Philosophy 405: Knowledge, Truth and Mathematics
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M, W: 1-2:15pm

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Class 13: Frege

I. Ontological reductions in mathematics

Kara was worried, in an earlier class, about Frege's focus on the natural numbers, alone.
Also, see Shapiro 113.
We have been talking quite a bit about the relations among different areas of mathematics.
I thought I should make some of these details explicit, tracing how to define the real numbers in terms of the natural numbers, and some set theory.
Recall that we spoke in our last class about two possible representations of natural numbers using sets, the von Neumann and Zermelo constructions.

Consider ℕ=0, 1, 2, 3...
I include 0, for ease of presentation; it is arbitrary and Frege includes 0.
We can define the integers in terms of the natural numbers by using subtraction.
-3 is 5-8.
We can define -3 as the ordered pair <5,8>, but it can also be defined as <17,20>.
We take the negative numbers to be equivalence classes of these ordered pairs: <a,b>~<c,d> when a+d=b+c.
Note, that to the definition of rationals in terms of integers, we add an identity clause.
"There is no entity without identity," Quine (*Theories and Things*: 102)

So, we can define **Z** = ...-3, -2, -1, 0, 1, 2, 3... in terms of ℕ.
We have already seen how the rationals, **Q**, can be defined in terms of the integers, **Z**, using the grid.
Again, we use ordered pairs of integers: a/b::<a,b>, where '<a,b>~<c,d> iff ad=bc' is the identity clause.

But, what about the reals, ℝ
Real numbers are on the number line.
Rational numbers are part of the number line.
But, the relation between the real numbers and the rational numbers was unclear in the 19th century.
Both the rationals and the reals are dense: between any two there is a third.
But, the reals are also continuous.
One important question was how to characterize this continuity.

In the early 19th century, worries about the infinite had put pressure on mathematicians.
Cantor had not yet produced his set theory, which founded his theory of transfinite numbers.
And, there was a growing pressure from analysis to provide a solid underpinning of calculus, and its infinitesimals.
Quote from Niels Abel, in Kline, 950
Karl Weierstrass, approaching the question of the relation between the reals and the rationals, was concerned, in part, with the geometric foundation of the calculus.
We will see Weierstrass on Wednesday, at the beginning of the Hilbert article.
Weierstrass's stated goal was the arithmetization of analysis.
Weierstrass's influence on Frege is a matter of historical debate.

The problem is that the geometric underpinning of the reals and their continuity is insufficient.
There is a mapping from the rationals to the number line, but not backwards.
That is, for every rational, we can construct a distance from an arbitrary origin to a point on the line.
But it is not the case that every distance on the line can be expressed as a rational.
That is the problem of incommensurables.
Read Dedekind quote from Gillies, p 4.

The definition of continuity, and indeed of what a function is, were in need of clarity.
It was in response to these worries that Bolzano, Cauchy, and Weierstrass pursued the arithmetization of analysis.
Part of their achievement was the epsilon-delta definition of continuous functions, due to Weierstrass in the 1860s, but based on ideas from Cauchy, 1821: a function f(x) is continuous at a if for any $\varepsilon > 0$ there is a $\delta > 0$ such that for all x such that $|x - a| < \delta$, $|f(x) - f(a)| < \varepsilon$.
From this definition, we can find a definition of limits: a function f(x) has a limit L at a if the same statement holds, with L replacing f(a).
The Bolzano-Cauchy-Weierstrass group proceeded to provide formal proofs in terms of these rigorous definitions, to replace the earlier, loose presentations.
The ε-δ definition of limits are arithmetic, rather than relying on the notion of an infinitesimal.

In addition to the arithmetic definitions of continuity and limits, Weierstrass and Dedekind, as well as Cantor, pursued more rigorous definition of the reals, in terms of the rationals.
I will mention only Dedekind's derivation, from 1872, though there are others; Cauchy's is different, and easier to work with, for some purposes.
The key concept is that of a cut, which has become known as a Dedekind cut.
The real numbers are identified with separations of the rationals, **Q**, into two sets, Q_1 and Q_2, such that every member of Q_1 is less than or equal to the real number and every member of Q_2 is greater.
So, even though $\sqrt{2}$ is not a rational, it divides the rationals into two such sets.
Not all cuts are produced by rational numbers.
So, we can distinguish the continuity of the reals from the discontinuity of the rationals on the basis of these cuts.
Real numbers, then, are defined in terms of sets of rationals, the set of rationals below the cut.
These sets have no largest member, since for any rational less than $\sqrt{2}$, for example, we can find another one larger.
But, they do have an upper bound in the reals.

