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Aristotle

I. A Footnote to Plato

Plato's central arguments for the existence of forms concerned their uses in explanations of commonalities and causes.
In Aristotle's language, Plato separated the forms, taking them to be separate from the sensible world.
Aristotle argued that their isolation entails that the forms can not explain sensible phenomena.
He also worried about Plato's multiplication of entities.
We saw Aristotle's worry about a preponderance of forms in the third man argument.
The same worry reappears in his criticism of Plato's view of mathematics.

> If besides the sensible solids there are to be other solids which are separate from them and prior to the sensible solids, it is plain that besides the planes [solids?] also there must be other and separate planes and points and lines; for consistency requires this. But if these exist, again besides the planes and lines and points of the mathematical solid there must be others which are separate...Again, there will be, belonging to these planes, lines, and prior to them there will have to be, by the same argument, other lines and points; and prior to these points in the prior lines there will have to be other points, though there ill be no others prior to these. Now the accumulation becomes absurd... (*Metaphysics* XIII.2: 1076b12-29).

Plato has reified mathematical objects, taking them as objects distinct (separate) from the sensible world.
Aristotle, in response, claims that we need not reify mathematics.
We can understand numbers, and shapes, as properties of objects.
Aristotle thus reinterprets mathematical terms as referring to physical objects qua their shape or number.
Mathematics is about magnitudes, which are properties of sensible objects.
In short, Aristotle presents an adjectival use of mathematical objects, indeed all properties.
Roundness (circularity) and twoness (from counting) are properties of primary substances, not substances themselves.

Aristotle's work in the philosophy of mathematics is turgid and obscure, like most of Aristotle's writings.
A lot of Aristotle's work remains extant, copied repeatedly through the middle ages because of the high esteem in which the Catholic Church held The Philosopher.
But, we have also lost a lot of his presumably more polished works.
What are left are mainly sets of difficult lecture notes.
The standard reading of Aristotle on the philosophy of mathematics, which Lear discusses in the beginning of his article, is that he presents a shallow and incoherent view: denying the existence of mathematical objects and the world of the forms while admitting that we have mathematical knowledge.

> This interpretation must view Aristotle as caught in the middle of a conjuring trick: trying to offer an apparently Platonic account of mathematical knowledge while refusing to allow the objects that the knowledge is knowledge of (Lear 161).

I follow Lear in finding a more subtle view in Aristotle's work.
In the end, Aristotle's view is incoherent, but not shallow.

II. Neither In the Objects nor Separate From Them

Books XIII and XIV of *Metaphysics* contain Aristotle's philosophy of mathematics, though the most important section is in Chapter 3 of Book XIII.
Much of the rest of the two books contains responses to other views.
There is also a helpful passage on mathematics in *Physics* II.2.
Our interest in Aristotle's criticisms of his contemporaries is limited, but it will help to look at Book XIII, Chapter 2, to set up Aristotle's positive account in Chapter 3.
In Chapter 2, Aristotle argues that mathematical objects are neither in sensible objects nor separate from them.
In other words, mathematical objects are not themselves perceivable, nor do they exist in a separate Platonic realm.
Aristotle also assumes the existence of mathematical objects, so we can see from the start that his account will have to be subtle: mathematical objects exist, but they are not part of the sensible world nor are they separate from it.

The argument from division at the beginning of Chapter 2 (1076a38-1076b12) is intended to show that the mathematical forms do not exist in sensible things.
Aristotle's claim, here, is fairly straightforward, even if he makes it obscurely.
Points, abstract mathematical objects, are indivisible.
If sensible bodies were made of points, then they could not be divided.
But, bodies are divisible.
So, they can not be made of mathematical points.
The argument from division merely establishes that mathematical objects can not be identical with any extension in matter: a ball is not a sphere, a piece of paper is not a plane.
We have to find a more subtle way of describing how mathematical objects exist in matter.

