Hamilton College
Russell Marcus

Class #5: September 13
Modern Rationalism I, Descartes and Leibniz

I. Definitions

Let's use 'Platonism' to refer the view that we ascribe to Plato and his followers, including the theory of forms.
We will use 'platonism' to refer to the beliefs that mathematical objects exist outside of space and time and that many mathematical statements are true.
So, Plato is both a Platonist and a platonist.
Descartes and Leibniz are platonists, but not Platonists.

II. Descartes and Mathematics

We are often tempted to think that sense experience is the source of most or all of our knowledge, and thus that mathematical knowledge should somehow derive from the senses.
Descartes inverts that order.
At the beginning of the *Meditations*, he discusses worries about some false beliefs.
He wonders about frail sense perception, how we know whether we are dreaming, and whether or not a powerful deceiver is putting false thoughts in our minds.
His real worry is that while some of our beliefs seem impervious to doubt, beliefs which are based on our senses are less secure.
As Plato worried about the veracity of beliefs about the sensible world, Descartes worries about the sense properties of physical objects, what came to be known as the secondary properties.
Locke is often credited with the primary/secondary distinction, but the primary/secondary distinction is present in much earlier scientists, including Galileo (1564-1642), who wrote:

> ...that external bodies, to excite in us these tastes, these odours, and these sounds, demand other than size, figure, number, and slow or rapid motion, I do not believe, and I judge that, if the ears, the tongue, and the nostrils were taken away, the figure, the numbers, and the motions would indeed remain, but not the odours, nor the tastes, nor the sounds, which, without the living animal, I do not believe are anything else than names (*Opere* IV, 336).

Descartes is writing as a member of, and in response to, the scientific revolution.
The Copernican system was seen, especially by the Church, as antagonistic to religion.
Galileo's censorship by the Inquisition forced Descartes to suppress publication of *Le Monde*, his work on the philosophical implications of the new science.
Descartes's real goal is a system which will accommodate the new science of Galileo while not undermining what he thought of as central Church doctrine.
The Catholic Church's world view was in many ways essentially Aristotelian.
The Church and Aristotle did differ about the nature of the soul.
For the Church, the soul was separate from the body.
As we have seen, Aristotle did not place the soul outside of the body, and thus had linked thought and sensation.
Descartes presents the *Meditations* as a proof of both the existence of God and the immortality of the soul.

By cleaving thought from sensation, Descartes can thus support the Church's view of the soul against Aristotelian accounts.
Our knowledge of mathematics, for Descartes, is the product of this pure, non-sensory, rational thought.
Leibniz follows Descartes, refining the rationalist view of mathematical knowledge.
For the rationalists, beliefs which depend on the senses are tenuous, while mathematical beliefs, unsullied by sensation, are elevated to pure truths.
We can know them certainly, because they are the result of pure reason.

Both Descartes and Leibniz were important mathematicians in addition to being seminal philosophers.
Descartes developed analytic geometry, applying algebraic methods to geometry, and facilitated the development of general theories of curves and functions.
The Cartesian plane, of course, is named after Descartes.
Leibniz developed the calculus, concurrently with, but independently from, Newton.
Both Descartes and Leibniz saw their mathematical developments as arising from their rationalist methods, as did others like Galileo, Huygens, and Newton.
Mathematical developments were spurred by empirical science, but even the methods of empirical science were as purely rational as they were experimental.
Scientists at the time presupposed simplicity for both mathematical and empirical theories, rather than finding it.
Concerning his claim that a stone falling from a ship's mast will drop in the same place whether or not the ship is moving, Galileo writes:

> "So, you have not made a hundred tests, or even one? And yet you so freely declare it to be certain?... Without experiment, I am sure that the effect will happen as I tell you, because it must happen that way" (Galileo, *Dialogue Concerning the Two Chief World Systems*, p 145.)

Descartes went further than most of his contemporaries in his view of the role of mathematics, and mathematical methods, in science.
Whereas most scientists of the seventeenth and eighteenth centuries were impressed by the advances that mathematics made for science, Descartes thought that the world was essentially mathematical.
For Descartes, the objectivity of science and the objectivity of mathematics were inseparable.