So, we can define the reals in terms of the natural numbers, and thus in terms of sets.
Such definitions do two things.
They make infinitesimals and reals accessible to finite (or at least denumerable) methods.
They make it plausible that we can reduce the problem of justifying our knowledge of mathematics to the problem of justifying our knowledge of just sets.
We have an ontological reduction, of the objects of analysis to the objects of set theory.
What about a methodological reduction, of the axioms of mathematics to those of set theory?
Look at the Peano axioms.
Most of them can easily be understood as set-theoretic axioms, if the objects over which they range are taken to be sets.
The only contentious one, really, is mathematical induction.
Note that Frege worries especially about induction; see p iv.
For, it seems like a mathematical process, not a logical one.

II. Frege against Kant

Frege's central claim is that arithmetic is analytic, not synthetic.
He criticizes Kant's analysis of analyticity.
Kant analyzes judgments all as linking subject concepts with object concepts.
To say that apples are sweet is to join the concept of apples with the concept of being sweet.
Since the concept of being sweet does not by itself include the concept of apples, the judgment is synthetic.
But, when we say that every mother has a child, the concept of having a child does include the concept of being a mother.
Read §88.

Against Kant's analysis, Frege first worries about when the sentence contains an individual object as the subject.
So, consider 'Matt is a cat'.
'Matt' seems not to stand for a concept, but for a thing.
Here, we might be tempted to say that the statement's analyticity conditions depend not on whether the subject concept is contained in the predicate concept, but whether the object to which 'Matt' refers necessarily has the property of being a cat.
Or, we might want to know if the concept of being cat contains an object, Matt.
But, Matt does not seem to be a concept.
This worry is at the foundation of twentieth-century philosophy of language.
But, it is ancillary here.

Second, Frege worries about existential statements, e.g. 'There are electrons'.
Such statements do not seem to fit into Kant's system.
One of Frege's advances in logic is to see a predicate not as a closed concept, but on analogy with a function, including the kind of hole, or holes, a function takes.
In 'Matt is a cat', the predicate is not catness, but '...is a cat'.
So, 'Clinton is between Syracuse and Albany' does not predicate a property of being between Syracuse and Albany of an object, Clinton.
Rather, it is to be understood as a three-place function, betweenness.
Frege was thus able to portray betweenness more generally.
It is the same predicate that holds among New York, Boston and Philadelphia.
On the Kantian notion of a predicate, being between Boston and Philadelphia was completely distinct from being between Albany and Syracuse.

Frege is also unhappy with Kant's notion of a concept, as a list waiting to be unpacked.
Frege claims that the elements of a concept are linked.
They are not merely appended to each other, but tied together.
And, we can unpack them, tracing them back to their justificatory grounds.

III. Frege's projects

Frege's work on the nature of propositions, Frege's logic, arises from his claim about mathematics.
In fact, all of Frege's work really traces back to the logicist project.
Logicism is the claim that mathematics is reducible to logic, that the truths of mathematics are really

logical truths.
Recall Leibniz's proposal to reduce all propositions to elementary identities.
Frege wants to trace all of mathematics to elementary logic.

Frege wrote three books.
The *Begriffsschrift* formulates his logical language.
The *Grundlagen*, which we have been reading presents a philosophical defense of the logicist project, and criticisms of Mill, Kant, and others.
The *Grundgesetze*, in two volumes, did the technical work promised in the *Grundlagen*.
(Russell sent Frege the paradox just as the second volume, self-published, was about to come out.)
Frege's work was almost entirely ignored when he initially published it, but became the cornerstone of twentieth-century English-language philosophy.

Read from the preface of the *Begriffsschrift*.
Note another similarity with Leibniz, on the distinction between origins and justification.
Recall my comments about Locke's commission of a genetic fallacy, confusing the origins of an idea with its justification.
In the introduction to the *Grundlagen*, Frege presents three principles.
The first principle says that we must distinguish the psychological from the justificatory, or logical.
The idea is that the way in which we come to learn about a mathematical claim is independent of the way we justify that claim.
Just as Leibniz used this principle against Locke, Frege accuses both Mill and Kant of confusing how we learn about something with the proof of that thing.
Read §3.

