Philosophy 405: Knowledge, Truth and Mathematics
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M, W: 1-2:15pm

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Class 11: Cantor

I. Counting sheep

II. A brief history of infinity

Until Cantor, mathematicians still maintained Aristotle's distinction between potential and actual infinity.

> I protest against the use of infinite magnitude as something completed, which is never permissible in mathematics. Infinity is merely a way of speaking, the true meaning being a limit which certain ratios approach indefinitely close, while others are permitted to increase without restriction. (Gauss, 1831, according to a random site on the internet!)

Aristotle had worried about Achilles and the tortoise, as well as other problems, like that of the rabbit and the food.
We saw that Aristotle dissolved the problem by distinguishing between actual and potential infinity.
In the middle ages, the actual infinity became aligned with the concept of God, inaccessible to human cognition.

The calculus itself had put some pressure on Aristotle's distinction.
For, it required working with an infinite number of infinitesimals.
But, Aristotle's distinction could be maintained by the concept of a limit.
If we consider the limit definition of the integral, we might maintain the potentiality of the infinite.
Still, we are completing the infinite sequence, just as Achilles does catch up to the Tortoise.

III. Moving towards actual infinity (following Tiles, Chapter 4)

In the nineteenth century, the rise of analysis put mathematical pressure on Aristotle's distinction.
Recall that the algebraization of geometry led to an inversion of what area of mathematics was seen as foundational.
From Euclid (and before) geometry was seen as the foundation of arithmetic.
Even Newton saw the calculus as essentially geometric.
But, Descartes flipped that view, taking algebra (and arithmetic) to be the foundation of geometry.

Consider, what is x^3?
For the Greeks, it is the volume of a cube with side length=x.
When we talk about cubic numbers, we are talking about geometric properties.
But what is x^5?
If we take the geometric view, it is a five-dimensional cube, a sort of mind-blowing creature.
If we take the analytic view, it is just another curve.
That is, we can plot $f(x)=x^5$ on the plane, and it is nothing more than a more rapidly growing curve in two-dimensions.
Thus, analysis, and the algebraization of geometry opened up mathematics to a wider, more general treatment of functions.

Any equation can be graphed.
The graph is complete, in the sense that it defines the curve for any values.
We thus see the shape, or curve, as containing all magnitudes, including incommensurables.
(Note, as Tiles does, the shift from seeing geometry as the study of closed figures to seeing it as the study of curves.)

Tiles discusses Bernoulli's equation for the motion of the vibrating string.
We have one equation which is the result of a superposition of an infinity of curves.
See Tiles 79.
The point here is that the function becomes increasingly fecund, and the graph becomes increasingly unable to represent the function.
Contrast Euler, who identified functions with their graphs, with D'Alembert, who identified functions with their algebraic expressions; see Tiles 79, also.

Further, we can move beyond the graphical representations to treat algebraic functions more broadly.
Tiles discusses some interesting pathological functions.
For example, Weierstrass explored an everywhere continuous but nowhere differentiable function; Tiles 81.
(This function seems related to fractals, in its fineness.)
The graph has lost its utility.
The function exists beyond the ability to picture it.

But, how far does the function go?
How deep is the discrepancy between the picture and the numbers over which the function ranges?
We know that there are more points on a line than rational numbers, since there are incommensurable numbers.
But, how many more points on the line?
How much more structure do the numbers have?
Cantor's continuum hypothesis is that there are $2^{\aleph_0}$ real numbers, and that $2^{\aleph_0} = \aleph_1$.
We will work up to it.

IV. The infinite hotel

To get in the mood for infinite numbers, we should consider the infinite hotel.
We can add one person, by shifting every one from room n to room n+1.
We can add an infinite busload of guests, by shifting every one from room n to room 2n, and putting the new guests in the odd-numbered rooms.
And we can add an infinite number of busloads of guests by shifting every one from room n to room 2^n.
We then place the people on the first bus in room numbers 3^n (for n people on the bus), the people in the second bus in rooms 5^n, the people in the third bus to rooms 7^n, and so on for each (prime number)^n.
Since there are an infinite number of prime numbers, there will be an infinite number of infinite such sequences.

The splitting headache which may arise from thinking about infinite numbers may correspond to a split between two ways to think about cardinal numbers.
We use them to measure size.
But, we also use one-one correspondence to characterize cardinal numbers.