We are concerned with two related Cartesian claims: a metaphysical claim and an epistemological one.
Descartes's metaphysical claim is that mathematical objects have a determinate, objective nature, independent of us.
Mathematical truths are necessary truths.
His argument for the first part of the metaphysical claim is:

DM
1. A thing's nature depends on me if I can make it any way I like.
2. A thing's nature is objective if I can not make it any way I like.
3. I can not make mathematical objects any way I would like.

So, mathematical objects are objective.

To show that mathematical truths are necessary truths, Descartes relies on the larger argument that error arises from the senses.
In fact, Descartes does not think that any truths are necessary in the sense that they are independent of God's will.
But the claim that the mathematical truths depend on God is complicated and idiosyncratic.

Even for Descartes, mathematical truths are necessary, as far as we can understand necessity.

Descartes's epistemological claim is that our knowledge of the truths of mathematics can not come from the senses.
So, it must be innate.
His argument for the epistemological claim is:

DE
1. All ideas must be invented, acquired, or innate.
2. Mathematical truths can not be invented, by the metaphysical claim.
3. Mathematical truths can not be acquired, by the chiliagon claim.
So, they must be innate.

We certainly do not learn mathematics before we have sense experience.
Descartes is distinguishing between the order of knowledge as it comes to us and the order as it is justified.
Leibniz makes this point explicitly.

> Although the senses are necessary for all our actual knowledge, they are not sufficient to provide it all, since they never give us anything but instances, that is particular or singular truths. But however many instances confirm a general truth, they do not suffice to establish its universal necessity; for it does not follow that what has happened will always happen in the same way (Leibniz, *New Essays on Human Understanding*, 49).

This may be the most important argument that we can take from Descartes and Leibniz.
There is a genetic fallacy in assuming that because evidence from the senses temporally precedes evidence for mathematics, the beliefs which are more closely connected to our sense experience are more secure than our mathematical beliefs.

Descartes's arguments for the objectivity and innateness of mathematical beliefs are only as good as his broader system.
Descartes's more general goal is to secure all of his beliefs from doubt.

> I realized that it was necessary, once in the course of my life, to demolish everything completely and start again right from the foundations if I wanted to establish anything at all in the sciences that was stable and likely to last (Meditation One, AT 17).

He uses Euclid's *Elements* as a model, though he believes that the method of the *Meditations* surpasses Euclid's work in its security.
Still, the geometric presentation, from the second set of replies, makes Descartes's underlying goal clear.
We call Descartes a foundationalist because he wants to ground our knowledge on basic, indubitable truths.
In the *Meditations*, it looks like his foundational truth is the cogito.
In the geometric presentation, it looks like his foundation is the existence and goodness of God.
But, in either case, Descartes's plan is to present an account of all of our knowledge that makes mathematical knowledge, if not primary, then certainly ahead of the beliefs based on our senses.
Leibniz, too, presents a foundational system.
For Leibniz, the foundations, or primary truths, are identities, known by pure (non-sensory) intuition.
All other truths reduce to primary truths by definitions.

III. Axiomatic Theories

Euclid's *Elements* was, in the seventeenth century, the only important axiomatic theory.
All of mathematics was presumed to be geometric.
Thus, new developments could be, theoretically, derived from Euclid's work.
In the late nineteenth century, spurred mainly by Frege's revolutionary work in logic, the method of axiomatization become central to mathematics.
Let us look briefly at a variety of axiomatic theories, to get us in the mood.
We will start with Hofstadter's simple MIU system

The MIU system

Any string of Ms Is and Us is a string of the MIU system.
MIU, UMI, and MMMUMUUUMUMMU are all strings.
Only some strings will be theorems.
The theorems will correspond to the true sentences of English.
Or, they could correspond to theorems of geometry.

Axioms and theorems

A theorem is any string which is either an axiom, or follows from the axioms by using some combination of the rules of inference.
The MIU system takes only one axiom: MI.

MI is our foundational truth, as the cogito, or God, is the foundation for Descartes's epistemology.

Four rules of inference

R1. If a string ends in I you can add U.
R2. From Mx, you can infer Mxx.
That is, you can repeat whatever follows an M.
R3. If III appears in that order, then you can replace the three Is with a U
R4. UU can be dropped from any theorem.