Why is an a priori error impossible?
Continue to p 4, and the implication that there is just one proof per proposition.
Compare with the Munchhausen comments in §8.

Back to the *Begriffsschrift*.
Frege constructs formal logic (both propositional logic and higher-order predicate logic) in order to establish the logicist claim, the criticism of Kant's analysis of mathematics as synthetic a priori.
There had been work in the nineteenth century, particularly by Boole and Schröder, unifying Aristotle's categorical logic with propositional logic, which traced back (at least) to the Stoics.
Frege made advances in predicate logic, including the use of quantifiers, and in the adoption of a unique, but cumbersome, notation.
The goal, remember, is to trace all of mathematics back, without any gaps.
That would be the Old Euclidean style of rigor to which Frege refers in §1.

IV. Set theory, logic, and the definitions of numbers

How far back are we going?
Kara's p 1: the primitive truths are the axioms of set theory.
Note that Kara is correct, at the end of that paragraph, to refer to the apriority of mathematics.
Frege argued for the apriority of mathematics against Mill.
But, his bigger quarry was Kant's analyticity; Kant and Frege agree on apriority.

Distinguish between naive set theory and axiomatic set theory.
Naive set theory includes an unrestricted axiom of comprehension.
Every predicate, every property, denotes a set.
Walter, in his Cantor paper, mentions one problem for naive set theory; the Burali-Forti paradox: there is no largest ordinal number.
The paradox is that the set of all ordinals is itself an ordinal, larger than any member of the set.
We learn that there can not be a set of all ordinals.
Similarly, Cantor's paradox showed that there can not be a set of all sets.
In current axiomatic set theory, we avoid such paradoxes by building sets iteratively.

For Frege, the set-theoretic axioms must be understood as logical axioms.
He builds the numbers out of sets, building all of mathematics out of logic, via set theory.
This derivation is guided by the three principles in the introduction.
The third principle says we should distinguish between concepts and objects.
Numbers apply to concepts, not to objects, §46.
But they are objects themselves, not properties.

Kara asks (p 2) about why Frege makes a big deal about numbers not being properties.
Mill had made the numbers into properties.
Recall, "Ten must mean ten bodies, or ten sounds, or ten beatings of the pulse" (Mill, 189).
Frege notes that we can rephrase "There are ten bodies" as "The number of bodies is ten".
Further, we should understand that 'is' as identity, not predication.
$(\exists x)(Nx \bullet x=10)$
There's an argument against taking numbers to be properties in §29:
$Pa \bullet Pb$ entails P(a and b), but not for numbers.
Take Px: x is a student, a and b as names of two students.
But, take Px: x is one, and take a and b the same.
The main point is that if number were a property, it would attach to objects.
But, number attaches to concepts.
(On non-spatial objects, §61.)

Part of the objectivity of numbers comes from the truth value of expressions which contain numbers.
So, the reason that 'snow is white' is true is that there is snow, and it has the property of being white.
By analogy, it seems that 'five is prime' should be true because there is a five, and it has the property of being prime.
See §26A, B, C, D (handout).
[I did not get to hand this out in class. If you are reading this, you should ask me for a copy.]

So, what are these objects?
The short answer is that they are sets, which Frege takes to be logical objects, just extensions of predicates.
The extension of a concept is the set of things which fall under that concept, or which have that property.

Frege starts by proposing definitions of when numbers apply to concepts.
See Kara's formulation (p2), and §55.
0 belongs to a concept if nothing falls under the concept.
1 is the number which applies to a concept such that it applies to at least one thing, and if it applies to two things, they are the same thing.

Recall, from logic: $(\exists!x)Fx :: (\exists x)[Fx \bullet (y)(Fy \supset y=x)]$

This first approach gives us the flavor of Frege's final formulation.
But, he does not accept the first approach, since it is open to the Julius Caesar objection.
The Julius Caesar objection is that this approach does not distinguish the numbers from other objects that might fall under the concept.
Both Julius Caesar and the number one seem to fall under the concept of the Roman emperor assassinated by Brutus, Cassius, and other Senators on March 15, 44B.C.E.
Further, the first approach makes the numbers look like properties.