In the finite realm, these two approaches converge.
But with transfinite numbers, as with the infinite hotel, the two concepts diverge.
The size of the integers seems to be bigger than the size of the even numbers.
But, they can be put into one-one correspondence with each other.

Cantor relies on the one-one correspondence notion to generate transfinites, as Walter discusses.
When we list the members of something, we are putting them into one-one correspondence with the natural numbers.
Cardinal numbers are the sizes of sets, the number we count to when we put the set in one-one correspondence with the natural numbers.
But, it turns out that we can not make certain lists.
For example, we can not list the real numbers, as I showed in the Aristotle class.
The argument I used there is called Cantor's diagonalization argument.
Tiles does the diagonalization argument for the reals much more neatly by using the binary representation of each real number, p 109.

Numbers actually have two different functions.
Cardinal numbers measure size.
Ordinal numbers measure rank.
Walter does a good job of framing the distinction, in his first paragraph.
Let's start with the cardinals.

V. Cardinal arithmetic

Cardinal numbers are sets which we use to measure the sizes of sets, by one-one correspondence.
For all cardinal numbers a, b, and c, whether finite or transfinite, the following hold:
1. $a+b=b+a$
2. $ab=ba$
3. $a + (b + c) = (a + b) + c$
4. $a \bullet (b \bullet c) = (a \bullet b) \bullet c$
5. $a \bullet (b + c) = ab + ac$
6. $a^{(b+c)} = a^b \bullet a^c$
7. $(ab)^c = a^c \bullet b^c$
8. $(a^b)^c = a^{bc}$
9. $2^a > a$

This last one is the key.
Notice that $a+1=a$, when a is transfinite.
And $2a=a$ holds as well.
Even $a \bullet a=a$
We can show these all by considering a bijective mapping from one set to the other.
We showed all of these facts in the infinite hotel.

In set-theoretic terms, the ninth claim is that $\wp(a) > a$.
'$\wp(a)$' refers to the power set of a, the set of all subsets of a set a.
Consider a set $A = \{2, 4, 6\}$
Then $\wp(A) = \{\{2\}, \{4\}, \{6\}, \{2, 4\}, \{2, 6\}, \{4, 6\}, \{2, 4, 6\}, \varnothing\}$

In general the power set of a set with n elements will have 2^n elements.
Since sets with n members are the same size as sets with n+1 members, or with 2n members, or with n^2 members, for infinite n, we might think that sets with n members are the same size as sets with 2^n members.
For, with infinite numbers, it is not always clear that what we think of as a larger set is in fact larger.

The claim that $\wp(a) > a$ has been called Cantor's paradox.
$\wp(a)>a$ is now taken to be Cantor's theorem.
The proof of the theorem is a set-theoretic version of the diagonalization argument.
We want to show that the cardinal number C of the power set of a set is strictly larger than the cardinal number of the set itself (i.e. $C(\wp(A)) > C(A)$).
To show that fact, it suffices to show that there is no function which maps A one-one and onto its power set.
A function is called one-one if it every element of the domain maps to a different element of the range.
A function maps a set A onto another set B if the range of the function is the entire set B, i.e. if no elements of B are left out of the mapping.

Assume that there is a function $f: A \Rightarrow \wp(A)$
Consider the set $B = \{x \mid x \in A \bullet x \notin f(x)\}$
B is a subset of A, since it consists only of members of A.
So, B is an element of $\wp(A)$, by definition of the power set.
That means that B itself is in the range of f.
Since, by assumption, f is one-one and onto, there must be an element of A, b, such that f(b) is B itself.
Is $b \in B$?
If it is, then there is a contradiction, since B is defined only to include sets which are not members of their images.
If it is not, then there is a contradiction, since B should include all elements which are not members of their images.
(This step of the proof does seem a lot like the self-referential Russell's paradox, as Walter noted.)
Either way, we have a contradiction.
So, our assumption fails, and there must be no such function.
$\wp(A) > A$
QED

Let's call the size of the natural numbers $\aleph_0$.
Then the real numbers, and the real plane, are the size of the power set of the natural numbers, $2^{\aleph_0}$.
We can proceed to generate larger and larger cardinals through the power set process.
Moreover, set theorists, by various ingenious methods, including addition of axioms which do not contradict the given axioms, generate even larger cardinals.
You can google 'inaccessible cardinals'.

