Knowledge, Truth, and Mathematics

Philosophy 405
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Class 9: Kant II

Necessity/Contingency and Apriority/Empiricality

- Kripke:
  - Necessary empirical: water is H_2O.
  - Contingent *a priori*: the standard meter is one meter long.
- Forget those.
- We are granting the controversial claim that all *a priori* claims are necessary.
- Collapsing the metaphysical and the epistemological

Epistemology and Semantics

- One might believe that all *a priori* claims must be analytic.
  - The only way to reason *a priori* is by analysis of concepts.
- Concomitantly, one might align contingency with empirical justification and syntheticity.
  - A claim is contingent when it is justified by appeal to sense experience and brings together concepts that are not necessarily related.
- Hume holds these two claims.
  - Mathematical propositions are necessary, relations of ideas which follow from *a priori* applications of the principle of contradiction.
  - Matters of fact are contingent, justified empirically (by tracing ideas back to initial impressions) and synthetic.

Hume's Rubric

Hume's Rubric	*A priori*	Empirical
Analytic	Relations of Ideas	--
Synthetic	--	Matters of Fact

Kant's Rubric

Hume's Rubric	*A priori*	Empirical
Analytic	Relations of Ideas	--
Synthetic	--	Matters of Fact

Kant's Rubric	*A priori*	Empirical
Analytic	Logic	--
Synthetic	Most Mathematics and Metaphysics, and Some Physics	Empirical Judgments

Synthetic *A Priori* Judgments

- Metaphysics: Every effect has a cause.
- Physics: Newton's third law of motion
  - For every action there is an equal and opposite reaction.
- Mathematics: 7+5=12.
- "Mathematical propositions, properly so called, are always *a priori* judgments rather than empirical ones; for they carry with them necessity, which we could never glean from experience...It is true that one might at first think that the proposition 7 + 5 = 12 is a merely analytic one that follows, by the principle of contradiction, from the concept of a sum of 7 and 5. Yet if we look more closely, we find that the concept of the sum of 7 and 5 contains nothing more than the union of the two numbers into one; but in [thinking] that union we are not thinking in any way at all what that single number is that unites the two. In thinking merely that union of 7 and 5, I have by no means already thought the concept of 12; and no matter how long I dissect my concept of such a possible sum, still I shall never find in it that 12. We must go beyond these concepts and avail ourselves of the intuition corresponding to one of the two..." (*Critique*, B14-5, p 5).

Daniel's Worries About Necessity

One Man's Modus Ponens...

...Is Another Man's Modus Tollens

- Hume agreed that universal physical laws could not be learned from experience.
  - We have no reason to believe that the future will be like the past, that the laws of nature are uniform.
  - The physical and metaphysical laws are not analytic.
  - We can not find the effects in the causes.
  - They are not learned by instances.
  - Hume inferred skepticism.
- Kant accepts that there are mathematical, metaphysical, and even physical laws that hold necessarily, that are known *a priori*.
- Working backwards, he argues that our cognitive abilities must be such that they allow us to know those principles *a priori*.
- "For experience would provide neither strict universality nor apodeictic certainty..." (*Critique*, A31/B47).
- No empirical experiences with undermine these claims.
  - Foxes and chickens
  - Salt and water

Are the Denials of Mathematical Theorems Contradictions?

- If mathematical propositions like ‘7+5=12’ were analytic, they would follow from conceptual analysis, or from the principle of contradiction.
- Conceptual analysis is more compelling than the principle of contradiction.
- The claim that we have to go beyond the seven and the five, and the concept of their sum, in order to think of the twelve, seems plausible.
- Kant also claims that no contradiction follows from the denial of some mathematical propositions.
  - “There is no contradiction in the concept of a figure which is enclosed within two straight lines, since the concepts of two straight lines and of their coming together contain no negation of a figure. The impossibility arises not from the concept in itself, but in connection with its construction in space, that is, from the conditions of space and of its determination” (*Critique* A220/B268, p 28).
- ‘$7+5\neq 12$’?
  - If the claim is synthetic, then it has to be logically possible for $7+5\neq 12$.
- It is only in the broadest sense which the denials of mathematical propositions are possible.