Derive MIIIII.
Try to derive MU.
For help, see Hofstadter's book, pp 259-261.

Let's turn to some more typically mathematical theories.
As those of you who studied logic with me know, the standard approach to presenting a formal theory is to first specify a language, including its syntax, and a definition of a well-formed-formula (or wff).
Then, one presents axioms, or basic assumptions, and rules of inference which allow one to derive theorems from the axioms.
Technically, the rules of inference are properties of the logical theory in which the particular theory is embedded.
We'll start with the most basic logical theory, propositional logic, in an elegant form.

Propositional Logic (PL), following Mendelson, *Introduction to Mathematical Logic*

The symbols are ~, ⊃, (,), and the statement letters A_i, for all positive integers i.
All statement letters are wffs.
If α and β are wffs, so are ~α and (α ⊃ β)
If α, β, and γ are wffs, then the following are axioms:
A1: (α ⊃ (β ⊃ α))
A2: ((α ⊃ (β ⊃ γ)) ⊃ ((α ⊃ β) ⊃ (α ⊃ γ)))
A3: ((~β ⊃ ~α) ⊃ ((~ β ⊃ α) ⊃ β))
β is a direct consequence of α and (α ⊃ β)

Predicate logic is rarely presented axiomatically.
Let's skip to the most widely-accepted foundational mathematical theory, set theory.
There are a variety of competing set theories, but ZF is standard.

Zermelo-Fraenkel Set Theory (ZF), again following Mendelson, but with adjustments
ZF may be written in the language of first-order logic, with one special predicate letter, ∈.

Substitutivity:	(x)(y)(z)[y=z ⊃ (y∈x ≡ z∈x)]
Pairing:	(x)(y)(∃z)(u)[u∈z ≡ (u = x ∨ u = y)]
Null Set:	(∃x)(y)~x∈y
Sum Set:	(x)(∃y)(z)[z∈y ≡ (∃v)(z∈v • v∈x)]
Power Set:	(x)(∃y)(z)[z∈y ≡ (u)(u∈z ⊃ u∈x)]
Selection:	(x)(∃y)(z)[z∈y ≡ (z∈x • $\mathscr{F}$u)], for any formula $\mathscr{F}$ not containing y free.
Infinity:	(∃x)(∅∈x • (y)(y∈x ⊃ Sy∈x), where 'Sy' stands for y∪{y}, the definitions for the components of which are standard.

Actually, most mathematicians would adopt a further axiom, called Choice, yielding a theory commonly known as **ZFC**.
Choice says that given any set of sets, there is a set which contains precisely one member of each of the subsets of the original set.
The axiom of choice has many equivalents, some of which are much less intuitively pleasing than the more simple formulations.
Further, Choice leads to some strange results.
We can discuss the topic later, when we look in more detail at set theory

It is widely accepted that all mathematical theories can be reduced to set theory.
A theory can be reduced to set theory if with the proper definitions, all the theorems of the higher-level theory can be written with just the language of set theory, and can be proved, in principle, with just the axioms of set theory.
Set theory is thus seen as a unifying framework for mathematics: all mathematical results can be brought together as complex set-theoretic statements.
Some philosophers and mathematicians now have given up on set theory as the most basic unifying mathematical theory, preferring a more abstract theory called category theory.

If our interests are more local, more mathematical, we can formulate axioms for particular mathematical theories, and skip the reduction to set theory altogether.
For example, we can construct axioms for number theory, or geometry, or topology, and just embed those

axioms in a logical theory.
Here is a classic formulation of number theory, called Peano arithmetic, which was developed by Richard Dedekind, but gets Peano's name.
(Peano himself credited Dedekind.)

Peano Arithmetic, again, following Mendelson with adjustments

P1: 0 is a number
P2: The successor (x') of every number (x) is a number
P3: 0 is not the successor of any number
P4: If x'=y' then x=y
P5: If P is a property that may (or may not) hold for any number, and if i. 0 has P; and ii. for any x, if x has P then x' has P; then all numbers have P.
P5 is mathematical induction, actually a schema of an infinite number of axioms.