On the other hand, Aristotle proceeds to argue that mathematical objects are not separate from sensible objects.
Here, he attacks Plato, and those who think that mathematical objects exist in a separate realm.
Most of his argument relies on the problem of multiplying entities, and that the accumulation of entities becomes absurd.
Since we have discussed Aristotle's worries about the forms, previously, I will not repeat those arguments.
In response to Aristotle, one might wonder why the multiplication of entities in a Platonic realm is troublesome.
But, we can accept that the view we would like to avoid positing a separate realm of forms and mathematical objects if possible.
We can proceed to Aristotle's positive account merely on the hope that he can provide a way of understanding mathematics to be true without taking mathematical objects to be identical either to physical objects or to separate objects in a separate realm.

III. Aristotle, Mathematics, and Magnitudes

Among anti-platonists, those who deny the separate existence of mathematical objects, there are revolutionaries and reinterpreters.
Revolutionaries think that mathematical statements are false and that mathematical objects do not exist.
Reinterpreters think that mathematical statements are true when reconstrued, and that while platonist mathematical objects do not exist, we can understand mathematical terms as shorthand for other kinds of objects.[1]
Aristotle is an anti-platonist, but he is a reinterpreter, not a revolutionary.

Since most of Aristotle's positive account of mathematics comes in Chapter 3 of Book M, we should take a moment with the first sentence of that chapter.
Here is Julia Annas's excellent translation

> Just as general propositions in mathematics are not about separate objects over and above magnitudes and numbers, but are about these, only not *as* having magnitude or being divisible, clearly it is also possible for there to be statements and proofs about perceptible magnitudes, but not *as* perceptible but as being of a certain kind (1077b18-22).

This sentences raises the questions of what a general proposition in mathematics is, and what a magnitude is.
Magnitudes are not just geometric lengths, as it might seem.
They are more general; anything that can be the subject of Eudoxus's theory of proportions:

$$(x)(y)(z)(w)\ \{(x : y :: z : w) \equiv (v)(u)[(vx > uy \supset vz > uw) \bullet (vx < uy \supset vz < uw) \bullet (vx = uy \supset vz = uw)]\}$$

Since :: is an equivalence relation, we often replace it with '='.
From the theory of proportions, a more familiar relation holds:

$$a{:}b{::}c{:}d \equiv ad{=}bc$$

Euclid presents the theory of proportions in Book V of the *Elements*, and it is generally thought to be one of Euclid's central achievements.
Even prior to Euclid, Eudoxus, who like Aristotle was a student of Plato, developed an axiomatic treatment.
Eudoxus developed the theory of proportions to handle the incommensurables we discussed when we talked about Pythagoreans.
The ratio, for example, of the length of a side of an isosceles right triangle to its diagonal, is incommensurable.
The Greeks could not imagine that such a division creates a number; irrational numbers appeared to be unacceptable, as complex numbers (which were originally called impossible) were avoided centuries later.
Given Eudoxus's work, the Greeks could work with incommensurable ratios without committing themselves to irrational numbers.
The quantifiers range over magnitudes, which could be lengths, weights, volumes, areas, or times.

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[1] See Burgess and Rosen's uses of 'revolutionary' and 'hermeneutic' nominalism in *A Subject with No Object*.

They do not range over numbers, for the Greeks, though we can see that they do.
Lear argues that the key to understanding the notion of a general proposition in mathematics is to understand Eudoxus's theory of proportions.

> The generalized theory of proportion need not commit us to the existence of any special objects - magnitudes - over and above numbers and spatial magnitudes (Lear, 167).

Thus, Aristotle's claim, in the first sentence of Chapter 3, is that we do not think there are magnitudes in addition to lengths, weights, times, and any other kind of measure to which the theory of proportions applies.
We should not reify magnitudes.
Mathematical objects are not substances.

Aristotle's discussion of health has the same point.

> And it is true to say of the other sciences too, without qualification, that they deal with such and such a subject - not with what is accidental to it (e.g. not with the white, if the white thing is healthy, and the science has the healthy as its subject), but with that which is the subject of each science - with the healthy if it treats things *qua* healthy, with man if *qua* man (*Metaphysics* XIII.3: 1077b34-1078a2).