No Innate Ideas, for Kant

- Kant argues that there are certain cognitive structures that impose an order to our possible experience.
- The mind has templates for judgments, which are imposed and can be known *a priori*.
- Our minds do not contain judgments themselves.
- If we look at our cognitive structures, turning our reasoning on itself, we can find the necessary structure of our reasoning, and grounds for synthetic *a priori* claims.
- Transcendental reasoning
  - We determine what the nature of our minds must be in order to support the most central, important claims.
  - What are the necessary conditions of the structure of our minds such that we get the necessary truths of mathematics, and laws of nature?
  - Like Hume, and Locke, we get a description of our subjective conceptual framework.
  - Unlike empiricism, the transcendental method allows us to show what holds necessarily for all possible experience.

Mathematics and Intuitions

- Kant takes the geometry and arithmetic as independent.
- Geometry arises out of spatial intuition.
- Arithmetic comes from the comibination of our temporal and spatial intuitions.

Mathematical Methodology

Let the geometrician take up [the questions what relation the sum of a triangle's angles bears to a right angle]. He at once begins by constructing a triangle. Since he knows that the sum of two right angles is exactly equal to the sum of all the adjacent angles which can be constructed from a single point on a straight line, he prolongs one side of his triangle and obtains two adjacent angles, which together are equal to two right angles. He then divides the external angle by drawing a line parallel to the opposite side of the triangle, and observes that he has thus obtained an external adjacent angle which is equal to an internal angle - and so on. In this fashion, though a chain of inference *guided throughout by intuition*, he arrives at a fully evident and universally valid solution of the problem (*Critique* A716-7/B744-5, p 15; emphasis added).

Extending from Definitions

- At the beginning of mathematical thought, the geometer is given some definitions.
- As we work through the construction, though, we extend those definitions.
- We extend a line, we draw new lines.
- “I must not restrict my attention to what I am actually thinking in my concept of a triangle (this is nothing more than the mere definition); I must pass beyond it to properties which are not contained In this concept, but yet belong to it” (*Critique* A718/B746).

Kant’s Psychological Account

“The assertion that 7+5 is equal to 12 is not an analytic proposition. For neither in the representation of 7, nor in that of 5, nor in the representation of the combination of both, do I think the number 12. (That I must do so in the *addition* of the two numbers is not to the point, since in the analytic proposition the question is only whether *I actually think* the predicate in the representation of the subject” (*Critique*, A164/B205, p 23; final emphasis added).

Constructing Objects of Mathematics in Intuition

- In our sense experience, we are given something, which we might call the passing show.
  - "The effect of an object on our capacity for representation, insofar as we are affected by the object, is *sensation*. Intuition that refers to the object through sensation is called *empirical* intuition. The undetermined object of an empirical intuition is called *appearance*" (A19-20/B34, p 8).
- Some intuitions are pure, about the structure (or form) of space and/or time themselves.
- But, all intuitions are particular, or singular.
- They represent particular objects, rather than general rules.

Empirical Intuitions

- In empirical intuitions we can divide the matter from the form.
  - The matter corresponds to sensation.
- This appearance has certain abstract properties, a form.
  - The particulars of the form of this appearance are unique to my specific experience.
  - The general properties of the form of appearances are properties of all such experiences.
- All experiences take place in space and in time.
- My experience of the pen is thus necessarily given in intuition in both space and time.

Pure Intuitions

- Some intuitions, pure intuitions, contain no empirical matter.
- “If from the representation of a body I separate what the understanding thinks in it, such as substance, force, divisibility, etc., and if I similarly separate from it what belongs to sensation in it, such as impenetrability, hardness, color, etc., I am still left with something from this empirical intuition, namely, extension and shape. These belong to pure intuition, which, even if there is no actual object of the senses or of sensation, has its place in the mind *a priori*, as a mere form of sensibility” (A20-1/B15, p 9).
- We arrive at our consideration of pure forms of intuition by a transcendental method with some similarities to abstraction.
- Kant does not claim that our knowledge of space (and time) is derived from abstraction.
- We are discovering that knowledge of space and time is necessarily presupposed in any empirical intuition.
  - He does not move forward, as Locke does, from experiences of two objects to beliefs about the number two.
  - He moves backward, from the claim that two and two are four to the forms of intuition which support such claims.