Frege wants two things: a definition of number that takes them as objects, and identity conditions for those objects.
Numbers are to be taken set-theoretically, as extensions of sets with certain properties.
They are sets of sets.

§74: 0 is the number which belongs to the concept 'not identical with itself'.
Note, that this definition of 0 presumes the concept of being a number which belongs to a concept.
He unpacks that concept as follows, in §68: the number which belongs to the concept F is the extension of the concept 'equal to the concept F'.
An extension of a concept is just a set of things which have the property assigned by the concept.
So, 0 is the set of all sets which are not identical to themselves (i.e. the number of x such that $x \neq x$).
1 then is defined as the number which belongs to the concept 'identical to 0', since there is only one concept 0.
The definitions of the rest of the numbers can be generated inductively, using the successor definition, §76.
More succinctly, 1 is the set of all 1-membered sets; 2 is the set of all 2-membered sets.

Kara wonders about Frege's argument against Kant.
The key is the idea about tracing back, §5, as Kara notes, on p 1.
If we have to trace back for the complex case, then why not trace back in all cases.
If we can trace back to pure logic, then we don't need intuition.
In fact, this is the meaning of Frege's second principle, perhaps his most influential.
The second principle is often called the context principle: never ask for the meaning of a term in isolation.
Note that Frege sees the context principle as solving the problems of the moderns, p x.
Frege argues that if we take the meanings of words out of context, we end up with meanings as pictures! See §58, and §60.
The connection between the context principle and the argument against Kant might not be obvious.
The central point here is the argument against any of the other reductive views about numbers.
Kant reduced numbers to intuition; Locke reduced them to abstract ideas, which are psychological, of course; and Mill reduced them to empirical objects.
Frege's logicism rejects all of these views: mathematical objects are logical.

V. Russell's paradox

So, how does the paradox arise?

Axiom 5: $\{x|Fx\}=\{x|Gx\} \equiv (\forall x)(Fx \equiv Gx)$

leads to

Proposition 91: $Fy \equiv y \in \{x|Fx\}$

(I have to find how he derives Proposition 91 from Axiom 5.
Axiom 5 presents identity conditions on sets: the set of Fs and the set of Gs are identical iff all Fs are Gs.
Proposition 91 just says that a predicate F holds of a term iff the object to which the term refers is an element of the set of Fs.
Both statements similarly assert the existence of a set of objects which corresponds to any predicate, though this claim could be made more explicitly with a higher-order quantification.)

Take F to be 'is not an element of itself'.
So, y is not element of itself is expressed:

$y \notin y$ (which is short for '$\sim y \in y$')

Now, take y to be the set of all sets that are not elements of themselves:

$\{x|x \notin x\}$

And substitute that expression for y in the above expression, to get Fy, the left side of proposition 91:

$\{x|x \notin x\} \notin \{x|x \notin x\}$

On the right of proposition 91, we get

$\{x|x \notin x\} \in \{x|x \notin x\}$

Put it together:

$\{x|x \notin x\} \notin \{x|x \notin x\} \equiv \{x|x \notin x\} \in \{x|x \notin x\}$

Which is of the form:

$\sim P \equiv P$

For those of you who like their contradictions in the form ‘p • ~p’, note that:

1. ~P ≡ P	
2. (~P ⊃ P) • (P ⊃ ~P)	1, Biconditional Exchange
3. ~P ⊃ P	2, Simplification
4. P ⊃ ~P	2, Simplification
5. ~P ∨ ~P	4, Conditional Exchange
6. P ∨ P	3, Conditional Exchange and Double Negation
7. ~P	5, Tautology
8. P	6, Tautology
9. P • ~P	8, 7, Conjunction

QED

VI. After the paradox

What do we have after the paradox?
Mathematics is reducible to mathematics, but not to logic, unless we take the axioms of set theory to be logical, which no one does.
Still, is mathematics plausibly analytic? Hmmm...
Return to the philosophical argument, at §88C.
Plant in the seeds vs beams in the house.

Frege did agree with Kant about geometry, §89.
But, how do we reconcile the Kantian claim about Euclidean geometry?
We can only intuit in Euclidean fashion.
See §14.
Is this right?