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Class 16: Intuitionism

I. Intuitionism

Cantor's transfinite numbers, and their underlying set theory, led to several oddities which created anxiety for mathematicians and philosophers in the early twentieth century.
We have seen the so-called Cantor's paradox, and Burali-Forti's paradox, as well as Russell's paradox.
Zermelo's 1904 proof that every set can be well-ordered (see below) was another worry about transfinite set theory, given Cantor's diagonal argument.
There was a consequent explosion of research in logic and the philosophy of mathematics.
Three distinct positions were explored, with many variations of those positions.
We have already looked at Frege's logicism, and mentioned Russell's version which was based on set theory with a theory of types.
We have also looked at formalism; Hilbert's programme is usually assimilated to formalism, though Hilbert was not a strict game-formalist of the type Frege attacked in the *Grundgesetze*.
Members of the third school were called intuitionists; Brouwer was the earliest and most prominent intuitionist.
The debates between Brouwer, and his followers, and Hilbert, and his followers, in the 1920s were intense and productive.
See Mancosu, *From Brouwer to Hilbert*, for more.

Both Hilbert and Brouwer wanted to rectify finite mathematicians with infinitary mathematics.
Hilbert takes the utility of infinitary statements to show the need for firmer grounding in mathematics.
He argues that ideal, infinitary, mathematics is not really about external objects, but about the systems themselves.
Hilbert gives up the idea that we are seeking mathematical truth, and retreats to the consistency of the systems we use.
But, he does not cede any mathematical results.

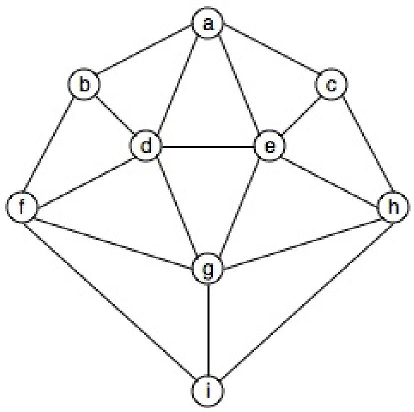
The intuitionists want to hold on to the idea that mathematics presents us with a body of verifiable truths.
They are willing to give up some results, especially those which involved infinite quantities.
Hilbert's defense of Cantor's paradise was really directed at the intuitionists, as was his comment that every mathematical problem has a solution.
Denying Hilbert's last claim, the intuitionists believe that mathematics is not discovered, that mathematics is not a body of transcendent truths.
Classical mathematics depends on untenable metaphysics; see Heyting 68.
The intuitionists believe that we construct mathematical objects in intuition.
Thus, mathematical objects are mental constructions.
Katherine asks about the essential two-oneness Brouwer mentions.
As far as I understand, Brouwer is referring to the unity in multiplicity, the transcendental apperception that Kant mentions.
As Katherine points out, we are deriving our knowledge of mathematics from temporal intuition.

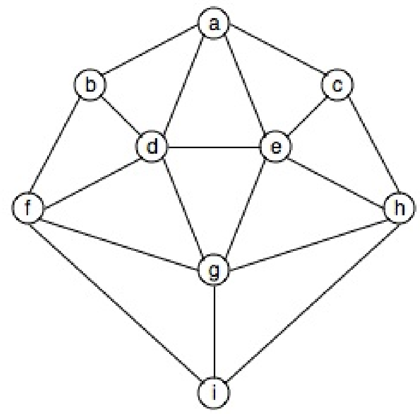
Intuitionists want constructive proofs.
Let's look more closely at the difference between constructive and non-constructive proofs.

II. A Constructive Proof

Definition: A coloring of a graph is an assignment of a color to each node of the graph.
Definition: A graph is 3-colorable if any coloring which uses only three colors does not assign the same color to any two nodes which share a branch.
Definition: A graph is 4-colorable if any coloring which uses only four colors does not assign the same color to any two nodes which share a branch.

Theorem: There are graphs which are 4-colorable but which are not 3-colorable.
Proof: In two stages. Present a graph which is not 3-colorable but which is 4-colorable. (See below.
Stage 1: Prove that the graph is not 3-colorable. Stage 2: Show that the graph is 4-colorable.





Stage 1: Call the three colors red, green, and blue.
Assign red to a (without loss of generality).
So, b and d must not be red, nor may they be the same color as each other.
So, assign green to b and blue to d, again without loss of generality.
Consider e, which now must be green.
Consider g, which now must be red.
Look at f. Uh-oh.
(Similarly for h, once we assign blue to c.)

Stage 2: Construct a four-coloring, use red, green, blue, yellow.

We tried a more complicated one, as well.

Note that at least stage 2 is clearly constructive.
We are constructing an object, a coloring of the graph, which satisfies the theorem.

III. A Non-Constructive Proof

Claim: There exist irrational numbers x and y such that x^y is rational.