Two Pure Forms of Intuition

space and time

- We represent objects as outside of us using our outer sense.
- All objects outside of us are represented as extended in space; space is the form of outer sense.
- We represent objects according to our inner sense as in time.
- The idea of a possible experience occurring outside of space or time is nonsense.
- Space and time are presuppositions we must impose on all our possible experience.
- Instead of despairing of learning of space and time from experiences which presuppose it, as Hume tried and failed, Kant inverts his account to make space and time subjective forms of intuition.
- They are ways in which we structure the world of things in themselves, not ways in which the world exists in itself.
- They are properties of appearances, which are the objects of our empirical intuition.
- Geometry is the study of the form of outer sense, of pure, *a priori* intuitions of space.
- Arithmetic, depends essentially on construing addition as successions in time.

Intuitions and Concepts

- The faculty of intuition gives us appearances.
  - Intuitions are just the nearly-raw data, the content, of experience.
  - Our intuitions are passive, and what is given in intuition is not structured by the understanding.
  - What we are given lacks conceptual structure.
- In order to go beyond those mere intuitions and develop mathematics, we have to think about them.
  - Appearances are structured; so the structure must come from somewhere.
  - We have to impose concepts on the intuitions.
  - We cognize using whatever conceptual apparatus we have.

The Synthesis of the Manifold

- The act of arranging what is given in intuition is what Kant calls synthesis of the manifold.
- This synthesis is cognized by the structured application of concepts in the understanding.
- If the synthesis is empirical, then we have an ordinary empirical cognition.
- If the synthesis is pure, then we can arrive at pure concepts of the understanding.
- Intuition and understanding thus work together to produce experience.
- "Thoughts without content are empty; intuitions without concepts are blind" (A51/B76).

Kant's Categories

derived transcendentally

- Quantity
  - Unity
  - Plurality
  - Totality
- Quality
  - Reality
  - Negation
  - Limitation
- Relation
  - Inherence and Subsistence (substance)
  - Causality
  - Community (Interaction)
- Modality
  - Possibility and Impossibility
  - Existence and Non-Existence
  - Necessity and Contingency

Two Stages of Transcendental Deduction

- In the first stage, Kant argues that the categories apply to any being with sensible intuition.
  - We proceed from a diverse manifold given in intuition to a single thought of a single, conscious person.
  - When we do so, we combine (either by synthesis or otherwise) the manifold.
  - This combination is an active function of our cognition, in contrast to the passivity of intuition.
  - Since our action is subjective, the application of the conditions of unity are subjective.
  - But the unity is also objective, since it determines objects for us.
- In the second stage, Kant shows that the categories necessarily apply to human sensibility.
  - We intuit the world in space and time.
  - But space and time, besides being forms of intuition, are also intuited themselves.
  - Since space and time are pure forms of intuition, they are presupposed in all experience.
  - Since any experience is already structured, or determined, space and time, as we experience them, are deeply embedded in those experiences.
  - Since any experience also presupposes the application of the categories, space and time themselves must be subject to the categories.
  - The categories apply to our intuition which is sensible in the forms of outer sense (space) and inner sense (time).

Defining Numbers

- The categories of quantity are directly relevant to our understanding of mathematics, especially arithmetic.
- "The pure image of all magnitudes for outer sense is space; that of all objects of the senses in general is time. But the pure *schema* of magnitude, as a concept of the understanding, *is number*, a representation which comprises the successive addition of homogeneous units. Number is therefore simply the unity of the synthesis of the manifold of a homogeneous intuition in general, a unity due to my generating time itself in the apprehension of the intuition" (*Critique* A142-3/B182, pp 11-12).
- We construct, in pure intuition, a figure.
  - For arithmetic, we construct stroke-symbols, and add them up.
- Then, we have rules for adding symbols which depend on those symbols.
  - "[Mathematics] abstracts completely from the properties of the object that is to be thought in terms of...a concept of magnitude. It then chooses a certain notation for all constructions of magnitude as such (numbers), that is, for addition, subtraction, extraction of roots, etc. Once it has adopted a notation for the general concept of magnitudes so far as their different relations are concerned, it exhibits in intuition, in accordance with certain universal rules, all the carious operations through which the magnitudes are produced and modified" (*Critique* A717/B745, p 15).

