

Some Sets of Mathematical Axioms

Propositional Logic, following Mendelson, *Introduction to Mathematical Logic*

The symbols are $\sim, \supset, (,)$, and the statement letters A_i , for all positive integers i .

All statement letters are wffs.

If α and β are wffs, so are $\sim\alpha$ and $(\alpha \supset \beta)$

If α, β , and γ are wffs, then the following are axioms:

$$A1: (\alpha \supset (\beta \supset \alpha))$$

$$A2: ((\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma)))$$

$$A3: ((\sim\beta \supset \sim\alpha) \supset ((\sim\beta \supset \alpha) \supset \beta))$$

β is a direct consequence of α and $(\alpha \supset \beta)$

Zermelo-Fraenkel Set Theory, again following Mendelson, but with adjustments

ZF may be written in the language of first-order logic, with one special predicate letter, \in .

Substitutivity: $(x)(y)(z)[y=z \supset (y \in x \equiv z \in x)]$

Pairing: $(x)(y)(\exists z)(u)[u \in z \equiv (u = x \vee u = y)]$

Null Set: $(\exists x)(y)\sim x \in y$

Note: the null set axiom ensures the existence of an empty set, so we can introduce a constant, \emptyset , such that $(x)\sim x \in \emptyset$.

Sum Set: $(x)(\exists y)(z)[z \in y \equiv (\exists v)(z \in v \cdot v \in x)]$

Power Set: $(x)(\exists y)(z)[z \in y \equiv (u)(u \in z \supset u \in x)]$

Selection: $(x)(\exists y)(z)[z \in y \equiv (z \in x \cdot \mathcal{F}u)]$, for any formula \mathcal{F} not containing y as a free variable.

Infinity: $(\exists x)(\emptyset \in x \cdot (y)(y \in x \supset \exists z \in x)z \in x)$

Note: 'Sy' stands for $y \cup \{y\}$, the definitions for the components of which are standard.

Peano Arithmetic, again following Mendelson with adjustments

P1: 0 is a number

P2: The successor (x') of every number (x) is a number

P3: 0 is not the successor of any number

P4: If $x'=y'$ then $x=y$

P5: If P is a property that may (or may not) hold for any number, and if

i. 0 has P; and

ii. for any x , if x has P then x' has P;

then all numbers have P.

Note: P5 is also called mathematical induction, and is actually a schema of an infinite number of axioms.

Birkhoff's Postulates for Geometry, following James Smart, *Modern Geometries*

Postulate I: Postulate of Line Measure. The points A, B,... of any line can be put into a 1:1 correspondence with the real numbers x so that $|x_B - x_A| = d(A, B)$ for all points A and B.

Postulate II: Point-Line Postulate. One and only one straight line l contains two given distinct points P and Q.

Postulate III: Postulate of Angle Measure. The half-lines l, m, \dots through any point O can be put into 1:1 correspondence with the real numbers $a \pmod{2\pi}$ so that if $A \neq 0$ and $B \neq 0$ are points on l and m , respectively, the difference $a_m - a_l \pmod{2\pi}$ is $\text{angle} \triangle AOB$. Further, if the point B on m varies continuously in a line r not containing the vertex O, the number a_m varies continuously also.

Postulate IV: Postulate of Similarity. If in two triangles $\triangle ABC$ and $\triangle A'B'C'$, and for some constant $k > 0$, $d(A', B') = kd(A, B)$, $d(A', C') = kd(A, C)$ and $\angle B'A'C' = \pm \angle BAC$, then $d(B', C') = kd(B, C)$, $\angle C'B'A' = \pm \angle CBA$, and $\angle A'C'B' = \pm \angle ACB$.