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Class 15: Hilbert and Gödel

I. Hilbert's programme

We have seen four different Hilberts:

- the term formalist (mathematical terms refer to inscriptions),
- the game formalist (ideal terms are meaningless),
- the deductivist (mathematics consists of deductions within consistent systems)
- the finitist (mathematics must proceed on finitary, but not foolishly so, basis).

Gödel's Theorems apply specifically to the deductivist Hilbert.
But, Gödel would not have pursued them without the term formalist's emphasis on terms, which lead to the deductivist's pursuit of meta-mathematics.
We have not talked much about the finitist Hilbert, which is the most accurate label for Hilbert's programme.

Hilbert makes a clear distinction between finite statements, and infinitary statements, which include reference to ideal elements.
Ideal elements allow generality in mathematical formulas, and require acknowledgment of infinitary statements.
See Hilbert 195-6.
When Hilbert mentions 'a+b=b+a', he is referring to a universally quantified formula: (x)(y)(x+y=y+x)
x and y range over all numbers, and so are not finitary.
See Shapiro 159 on bounded and unbounded quantifiers.
Hilbert's blocky notation refers to bounded quantifiers.
Note that a universal statement, taken as finitary, is incapable of negation, since it becomes an infinite statement.
(∃x)Px is a perfectly finitary statement
~(∃x)Px is equivalent to (x)~Px, which is infinitary - uh-oh!

The admission of ideal elements begs questions of the meanings of terms in ideal statements.
To what do the a and the b in 'a+b=b+a' refer?
See Hilbert 194.
The mathematical statements are legitimate, but the terms referring to ideal elements are, in some sense, meaningless.
Matt relates (p 2) the move to meaninglessness to Kant.
The game-formalist Hilbert ascribes real meaning to the finite elements, but no meaning to the ideal elements of mathematics.
The Kantian reference is good; see the dedication in Hilbert's axiomatization of geometry.
We will return to Kantianism with the intuitionists.
If the terms are meaningless, though, what relevance can they have for us?
Why should we care about meaningless statements?
One possible answer to that question involves their utility.
A system won't be useful, though, if it leads us astray.
So, Hilbert had to focus on the system itself.
Hilbert's deductivism is thus intimately related to his demand for consistency, p 199.
See also Shapiro 150.

II. Meta-theories

Von Neumann 61-2 discusses the goals for the Hilbert programme.
Note also, Hilbert 200, where he claims that every mathematical question is solvable.
Shapiro mentions that some contemporary philosophers argue that mathematics is a conservative extension of physical theory, p 163.
We will read Field, who holds this view, later in the term.

Hilbert's programme led to the rise of metalogic, and meta-mathematics.
A meta-theory is a theory in which you discuss a different theory, which is called an object theory.
We can construct metalogical theories to explore logical systems, to find the limits of the object theory, say first-order logic, or predicate logic.
We can explore mathematical theories by doing meta-mathematics.
(Metaphysics is not, generally, a theory about physics, though.
'Metaphysics' is an older term, with an independent etymology.)
Metalogic, with its central notions of proof theory and model theory, blossomed in the twentieth century, as a new and fruitful way of doing mathematics.
Model theory gave mathematicians novel tools for their proofs.

Von Neumann 63 discusses the key steps of meta-mathematics.
The first three steps are the designation of syntax.
For those of you who have taken logic, we are just providing formation rules (steps 1 and 2) and inference rules (step 3).
Step 4 is the key to Hilbert's programme.

There are two notions of meta-theoretics important for us, here.
A system is called *consistent*, if no contradiction is derivable from the axioms of that system.
We can prove that a system is consistent if there is some formula that is not provable from the system.
For, if a contradiction is present in the system, then any formula is derivable.
If one can prove that some statement, say '0=1', is not provable in a system, then the system must be consistent.
(That result only holds in classical logic, which contains the law of the excluded middle.
The intuitionists will reject this law, as we will see later.)
See von Neumann 65.

Consistency is minimal condition on a theory's utility.
In logic, we want only the logical truths to be derivable.
In mathematics, we want only true theorems to be provable.

(Consistency is related to *soundness*.
In a sound theory, every formula that is provable is true.
An inconsistent system is automatically unsound.
But, we might want more of our system than consistency.
And, we might want to avoid all talk of truth when determining consistency.
That is, consistency is a syntactic notion, independent of truth, whereas soundness is a semantic notion, involving truth.
Hilbert did not make the distinction between syntactic and semantic properties, and so ran the notions of soundness and consistency together.)

One clear goal of the Hilbert programme was a consistency proof for mathematics.
Hilbert, in his 1899 axiomatization of Euclidean geometry had shown an arithmetic interpretation of the geometric system.
Thus, geometry is consistent only if arithmetic was consistent.
We know Euclidean geometry to be relatively consistent, but not absolutely consistent.
See Hilbert 200.
James alludes to this proof on p 1.
But, Hilbert's goal was stronger than James portrays: we want the consistency of arithmetic itself!

The converse of soundness is called *completeness*: Every true formula is provable.
(Note that inconsistent systems are trivially complete!
For, if we can prove every statement, then among those statements will be the true ones.)
A goal that one might have for mathematics is a completeness proof.
We might interpret step 4 in the Von Neumann as the goal of soundness and completeness: statements are true if, and only if, they are provable.

The logicists were also interested in the union of proof and truth.
The logicists did not worry about consistency, since mathematical statements were all clearly true, logical truths.
That is, they assumed the consistency of mathematics from its truth.
Hilbert derived truth (and existence, speaking like a mathematician) from consistency.
See Shapiro 156!
The difference between Hilbert and the logicisists is that the logicist wants a formal system to show that we can derive all of mathematics from logical truths, while Hilbert gives up the idea that we have proven any truths, and just wants to show that the system we use for our mathematics is healthy (i.e. consistent.)
All we are doing is proving that a system is a fully acceptable tool.