Proof: Let $z = \sqrt{2}^{\sqrt{2}}$. If z is rational then z is our desired number with $x = \sqrt{2}$ and $y = \sqrt{2}$.
Suppose that z is irrational. Then, let x = z and $y = \sqrt{2}$.

$$x^y = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2} \cdot \sqrt{2})} = \sqrt{2}^{2} = 2.$$

In this case, x^y is again rational.
So, whether z is rational or irrational, we have shown the existence of irrational numbers x and y such that x^y is rational.
QED

IV. Intuitionism and the rejection of excluded middle

The reason that the above proof is not constructive is that we do not know whether z is rational or irrational.
Von Neumann provides another example of a non-constructive existence proof, pp 62-3.

Heyting 67 contains the intuitionist's argument against the law of the excluded middle, on the basis of a similar case, but this time a non-constructive definition of a number.
Katherine discusses the argument in her paper.
Heyting compares a definition which yields a specific number with one that depends on other factors.
Here is the classical argument supporting Heyting's second definition:

1. The sequence of twin primes is either finite or infinite.
2. If it is finite, then x is the larger element of the largest pair.
3. If it is infinite, then x is 1.

Conclusion: x is some integer.

I mentioned that classical logic holds the law of the excluded middle: $p \vee {\sim}p$.
We see the law of the excluded middle at step 1 of the classical argument.
Since step 1 is accepted by the classical mathematician, she finds the definition legitimate.
The intuitionist does not accept step 1, though.

Walter expressed a certain unease with the intuitionist's denial of step 1.
He challenged the intuitionist to provide another option for the sequence of twin primes.
Walter assumed the classical mathematician's point of view.
The intuitionist, responds that neither option has been demonstrated, and so we can not disjoin the options, even if they are exhaustive.
We have not shown that the sequence of twin primes is finite.
We have not shown that the sequence of twin primes is infinite.
So, we can not conclude, says that intuitionist, that the sequence of twin primes must be either one or the other.

V. Excluded middle and graph coloring

Given the rejection of the excluded middle, go back to Stage 1 of the graph-coloring.
Is our procedure constructive?
It looks like it relies on the law of the excluded middle.
We make assumptions, and show that they lead to contradictions.
For the intuitionist, '~A' means that the assumption of A leads to a contradiction.
See Brown 126.

On the one hand, we have exhausted all the option.
On the other hand, we do seem to be using excluded middle.
I think the coloring is constructive.
So, the use of excluded middle is not always unacceptable to the intuitionist.

The real problem for the intuitionist is the infinitary nature of the proofs which are disallowed.
Statements which do not produce a number, or a construction of some sort may be neither proven nor refuted.
Statements do not have transcendental truth values.
They are made true or false by our proofs, by what axioms, or theorems, we accept.
These statements are to be taken as neither true nor false.

VI. Intuitionist logics

Statements such as step 1 in the classical argument for the Heyting definition are to be taken as neither true nor false.
Thus, the intuitionist needs need a three-valued logic.
I will put a link on the website to some three-valued semantics.
The Mancosu book has a nice section on intuitionist logics.

Heyting tried to formalize intuitionist logic, but Brouwer did not approve.
Formalism makes mathematics about the rules of a formal system.
Brouwer wanted intuitionism to be informal, about mental constructs, not formal systems.
The Kantian roots of Brouwer's work are obvious.
There are also Lockean roots, in abstraction, which might be worth exploring.

VII. On well-ordering

I mentioned Zermelo's proof that every set can be well-ordered.
Here is a definition of a well-ordering:

1. We never have $a < a$.
2. Whenever $a < b$ and $b < c$, we have $a < c$.
3. For any a and b, either $a < b$, $a = b$, or $b < a$.
4. Every nonempty subset has a smallest element.

The natural numbers are well-ordered by their usual sequence.

The reals are not well-ordered: consider the set of reals greater than 0!
There is a proof that the reals are well-ordered.
But, it does not produce the ordering.
In fact, we know that we could not, in principle, produce such an ordering, by Cantor's diagonal theorem.

The oddity of the well-ordering of the reals paired with the impossibility of listing them has bugged mathematicians for a long while, and has been the basis of some Constructivism worries about set theory. I haven't talked about the axiom of choice, really, in class, but it is an important topic. The well-ordering theorem I mentioned today, from Zermelo, is linked to the axiom, as Joe Mileti, a mathematician at Dartmouth, points out in a nice, and fairly easy-to-read, piece on his website:
http://www.math.dartmouth.edu/~mileti/museum/choice.html

Above, I alluded to Zermelo's proof, using the Axiom of Choice, that every set could be well-ordered. This was a defining moment in the history of the Axiom of Choice, because soon thereafter many mathematicians began to have misgivings about it. If the argument is correct, then we should be able to well-order the real numbers. That is, we should be able to list them in a manner that goes into the transfinite. Did Zermelo show mathematicians a clear way to do this? No. Did he give a clear method that, when applied, put the real numbers into a nice transfinite list? No. Zermelo's result simply claims that this could be done without giving any constructive way to do it, and this made mathematicians uneasy. If you examine the Axiom of Choice closely, you can see where this nonconstructive aspect enters the picture. Recall the Axiom of Choice says that given an arbitrary collection of nonempty sets, we may choose an element from each of them, without any sort of rule, method, or scheme to do so. Thus, by adopting the Axiom of Choice, we commit ourselves to a new kind of nonconstructive mathematics.

You might find the entire piece illuminating; check it out.