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# Class 14: Hilbert

I. On Abstracts, due next Wednesday:

Writing a good abstract is extremely difficult. Ideally, you have the whole paper written, then you distill it to its essence. Obviously, you are not expected to do that for next week. But, if you have an idea of an outline, it will help. If you toss it off an hour before class, I expect you will not be happy with my response.

Your papers should not be just a bare extension of a seminar paper. You can compare and contrast different authors: Kant and Frege, Mill and Frege, Locke and Leibniz, etc. The content should be critical, not just expository. For topics, you might look at different definitions of numbers, different positions regarding the infinite, necessity, or innateness. Benacerraf's "Mathematical Truth" might be useful in getting a big picture of the central problem as it stands. There are also more technical topics: working through Gödel's theorems, Cantor's transfinites, or the arithmetization of analysis. Feel free to talk with me about a choice of topics.

II. Recursive functions and Church's Thesis

What is an algorithm?
An algorithm is anything computable, by means that Hilbert would have called finitistic.
Still, the notion of computability is an ordinary-language notion.
Church's Thesis says that we can get a precise mathematical explication of this loose concept.
It says that the computable functions are the recursive functions.
The name 'recursive function' comes from Gödel, in his incompleteness paper.
The subclass of primitive recursive functions are those obtainable without the use of the μ-operation.
The superclass of partial recursive functions are those obtainable by a weaker μ-operation.
The following presentation comes from Hunter, *Metalogic*, p 232 et seq.
For an alternative presentation, see Mendelson, *Introduction to Mathematical Logic*, p 174 et seq.

A recursive definition of a function f(x) is one that is given by mathematical induction.
We give the value of the function for x=0.
Then, we give the value of the function for x=n+1 in terms of the value for x=n.
So, we can just churn out any values of the function that we could want, recursively.
Now, we would like to know which functions are the recursive functions.
We would like to specify all the computable functions.
We can do so by specifying a list of computable functions, and how to generate other recursive functions from them.

<u>Initial functions</u> (functions from ordered n-tuples on ℕ to ℕ)
(n-tuples are just ordered pairs, triples, etc.)
Successor: $f_1(x) = x+1$
Sum: $f_2(x,y) = x+y$
Product: $f_3(x,y) = x \bullet y$
Power: $f_4(x,y) = x^y$ (calling $0^0 = 1$, in order to have all values defined)
Arithmetic difference: $f_5(x,y) = x \dot{-} y$, where $x \dot{-} y = x-y$, if $x>y$, and $x \dot{-} y=0$ if $y \geq x$
(Recall, that these are functions from ordered n-tuples on ℕ to ℕ.)

Operations on Computable Functions

Combination: Any combination of computable functions is computable

The μ-operation: Let $f(x_1,...x_n,y)$ be a computable function such that for each n-tuple of natural numbers $<x_1,...x_n>$, there is a natural number y, such that $f(x_1,...x_n,y)=0$. The μ-operation returns the least such y. That is, the function $g(x_1,...x_n,y)=\mu y[f(x_1,...x_n,y)=0]$ is given by the μ-operation. Functions obtainable by the μ-operation (given the conditions in the first sentence) are computable.

Recursive Functions

Any function which is obtainable from the initial functions by a finite number of steps, using combination or the μ-operation is recursive.

The set of recursive functions is provably equivalent to what a Turing machine can calculate.
There are other ways to characterize the recursive functions, due to Post (like Turing's) and Markov (algebraic).
Church's Thesis: The recursive functions are exactly the computable ones.
In one direction, Church's Thesis is obvious: all recursive functions are computable.
In the other, there is a question: are there computable functions that are not recursive?
Church's thesis is pretty well accepted today.

III. Hilbert's finitism

How does recursion theory relate to Hilbert?
Recall the arithmetization of analysis.
The worries about infinitesimals and infinites were not assuaged by the discoveries of the paradoxes of the naive set theories of Frege and Cantor.
Cantor's theorem showed that there was no set of all sets.
Cantor even proposed to work with "inconsistent multitudes", which today we call classes.
Frege's definitions of numbers, sets of all n-membered sets, are among these inconsistent multitudes.

Further, as Matt points out, the world does not seem infinite, either in the large or in the small.
Natura non facit saltus: ask Matt about nature being quantized
In the large, we might think that the universe is finite.
Or, we might think that it anyway was no more than denumerably infinite, or no more than the continuum.
So, the Millian approach, of deriving our knowledge of mathematics from sense experience seems doomed.