At the time of the von Neumann article, 1931, Hilbert's programme had some hope.
Before Gödel proved his incompleteness theorems, in 1931, he proved the completeness of first-order logic.
Von Neumann alludes to consistency proofs for small, non-classical systems, p 65.
(I have a short-ish paper on the topic of the hopes for formal theories on my website, called "Formalities".)

III. Gödel first theorem

The details of Gödel's incompleteness proofs are left for those of a mathematical bent.
Smullyan's account is nice, but it is a bit technical still.
There are lots of good approaches.
One other that I recommend is in Boolos and Jeffrey, *Computability and Logic*.
Boolos and Jeffrey approach the same concepts through the notions of Turing machines and computation.

Gödel provided two theorems.
The first shows that completeness is impossible, in a sufficiently strong consistent theory.
See just below for a discussion of what it means to be sufficiently strong.
Gödel's original proof uses a version of the liar paradox: we construct a sentence within the formal theory that says of itself that it is not provable.

James's representation of Gödel's argument is essentially correct.
But, how do we get to step 2, how do we get to a sentence which says of itself that it is not provable?

The key to Gödel's theorems is the arithmetization of proof procedures.
This is called Gödel-numbering.
We assign numerical values to all the expressions of the system.
To the rules of inference, we correlate arithmetic equivalents.
Thus, deriving a statement becomes equivalent to performing some arithmetic operation.
See handout.
We know that the Gödel-sentence is not provable by a version of the diagonalization argument.
For a theory to be sufficiently strong, it must be able to Gödel-number.
For Gödel's theorem to hold, the theory must have names for each of the numbers.

There are other requirements on a theory.
The following selection on what it takes for a theory to be sufficiently strong to admit of the Gödel results is from Peter Suber, http://www.earlham.edu/~peters/courses/logsys/g-proof.htm

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Remember that Gödel's theorem does not apply to all systems of arithmetic, only to those that are "sufficiently powerful." This is what creates the dilemma of incompleteness: either a system is incomplete because it is too weak for Gödel's theorem to apply, or it is incomplete because the theorem does apply. Hofstadter uses a good analogy here: if we imagine a thief who only robs the "sufficiently rich" and who accosts all travelers on a certain road, then we know that travelers on that road will always be poor: either because they are not sufficiently rich, or because they have been robbed. Here in summary are the conditions of eligibility that describe when a system is "sufficiently powerful" —or when it is rich enough to be robbed by G.

1. It must be a formal system of arithmetic.
    1. On its intended interpretation, some of its theorems must express truths of arithmetic.
    2. The formal language of the system is capable of naming each of the natural numbers, and does so on its intended interpretation.
    3. The formal language of the system has a finite alphabet, and all wffs are only finitely long.

2. It must be a "respectable" system of arithmetic.
    1. It must be consistent.
    2. It must represent (in the technical sense) every decidable set of natural numbers.
    3. It must be the case that each of its wffs with free variables is a theorem iff some closure of it is a theorem.

3. It must be a "well-made" system of arithmetic.
    1. Its set of axioms must be decidable.
    2. Its set of rules of inference must be decidable.

The first wave of conditions means that the system must be a system of finitary polyadic predicate logic, extended with proper (as opposed to logical) axioms so that it is capable of proving at least some arithmetical wffs as theorems. The second wave of conditions makes it powerful. The third wave ensures that its set of proofs is decidable, which gives it an effective test of proofhood, which allows the predicate for proof-pairhood to be decidable.

Weak systems, ones which can not arithmetize their vocabulary, are not susceptible to the Gödel theorems.
That is why first-order logic can be complete.
There are even some version of arithmetic, like Presburger arithmetic, which are complete.
Unfortunately, Presburger arithmetic omits multiplication!

IV. Gödel's second theorem

Gödel's second theorem shows that proofs of consistency are impossible, within a single system.

Here is an argument, following Shapiro, parallel to the one James presents:

1. For any sufficiently strong theory, T, we can regiment predicates for consistency and derivability, and a Gödel sentence, G, which can not be proven within the theory.
2. So, we can write a sentence within a theory, that says: If T is consistent then G is not derivable in T.
3. So, if T is consistent, then G, and we can write this within the theory.
4. Assume T is consistent.
5. Then, we can derive G.
6. But, we know that ~Der(G)
7. So, no consistent theory can prove its own consistency.

1. $G \equiv \sim Der_T(G)$	By definition
2. $Con(T) \supset \sim Der_T(G)$	By the first theorem
3. $Con(T) \supset G$	fr 1, 2
4. Provable: Con(T)	
5. Provable: G	
6. Not provable: G	
7. So not provable Con(T)	

V. A way out?

James proposes a solution for Hilbert.
He suggests that we need not demand completeness from our theories.
Similarly, Shapiro thinks that Hilbert can back-peddle from a desire for completeness, p 166.
Perhaps James's suggestion is that we can rest assured with just soundness.
James is correct that the Gödel sentences are highly technical.
Perhaps we could accept the omission of them from our mathematical theories.
But, there are other sentences that the theory omits.

The central problem with limiting ourselves to soundness is that many weak systems are sound.
Presburger arithmetic is sound, as well as complete.
But, it is not sufficient for full mathematical purposes.
Remember, there is an almost-universal presumption that all of mathematics reduces, in some sense, to set theory via arithmetic.