The first modern axiomatization of geometry is due to Hilbert in the late nineteenth century.
Hilbert's axiomatization is notable for its completely pure geometric form, which eschews all number theory.
Here is an more elegant axiomatization of geometry, but one which uses real numbers.

Birkhoff's Postulates for Geometry, following James Smart, *Modern Geometries*

Postulate I: Postulate of Line Measure. The points A, B,... of any line can be put into a 1:1 correspondence with the real numbers x so that $|x_B - x_A| = d(A,B)$ for all points A and B.
Postulate II: Point-Line Postulate. One and only one straight line l contains two given distinct points P and Q.
Postulate III: Postulate of Angle Measure. The half-lines l, m... through any point O can be put into 1:1 correspondence with the real numbers a(mod 2π) so that if A≠0 and B≠0 are points on l and m, respectively, the difference $a_m - a_l$ (mod 2π) is angle△AOB. Further, if the point B on m varies continuously in a line r not containing the vertex O, the number a_m varies continuously also.
Postulate IV: Postulate of Similarity. If in two triangles △ABC and △A'B'C', and for some constant k>0, d(A', B') = kd(A, B), d(A', C')=kd(A, C) and △B'A'C'=±△BAC, then d(B', C')=kd(B,C), △C'B'A'=±△CBA, and △A'C'B'=±△ACB.

IV. Axioms and God

One advantage to encapsulating a vast system of knowledge, like a mathematical theory, in a simple form is that it can focus our most basic questions about the larger theory.
In particular, questions about formal systems of mathematics may be focused on their two central components: axioms and rules of inference.

In the *Meditations*, as a response to his First-Meditation doubts, Descartes presents the cogito as the foundation of our knowledge, as an axiom.
The Cogito is secure.
But it is not much of an axiom.
Further progress in the *Meditations* requires an argument for the existence of God to secure the criterion

of clear and distinct perception.
Moreover, in the geometric presentation, Descartes's first proposition is the existence of God.
Indeed, the Cogito is only mentioned in passing in the geometric presentation.

The *Meditations* contains two arguments for the existence of God.
The first one, the causal argument, is in the remainder of the third meditation, which I did not include in our selection.
The ontological argument is in the Fifth Meditation.
Both of these arguments appear up front in the geometric presentation.
Descartes's ontological argument, which derives from Anselm's eleventh-century ontological argument, is simple.
Anselm argued that an object which corresponds to the concept 'something greater than which can not be thought' must exist.
For, if we thought that the object which corresponded to that concept did not exist, then it would not be the object which corresponded to that concept.
There would be something greater, i.e. the object which does exist.
So, we give the name 'God' to that best possible object.
Descartes's version does not depend on our actual conception, or on our ability to conceive.
He merely notes that existence is part of the essence of 'God'.
This conceptual containment is similar to the way that having angles whose measures add up to 180 degrees is part of the essence of a 'triangle'.
Or, as Descartes notes, like the concept of a mountain necessarily entails a valley.

The essence of an object is all the properties that necessarily belong to that object.
They are the necessary and sufficient conditions for being that object, or one of that type.
Something that has all these properties is one.
Something that lacks any of these properties is not one.
A chair's essence (approximately) is to be an item of furniture for sitting, with a back, made of durable material.
The essence of being a bachelor is being an unmarried man.
A human person is essentially a body and a mind.
The essence of God is the three omnis, and existence.

Descartes's ontological argument starts by noting that the concept 'God' is that of a being with all perfections.
Since it is more perfect to exist than not to exist, the concept must include existence.
And if the concept includes existence, the object to which it corresponds must exist.
You can have the concept of a non-existing object just like God, but which does not exist.
But this would not be the concept 'God', by definition.

In the two centuries following Descartes's work, the ontological argument was examined in great detail.
Caterus, a Dutch philosopher, noted that the concept of a necessarily existing lion has existence as part of its essence, but it entails no actual lions.
We must distinguish more carefully between concepts and objects.
Even if the concept contains existence, it is still just a concept.
Kant, who coined the term 'ontological argument' claimed that existence is not a property, the way that perfections are properties.
Existence can not be part of an essence, since it is not a property.

Kant's point had been stated, though less elegantly, by Gassendi in the Fifth Objections.