We do not reify health, in addition to the healthy or unhealthy person.

So, there are magnitudes; they are the subject of the theory of proportions.
In contrast, there are no things that we call magnitudes.
There are just lengths, and weights, and times, and volumes of solids.
We need not think that there is a shape of the book over and above the book itself.
There is just the book itself, considered more abstractly.
There are mathematical objects, in the sense that the book has a shape.
But, there are no mathematical objects separate from the sensible objects which have shapes, and other magnitudes.

Aristotle is presenting an adjectival view of properties, rather than a substantival view.
Mathematical objects are predicated of actual objects, but they are not themselves objects.
To help us understand Aristotle's account of mathematics, it might be useful to consider his account of the soul.

IV. Matter, Form, and the Soul

For Aristotle, every living thing, indeed every thing that we can name, has matter and form.
The matter is, roughly, the stuff out of which it is made.
The form is, roughly, the shape or function of the object.

Consider, the difference between a lump of clay, and a similar lump made into a statue.
The two lumps are made of the same kind of stuff, but have a different shape.
Plato would say that the lump that looks like a statue participates in the abstract form of the statue.
Aristotle calls the shape of the sensible statue itself its form.

Matter itself is mere potentiality; it is nothing in itself unless it has some form.
The form is what makes it what it is.
We call an object by a particular name according to its form.
Forms, for Aristotle, are thus just one aspect of a substance.
In the case of the statue, the form is related to its shape, though the form of something need not be merely its shape.

Consider an eye.
It has matter, which it can share with a dead eye.
It has some properties in common with an eye of a statue, like its shape.
But, the real eye is able to see.
The function of seeing is what makes an eye a real eye.
So, the form of the eye is related to its function.
Similarly the form of my hand, which has particular functions, is not merely its shape.
All the parts of me: my heart, my lungs, my toes, have functions, and so both matter and form.

When we put all of these pieces together, we get a person.
We are all made out of the same kind of matter.
But, we have different properties.
The properties which make me what I am are my form.
Aristotle calls the form of a person his or her soul.
Since the form of something is what makes it what it is, the soul includes our biological aspects, like sensation and locomotion, as well as reason.
The soul is thus not separable from the body, though it is different from just the matter of the body.
Aristotle is thus a monist: there is only one realm.

Aristotle's account of the soul as the form of the human body makes the soul of a person seem a lot like the soul of an animal or plant.
For, plants and animals also have a matter and a form.
Each of these, thus, has a soul.
Plants have nutritive souls.
Animals also have sensitive souls.
While Plato identified several parts of the human soul, Aristotle mentions six faculties, though these are not to be taken as parts: nutrition and reproduction, sensation, desire (which cuts across all three parts of Plato's soul), locomotion, imagination (which we share with some animals), and reason.
Only humans have rational souls.
Thus, Aristotle defines human beings according the functions of their souls: rational animal.

V. Mathematics, Abstraction, and the *Qua* Operator

Just as Aristotle believes that there is a soul, but that it is just the form of the body, he believes that there are mathematical objects, as aspects of physical objects, as physical objects taken in a particular way.
In each case, Aristotle denies that there is a separate realm.
He is, essentially, a natural scientist about both questions.
The soul is an aspect of a person, apart from his or her matter, but tied to his or her functions.
Mathematical objects are just aspects of physical objects.

To see sensible objects in the mathematical way, we can abstract from their sensible properties.
The word 'abstract' may be used in at least two ways.
In one way, we refer to objects outside of space-time, or outside the sensible realm, as abstract objects.
That is a metaphysical interpretation of 'abstract'.
For Aristotle, abstraction is an epistemological notion.
Abstraction is a process we use on ordinary objects, to consider them as mathematical.
Elsewhere, Aristotle uses the term explicitly.

> Of what does a demonstration hold universally? Clearly whenever after abstraction it belongs primitively - e.g. 'two right angles' will belong to 'bronze isosceles triangle', but also when being bronze and being isosceles have been abstracted (*Posterior Analytics* I.5: 74b1).