Returning to the Cartesian View

- Descartes had separated thought from sensation.
  - “Many properties can be very clearly and very distinctly demonstrated of [a chiliagon], which could certainly not happen if we perceived it only in a confused manner or...only in a verbal way. In fact, we have a clear understanding of the whole figure, even though we cannot imagine it in its entirety all at once. And it is clear from this that the powers of understanding and imagining do not differ merely in degree but are two quite different kinds of mental operation. For in understanding the mind employs only itself, while in imagination it contemplates a corporeal form” (Descartes, *Fifth Replies* AT 384-5).
- For Kant, a pure intuition need not be like a picture.
  - “If five points be set alongside one another, thus, ..., I have an image of the number five. But if, on the other hand, I think only a number in general, whether it be five or a hundred, this thought is rather the representation of a method whereby a multiplicity, for instance a thousand, may be represented in an image in conformity with a certain concept, than the image itself. For with such a number as a thousand the image can hardly be surveyed and compared with the concept” (*Critique* A140/B179, p 11).

Pure Intuitions are Formal, Rather than Material

In respect to this material element, which can never be given in any determinate fashion otherwise than empirically, we can have nothing *a priori* except indeterminate concepts of the synthesis of possible sensations...As regards the formal element, we can determine our concepts in *a priori* intuition, inasmuch as we create for ourselves, in space and time, through a homogeneous synthesis, the objects themselves - these objects being viewed simply as *quanta*... The determination of an intuition *a priori* in space (figure), the division of time (duration), or even just the knowledge of the universal element in the synthesis of one and the same thing in time and space, and the magnitude of an intuition that is thereby generated (number), - all this is the work of reason through construction of concepts, and is called *mathematical* (*Critique* A723-4/B751-2, p 17).

Pure Intuitions and Rules

- What is important about a pure intuition is the rule it dictates, not the picture in the mind that accompanies it.
- The matter of the intuition is empirical.
- But the form of the intuition is known *a priori*, and is independent of its matter.
- Thus, general arithmetical concepts can be applied to particular spatial and temporal intuitions (to magnitudes, considered abstractly), and geometrical concepts can be applied to spatial intuitions, without taking those intuitions to be empirical.

Another Argument for Syntheticity

- Consider drawing a diagram (say, of an isosceles triangle with an altitude) to accompany a proof (say, that the altitude is also an angle bisector).
- We use such pictures to evoke concepts.
- But we do not take those pictures to be the subjects of our mathematical theorems.
- We take the drawing to be an example, and instance of a more general rule.
- The empirical intuition does not exhaust the reference of the proof.
- But, the proof remains about intuitions.
- If we were just analyzing concepts, definitions, we would have no need for such intuitions.
- Mathematical work is synthetic, but it remains *a priori*.
- The inferences are conceptual, but guided by intuition (A717/B745, p 15).

Locke : Berkeley :: Kant : Berkeley*

- Kant's appeals to the conceptualizing of our pure forms of intuition are like Locke's appeals to abstraction.
- For Locke, mathematical theorems applied to abstract ideas which might be triggered by empirical experiences, but which were mental constructs.
- Berkeley criticized Locke's claim that we could have an abstract idea of a triangle.
- An abstract idea is supposed to apply to all kinds of triangles, and some triangles have properties which contradict the properties of other triangles.
- Kant's claim that mathematical ideas are constructions in pure intuition seems liable to the same kind of criticism.