VI. Ordinal numbers

Let's start counting.
By adding one, here, we normally mean taking the successor of 1.
So, $\omega+1$ will be the successor of ω.
Ordinals generated in this way are called successor ordinals.

In transfinite set theory, there are also sets which are called limit elements.
We get them by taking the union of all the members of a set.
Ordinal numbers, set-theoretically, are just special kinds of sets, well-ordered sets.
A set is well-ordered if, basically, we can find an ordering relation on the set, and it has a first element.
If we consider all the sets that correspond to the finite ordinals, and combine them into a whole, we can get another well-ordered set.
This will be a new ordinal, and it will be larger than all of the ordinals in it.
So, there are two kinds of ordinals: successor ordinals and limit ordinals.

$1, 2, 3, \ldots \omega$
$\omega+1, \omega+2, \omega+3 \ldots 2\omega$
$2\omega+1, 2\omega+2, 2\omega+3 \ldots 3\omega$
$4\omega, 5\omega, 6\omega \ldots \omega^2$
$\omega^2, \omega^3, \omega^4 \ldots \omega^\omega$
$\omega^\omega, (\omega^\omega)^\omega, ((\omega^\omega)^\omega)^\omega, \ldots \varepsilon^0$

The limit ordinals are the ones found after the ellipses on each line.
Large ordinals correspond to the large cardinals.

VII. Definitions of natural numbers

We identify the natural numbers with certain sets, in order to link number theory with set theory.
Zermelo (1908)

$0 = \{\}$
$1 = \{\{\}\}$
$2 = \{\{\{\}\}\}$
...

Von Neumann (later)

$0 = \{\}$
$1 = \{\{\}\}$
$2 = \{\{\}, \{\{\}\}\}$
$3 = \{\{\}, \{\{\}\}, \{\{\}, \{\{\}\}\}\}$
...

Those are a bit less elegant, but more efficient.
I was unclear in class, in response to the question of whether these are ordinals or cardinals.
In set theory, both ordinal and cardinal numbers are certain kinds of sets.
I mentioned these definitions of natural numbers, which were not part of Cantor's theory, or any set theory, just to show you how one might embed number theory within set theory.
Frege, too, has a different way of picking out the natural numbers from the sets.

VIII. The continuum hypothesis

We did not discuss the continuum hypothesis, the material in this section, except in passing, in class.
We will return to it when we get to Gödel's paper on the topic.

Certain questions in the history of mathematics have proven difficult to answer.

Is the parallel postulate true?
Is Goldbach's conjecture true?
Is Fermat's theorem true?

Some questions are clearly answered affirmatively, like Fermat's conjecture.
We expect the same kind of answer for Goldbach.
The parallel postulate is a bit more interesting.
We see that it can fail, but it can also be true.
We see that the question is ill-formed.
There are different kinds of spaces, and they are each defined by a different answer.
There is not one true answer!

How about the continuum hypothesis?
The continuum hypothesis says that $\aleph_1 = 2^{\aleph}{}_0$.
Cantor thought the CH was true.

This is a wonderful topic, and deserves some time.
To understand the problem, we have to think a bit more about the progress of set theory.

Walter discusses the Burali-Forti paradox.
The point of that problem, and Cantor's paradox, is that some sets are just too large.
Naive set theory begins with the supposition that any collection determines a set.
This is called the axiom of comprehension, or the axiom of abstraction.
But, then we get to sets of sets that don't contain themselves, or the set of all ordinals.
The set of all ordinals will itself be an ordinal larger than itself.
Oops.
We now take these paradoxes to be theorems, showing that there are no functions mapping a set to its power set one-one and onto(Cantor) or set to which every ordinal number belongs (Burali-Forti).

Frege used an unrestricted axiom of comprehension, similar to that of Cantor.
As Tiles points out, Cantor proposed a distinction, parallel to that of Aristotle, of an absolute infinity.

Ordinarily, these days, we avoid the paradoxes of unrestricted comprehension by presenting a piecemeal axiomatization of set theory.
We start with a few basic axioms, and build up the rest from those.
We thus use what is called an iterative concept of set.
We iterate, or list, the sets from the beginning.

Cohen (1963) showed that the CH is independent of the standard axioms of set theory.
In fact, its negation is consistent with the axioms of set theory.
We can consistently consider the continuum to be of all different sizes.
The CH remains open.