Leibniz objects that Descartes's argument must first show the concept of God to be possible.

> One must realize that from this argument we can conclude only that, if God is possible, then it follows that he exists. For we cannot safely use definitions for drawing conclusions unless we know first that they are real definitions, that is, that they include no contradictions, because we can draw contradictory conclusions from notions that include contradictions, which is absurd (Leibniz, *Meditations on Knowledge, Truth, and Ideas*, 25).

Imagine the fastest motion.
It seems like 'the fastest motion' might be a consistent concept.
But, we can construct a faster motion than any real motion.
Consider a wheel spinning at the fastest motion.
Now, consider a point extended out beyond the rim of the wheel.
The extension will be moving at a faster speed than any point on the wheel.
Leibniz is worried that the concept of God might be self-contradictory, when analyzed appropriately.

Relativity theory undermines the details of Leibniz's response to Descartes, but not his more general point.
According to the theory of relativity, there is indeed a fastest motion: the speed of light.
Leibniz's thought experiment, according to special relativity, is itself problematic.
But the general point still stands.
It just turns out that the impossible notion is the impossibility of a fastest motion!

While the arguments for the existence of God are not our concern, the underlying point of those arguments is essential to understand.
Plato denied the reality of the sensible world; it was a world of mere belief, not a world of truth.
Descartes only denies that our access to truth can come from the senses.
So, his arguments for the existence of God are *a priori*, from pure thought.

Still, his arguments about mathematics depend on his axioms concerning the existence of God.
If these arguments are not self-evidently secure, then his whole system can not function the way in which it is supposed to.
Thus, the first problem for Descartes's account of mathematics is the insecurity of his axioms.

V. Clear and Distinct Ideas

In addition to axioms, any formal theory must include a procedure by which one infers theorems from axioms.
Specific formal theories are usually presumed to be embedded in more general logical theories, which include rules of inference, like modus ponens.
Descartes's method of clear and distinct ideas is then like a rule of inference used in a formal system.
The sixth and seventh postulates of the geometric presentation correspond to the method of clear and distinct ideas.

> I ask my readers to ponder on all the examples that I went through in my *Meditations*, both of clear and distinct perception, and of obscure and confused perception, and thereby accustom themselves to distinguishing what is clearly known from what is obscure. This is something that it is easier to learn by examples than by rules, and I think that in the *Meditations* I explained, or at least touched on, all the relevant examples...
>
> When they notice that they have never detected any falsity in their clear perceptions, while by contrast they have never, except by accident, found any truth in matters which they grasp only obscurely, I ask them to conclude that it is quite irrational to cast doubt on the clear and distinct perceptions of the pure intellect merely because of preconceived opinions based on the senses, or because of mere hypotheses which contain an element of the unknown. And as a result they will readily accept the following axioms as true and free of doubt (Descartes, Second Replies, ATVII.164).

A logical system, and any foundational theory, is only as secure as its rules of inference.
In the mid-nineteenth century, mathematics was shaken by some unsettling results in geometry, with odd results concerning non-Euclidean spaces; in number theory, with odd results concerning infinity; and in analysis, with persistent worries about the methods of the calculus and its reliance on infinitesimals.
Frege's work in logic was impelled by a desire to make sure that every inference in a mathematical proof was secure.
To do so, he formulated a mathematical theory of logical consequence.

> The course I took was first to seek to reduce the concept of ordering in a series to that of *logical* consequence, in order then to progress to the concept of number. So that nothing intuitive could intrude here unnoticed, everything had to depend on the chain of inference being free of gaps (Frege, *Begriffsschrift*, IV).

For modern logical theories, rules of inference are supposed to be syntactic, defined by the shape of the symbols used.
For older theories, though, rules of inference were still identified by their form.

> At the very least, the argument must reach its conclusion by virtue of its form (Leibniz, *Meditations on Knowledge, Truth, and Ideas*, 27).

Descartes's rule of inference, the method of clear and distinct ideas, has seemed to many subsequent philosophers to lack the security that Descartes imputed to it.

> [O]ften what is obscure and confused seems clear and distinct to people careless in judgment (Leibniz, *Meditations on Knowledge, Truth, and Ideas*, 26).

