Philosophy 405: Knowledge, Truth and Mathematics
Spring 2008
M, W: 1-2:15pm

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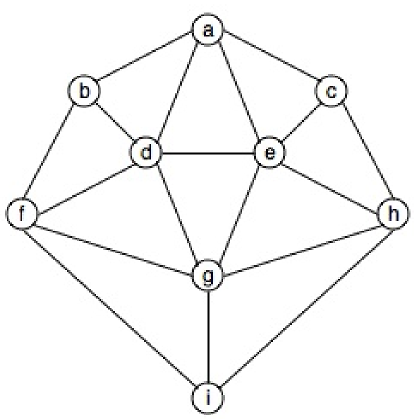
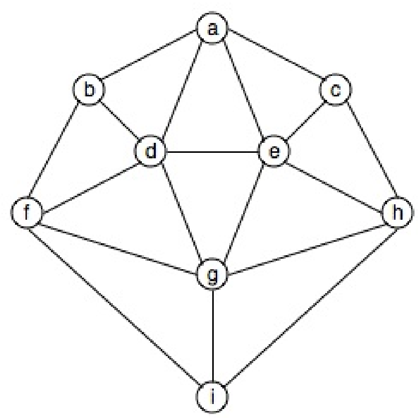
Constructive and Non-Constructive Proofs

A Constructive Proof:

Definition: A coloring of a graph is an assignment of a color to each node of the graph.
Definition: A graph is 3-colorable if any coloring which uses only three colors does not assign the same color to any two nodes which share a branch.
Definition: A graph is 4-colorable if any coloring which uses only four colors does not assign the same color to any two nodes which share a branch.

Theorem: There are graphs which are 4-colorable but which are not 3-colorable.
Proof: In two stages. Present a graph which is not 3-colorable but which is 4-colorable. (See below.
Stage 1: Prove that the graph is not 3-colorable. Stage 2: Show that the graph is 4-colorable.





A Non-Constructive Proof:

Claim: There exist irrational numbers x and y such that x^y is rational.

Proof: Let $z = \sqrt{2}^{\sqrt{2}}$. If z is rational then z is our desired number with $x = \sqrt{2}$ and $y = \sqrt{2}$.
Suppose that z is irrational. Then, let $x = z$ and $y = \sqrt{2}$.

$$x^y = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2} \cdot \sqrt{2})} = \sqrt{2}^{2} = 2.$$

In this case, x^y is again rational.

So, whether z is rational or irrational, we have shown the existence of irrational numbers x and y such that x^y is rational.
But, we do not know whether z is rational or irrational, so we do not have a constructive proof.