments merely pragmatic? How does Gödel bring these considerations to the question of the continuum hypothesis? (See 264.) Consider his use of them for the axiom of choice, in fn 2.

Comments:

Sarah (p 2) rightly highlights one kind of fruition that Gödel mentions: the addition as a new axiom a statement which facilitates the derivation of theorems that are already proven. If the new axiom makes the derivations easier, that might be a good reason to adopt the axioms.

The other kind of fruition is when a new axiom allows us to derive theorems that we have not yet been able to derive. Gödel is hoping for axioms that would decide the continuum hypothesis, while at the same time yielding other, independent reasons for us to accept it.

Contrast Gödel's view here to the standard view that we prove the theorems, but take the axioms as assumptions. What would be the status of these assumptions? Hilbert took them as empty, or meaningless, but they do not lack content. Frege took them as logical truths, but we saw that their derivations required set theory, which, most mathematicians agree, is not merely logic. Gödel is offering to use arguments about success to argue for their truth.

Gödel uses exactly the criteria of success and fruitfulness in arguing for the axiom of choice. The axiom of choice has both seemed obviously true to set theorists, and seemed obviously false. On the obviously true side, consider the version that says that, given a set of sets, we can construct (or there is) another set which contains one member from each of the member of the original set. For a simple example, consider the set:

A: {{2, 4}, {1, 5}, {7, Hillary Clinton}}

The axiom of choice says the existence of A ensures that there is a set:

B: {2, 1, Hillary Clinton}

Who could argue? But, the axiom of choice entails the theorem, first proved by Zermelo in 1904, that every set can be well-ordered. In particular, the set of all real numbers can be well-ordered if the axiom of choice is assumed. And, that seems wrong.

As Gödel notes, the axiom of choice is consistent with the other axioms of set theory. More importantly, it is "just as evident as the other set-theoretical axioms..." (255).

4. How has the question of the status of the parallel postulate lost its meaning? How, according to Gödel, is the question of the size of the continuum different? Consider the method Gödel uses to argue for axioms asserting the existence of inaccessible numbers. Also, consider the "epistemological point of view" (267). In what sense has the parallel postulate retained its meaning?

Comments:

Mathematicians set out to prove that the parallel postulate could be derived from the other Euclidean axioms. It turned out that it and the two forms of its negation were all consistent with the other axioms. So, the question of which version is right became moot: each describes a distinct, consistent space.

Gödel argues that the situation is different in the case of the continuum hypothesis, as Sarah notes, on pp 2-3. He uses criteria of success to argue that the continuum is likely to have one, and only one, acceptable size, just as the axioms for inaccessible numbers. ("The generalized continuum hypothesis... can be shown to be sterile for number theory..." (267).)

Gödel also points out that the question of the geometry of space has not lost its meaning, despite the results concerning the parallel postulate for abstract space. That is, it is still a fact that space-time is not flat, like Euclidean geometry, but curved, like hyperbolic geometry.

5. What is mathematical intuition? How is it like sense perception?

Comments:

Gödel is presenting an analogy between mathematical intuition and sense perception, grounded in the analogy between the way that we are forced to believe in the objects that we sense and the way that mathematical axioms force themselves on us. In both cases, we are liable to error: errors of illusion or hallucination for the senses; a priori errors like those which led to the axiom of comprehension in set theory.

6. How does Gödel argue that something besides sensations is immediately given? Is he a Platonist, as well as a platonist?

Comments:

When I first read the second paragraph on p 268, I thought I saw a connection with Plato. Now, I don't see it quite so well. There does seem to be something Kantian about Gödel's belief that the idea of the object itself is presumed, rather than given in sense experience. Perhaps I thought the following sentences sounded like an allusion to anamnesis.