Kant Avoiding the Berkeley Criticism

- Kant avoids some of Berkeley's criticism by claiming that our intuitions are not pictures.
- In determining how a theorem applies, we rely on the rule which guides the construction in intuition, rather than the intuition itself
- Depending on which kind of triangle we are considering, we appeal to different rules.
- Thus, though Kant avoids Berkeley's criticism, he must distinguish among the properties which belong to all triangles, the concepts of which will accompany any construction of a triangle, and those which belong only to some triangles, and which we are free to add to our construction or not.

Philip Kitcher Distinguishes Three Types of Properties

- R-properties, which all triangles have, analytically, e.g. having three angles;
- S-properties, which are synthetic, arising from the construction in intuition for any triangle, e.g. the sum of two sides is greater than the third;
- A-properties, like being scalene, which need not apply to every triangle.
- "R-properties are just the properties which the schema alone determines; for the triangle an example of an R-property would be the property of having three sides. S-properties are those properties which the schema and the structure of space together determine; the side-sum property and the property of having the internal angle-sum equal to 180 degrees are both supposed to be S-properties. Finally, there are the A-properties, peculiarities of the particular figure drawn, such as the scaleneness of the triangle" (Kitcher, "Kant on Mathematics" 44).

The R, the A, and the S

- In pure intuition, any construction of a figure or number will have to have its R-properties.
- We are free to construct objects with any A-properties we like.
  - They do not follow from the nature of, say, the triangle.
  - Their denials are logically possible.
  - We can construct a scalene triangle, or an equilateral triangle, or a blue triangle.
- S-properties do not follow from the nature of the triangle, and their denials are also logically possible.
  - But, we are only free to construct objects with or without these properties insofar as we are free to construct different spatial structures.

Non-Euclidean Space and Kant's Post-Berkeley Problem

- Kant assures us that all space is necessarily Euclidean.
  - "Our exposition...establishes the *reality*, that is, the objective validity, of space in respect of whatever can be presented to us outwardly as object" (*Critique* B44/A28).
  - We construct our intuitions in Euclidean space.
  - Our knowledge of geometry is *a priori* knowledge of the necessary structure of space.
  - Our knowledge of arithmetic is *a priori* knowledge of the necessary structure of "combinatorial" aspects of space and time.
- But there are different kinds of space: Euclidean and non-Euclidean.
  - Consider an interstellar triangle.
  - The sum of its angles will not be 180°, due to the curvatures of space-time corresponding to the gravitational pull of the stars, and other large objects.
  - Space-time is not Euclidean, but hyperbolic.
- Given the different structures of space, Kant would have to argue that we can know, *a priori*, which space we are using in our intuition.
  - We have to know whether we are constructing an A-property or an S-property.

Kitcher's Claim

- Kant has no way to distinguish S-properties from A-properties without already knowing the properties of space which underlie my intuitions.
- "Let G be a basic geometrical truth. G is supposed to be synthetic *a priori*. Its synthetic status arises because its truth value is, partially, determined by the structure of space. It is logically possible that space have a structure such that G be false. (Otherwise, G would be analytic.) Further, it is logically possible that G might have been false in such a way that many figures actually had the property ascribed to them by G. How could we have determined from inspection of such a figure that the property was only an A-property and that we should not therefore generalize over it? We can answer this question only if we can decide what counts as the application of a rule on the structure of space and what was our free decision in drawing the figure. Yet to distinguish S-properties from A-properties is just to recognize the structure of space. We could not therefore come to know G in the way which Kant describes" (Kitcher, "Kant on Mathematics", 45-6).
- Kant's claims that Euclidean geometry is known *a priori* and that *a priori* knowledge is infallible thus can not both hold.

Looking Forward

- We have seen an interplay between metaphysics and epistemology.
- Those philosophers who provided a substantial mathematical ontology (e.g. Plato, Descartes) seemed committed to an unacceptable epistemology.
- Those philosophers who started with commonsense epistemologies (e.g. Aristotle, Locke) seemed committed to an unacceptably anemic mathematical ontology.
- Where Kant denied that the empirical intuition constituted or sufficed for mathematical knowledge, Mill will deny that we can get anything more than that.
- Mill severely restricts mathematical necessity and apriority.
- But, he provides a much less controversial epistemology.