Philosophy 405: Knowledge, Truth and Mathematics
Spring 2008
M, W: 1-2:15pm

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Recursive Functions (following Hunter, *Metalogic*, p 232 et seq.)

Initial functions (functions from ordered n-tuples on ℕ to ℕ)
(n-tuples are just ordered pairs, triples, etc.)

Successor: $f_1(x) = x+1$
Sum: $f_2(x,y) = x+y$
Product: $f_3(x,y) = x \bullet y$
Power: $f_4(x,y) = x^y$ (calling $0^0 = 1$, in order to have all values defined)
Arithmetic difference: $f_5(x,y) = x \dot{-} y$, where $x \dot{-} y = x-y$, if $x>y$, and $x \dot{-} y=0$ if $y \geq x$

Operations on Computable Functions

Combination: Any combination of computable functions is computable
The μ-operation: Let $f(x_1,...x_n,y)$ be a computable function such that for each n-tuple of natural numbers $<x_1,...x_n>$, there is a natural number y, such that $f(x_1,...x_n,y)=0$. The μ-operation returns the least such y. That is, the function $g(x_1,...x_n,y)=\mu y[f(x_1,...x_n,y)=0]$ is given by the μ-operation. Functions obtainable by the μ-operation (given the conditions in the first sentence) are computable.

Recursive Functions

Any function which is obtainable from the initial functions by a finite number of steps, using combination or the μ-operation is recursive.

The set of recursive functions is provably equivalent to what a Turing machine can calculate.

Church's Thesis: The recursive functions are exactly the computable ones.

In one direction, this is obvious: all recursive functions are computable.
In the other, there is a question: are there computable functions that are not recursive?
Church's thesis is pretty well accepted today.
But, there are interesting debates about whether it is provable.

Notes
The name 'recursive function' comes from Gödel, in his incompleteness paper.
The subclass of primitive recursive functions are those obtainable without the use of the μ-operation.
The superclass of partial recursive functions are those obtainable by a weaker μ-operation.
For an alternative presentation, see Mendelson, *Introduction to Mathematical Logic*, p 174 et seq.
Full references to Hunter and Mendelson are in the course bibliography.