So, the second problem the rationalists face is to secure the method of inference
Leibniz's solution to the problem is to refine the notions of clarity and distinctness.

VI. Leibniz

Like Descartes, Leibniz thought that we could trace all complex ideas back to foundational ones.
Leibniz also reduces knowledge to simple principles.
In addition, Leibniz thought that there were foundational objects.
That is, he provides twin reductions: an epistemological reduction and its sister metaphysical reduction.

The metaphysical reduction is not really our concern, so I'll keep it brief.
Descartes thought that matter was just geometry, made concrete.
The essential property of matter is its extension.
Thus, according to Descartes, matter should be infinitely divisible.
Leibniz realized that if matter were infinitely divisible, then there could be no utterly simple objects to serve as the basic building blocks.
In other words, the infinite divisibility of matter blocks metaphysical reductionism.
Leibniz thus posited foundational objects which were, like Aristotelian substances, active.
Leibniz called the foundational substances monads.
Monads are soul-like, and they reflect the entire state of the universe at each moment.
Leibniz's metaphysical foundationalism is ancillary to the mathematical question, but it's always worth quoting Voltaire on Leibniz.

> Can you really believe that a drop of urine is an infinity of monads, and that each of these has ideas, however obscure, of the universe as a whole? (Voltaire, *Oeuvres complètes*, Vol. 22, p. 434).

Our interest is more in Leibniz's attempt to refine Descartes's rule of inference, his criterion of clear and distinct perception.
To refine Descartes's criterion, Leibniz presents a variety of distinctions among kinds of knowledge.
Obscure knowledge does not allow us even to identify a thing.
For example, I see that something is a leaf, but I don't know what kind of tree it came from.
Obscure knowledge should not even be called knowledge, in our sense of the term.
It is mere belief.

Clear knowledge gives us a "means for recognizing the thing represented" (24).
Clear knowledge may be divided into confused or distinct knowledge, which are distinguished by our ability to distinguish things from each other.

Confused knowledge is working knowledge, like that of color.
Leibniz says that we know how colors appear to us, but not so well how they work, how they are composed.
Perhaps we have a better understanding of the physical bases for colors.
But, consider chicken sexers, or musicians.
We can not communicate confused knowledge well.

Distinct knowledge is connected with marks to distinguish an object from others.
We can communicate it, and start to discuss its component parts.
Still, distinct knowledge may be adequate or not, depending on how many of the component parts we understand.

Inadequate knowledge is when we do not know, and can not communicate, all of the component notions of a thing.
The assayer may know how to distinguish gold from iron pyrite, and aluminum from molybdenum.
But, part of that distinction has to do with atomic weight.
And, the assayer may know how to test for atomic weight, but not know what it is.

If I have adequate knowledge of p, then I have adequate knowledge of all components of p, all components of components of p, etc.
This seems like a tall order, and Leibniz admits that we may not have any adequate knowledge.
Still, this is the presumed domain of mathematical knowledge.

> I don't know whether humans can provide a perfect example of [adequate knowledge], although the knowledge of numbers certainly approaches it (Leibniz, *Meditations on Knowledge, Truth, and Ideas* 24).

Leibniz, like Descartes, is thinking of the mathematical method, the axiomatic method.
In mathematics, we can trace any claim, via its proof, back to the axioms.
But, even adequate knowledge is not the ultimate foundation, since we have to justify knowledge of the axioms.
The mathematician uses definitions to make his work perspicuous.
When we give a proof in, say, linear algebra, we do not present it in its set-theoretic form.
Our finite minds have limited abilities to comprehend all the steps in a long and complex proof, or proposition.

Symbolic knowledge, then, is adequate knowledge which appeals to signs (definitions) to represent our knowledge of components.
The use of definitions prevents our knowledge from being fully intuitive.

Intuitive knowledge is of distinct primitive notions.
An infinite mind would be able to have intuitive knowledge of all propositions.

For Leibniz, the foundational truths are identities, laws of logic.
These would be known intuitively, or directly.
We can consider all the component notions of the most perfect knowledge at the same time.
Note that the most perfect knowledge, intuitive and adequate knowledge, would be *a priori*, traced back to the component parts of its real definition (not just its nominal one, p 26).