Still, the infinite numbers were mathematically useful; see the success argument at Hilbert 184B.
Mathematicians did not want to let go of the transfinites, despite these worries.
Matt thinks that Hilbert is out of touch; p 195.
See Hilbert 191, for his goals.
What today has become known as the Hilbert programme included, most centrally, the requirement to reduce the infinite to the finite, in some sense.
Hilbert distinguishes between ideal mathematical statements and finitary (or real) mathematical statements.
Shapiro 161: finitary methods are the primitive recursive functions.

IV. Hilbert's formalism

Hilbert's finitism is only one aspect of his work; and it is a broad kind of finitism, and so the term is misleading.
Hilbert is most often classified as a formalist.
There are different kinds of formalism, and Hilbert, at different stages accepted different aspects of formalism.
Shapiro distinguishes term formalism, game formalism, and deductivism.

Term formalism is the claim that the statements of mathematics actually concern the symbols themselves.
See Hilbert 192.
Note the roots in Kantian intuition.

But, term formalism has its limits.
Shapiro notes that the term formalist trips on the real numbers.
One lesson of Cantor's diagonal argument is that the real numbers do not each have names.
Thus, Hilbert, motivated by the success of transfinite mathematics and the success argument, introduced ideal elements, in contrast to real elements of mathematics.

Matt wonders, p 2, what 'real' refers to, in Hilbert.
Hilbert means something physical, something empirical.
See Hilbert 186.

Ideal elements allowed generality in mathematical formulas.
They require acknowledgment of infinitary statements.
See Hilbert 195-6.
When Hilbert mentions 'a+b=b+a', he is referring to a universally quantified formula: (x)(y)(x+y=y+x)
x and y range over all numbers, and so are not finitary.
See Shapiro 159 on bounded and unbounded quantifiers.
Hilbert's blocky notation refers to bounded quantifiers.
Note that a universal statement, taken as finitary, is incapable of negation, since it becomes an infinite statement.
(∃x)Px is a perfectly finitary statement
~(∃x)Px is equivalent to (x)~Px, which is infinitary.
Uh-oh!

Relate to Matt, p1, on Hilbert's claim that the infinitary has not been completely clarified.
Hilbert is still worrying about finite minds having access to infinities.
But Hilbert doesn't want to be too much of a finitist, p 184A.

The admission of ideal elements begs questions of the meanings of terms in ideal statements.
To what do the a and the b in 'a+b=b+a' refer?
See Hilbert 194A.
These are legitimate mathematical statements.

Matt asks how Hilbert derives the transfinite axioms from his choice function.
The short answer is that I don't know.
I think the point is that all we need to move from predicate logic to set theory is the ∈ of set inclusion and

some axioms governing that.
See the ZF axioms I presented.
Matt also asks how Hilbert deals with Russell's paradox: we need a better set theory, like ZF.
Hilbert's axiom seems perilously close to Frege's Rule V, so I'm not sure about Hilbert's solution.
By 1925, Zermelo had axiomatized set theory (1908) in a way to avoid the paradoxes.

There are many different versions of the axiom of choice.
In our list of axioms, I presented ZF without Choice.
Here's one version: For every set S of disjoint sets, there is another set which includes exactly one member of each member of S.
(Compare to selecting a governor for each state.)
Enderton's *Elements of Set Theory*, has six equivalent versions of the axiom of choice, p 151.
Mendelson's *Introduction to Mathematical Logic*, has five versions on pp 275-6, and six more on p 277.

We are moving to game formalism.
Matt relates (p 2) the move to meaninglessness to Kant.
The game-formalist Hilbert ascribes real meaning to the real elements, but no meaning at all to the ideal elements of mathematics.

## V. Deductivism and consistency

The formalist turned our interest to the study of mathematical systems themselves.
Here we see Hilbert's deductivism, which is related to his demand for consistency, p 199.
Shapiro 150; but 156!
Hilbert thus opened up the world of meta-mathematics.
Note also, Hilbert 200 - Every mathematical question is solvable.
Shapiro and conservative extensions, p 163.

James will tell us more about the desire for consistency proofs and the problems which arose for them, on Monday.

We have seen four different Hilberts: the term formalist (terms refer to inscriptions), the game formalist (ideal terms are meaningless), the deductivist (mathematics consists of deductions within consistent systems) and the finitist (mathematics must proceed on finitary, but not foolishly so, basis).
Gödel's Theorems apply specifically to the deductivist Hilbert, but could not have arisen without the emphasis on terms which led to meta-mathematics.