To explain the process of abstraction, Lear introduces a qua operator.
Aristotle's qua operator may be formalized, a bit more simply than Lear presents it, as:

LQ $\quad G(b \text{ qua } F) \equiv F(b) \bullet (x)(Fx \supset Gx)$

So,

> Haman's hat (taken as a triangle) has angles that sum to 180 degrees if and only if Haman's hat is a triangle and all triangles have angles which sum to 180 degrees.

The qua operator formalizes the process of abstraction that underlies Aristotle's positive account.
Lear says it acts as a filter to get rid of incidental, or accidental, properties.

> To use the *qua*-operator is to place ourselves behind a veil of ignorance: we allow ourselves to know only that *b* is *F* and then determine on the basis of that knowledge alone what other properties must hold of it (Lear 168).

Note that it is not that case that G(b qua hat) (i.e. Haman's hat, taken as a hat, does not have angles that add up to 180 degrees).
For, it is not the case that all hats have angles which sum to 180 degrees.
So, the sensible object is not to be taken as having mathematical properties absolutely; the mathematical objects are not in the sensible objects.

Aristotle's approach solves the problem of applicability.
The problem of applicability is to explain how objects in a separate realm can have any relevance to the sensible realm.
Plato's theory of forms and mathematical objects incurs a problem of applicability, as do most contemporary platonist accounts.
It is difficult to see how a separate form can have any effect on or interaction with the sensible world.
This problem led Plato to denigrate the sensible world, as a world of mere becoming, and not actual being.
Aristotle, the natural scientist, takes a different moral from the argument.
(One person's modus ponens is another person's modus tollens.)
For Aristotle, we can study a perceptible triangle, like Haman's hat, qua triangle because it actually is a triangle.
It does not merely approach triangularity.

Aristotle has given us a subtle doctrine.

> If [geometry's] subjects happen to be sensible, though it does not treat them *qua* sensible, the mathematical sciences will not for that reason be sciences of sensibles - nor, on the other hand, of other things separate from sensibles (*Metaphysics* XIII.3: 1078a2-4).

Mathematical platonists do not speak falsely.

> It is true...to say, without qualification, that the objects of mathematics exist, and with the character ascribed to them by mathematicians...If we suppose things separated from their attributes and make any inquiry concerning them as such, we shall not for this reason be in error, any more than when one draws a line on the ground and calls it a foot long when it is not; for the error is not included in the propositions (*Metaphysics* XIII.3: 1077b31 - 1078a17).

There are no separable mathematical objects; the platonist is contriving a fiction.
But platonists do not get mathematics wrong, so the fiction is harmless.
Most importantly, supposing that there are mathematical objects does not lead to errors in physics.

> The best way of studying geometry is to separate the geometrical properties of objects and to posit objects that satisfy these properties alone...Though this is a fiction, it is a helpful fiction rather than a harmful one: for, at bottom, geometers are talking about existing things and properties they really have... (Lear 175).

Lear compares Aristotle's account with a recent contemporary view of mathematics, Hartry Field's fictionalism.
We will look closely at Field's fictionalism later in the term.
Briefly, here, Lear's claim is that both Aristotle and Field take mathematics to be conservative over physical theory.
That just means that strictly speaking, there are no mathematical objects.
But, if we add mathematical axioms to those of our best physical theory, we will not derive any further physical consequences.
Mathematics is a useful tool for reasoning about the physical world.
But, it is no more than a tool.

VI. Problems with Aristotle's Account

I'll mention a few problems with Aristotelian accounts of mathematics.

Any platonist account of mathematics leads to difficulties about access and application: how can we know of this separate world, and why does it have application here?
Aristotle captures our intuitions that a separate realm can have no causal effect on the sensible world.
But we have opposing intuitions as well.
Mathematical statements seem to transcend the physical world.
According to Aristotle, if matter were to disappear, there would be no more mathematics.
The problem here is that we have opposing, inconsistent intuitions that must be resolved.
Aristotle's theory captures only one set of the opposing intuitions.

A related problem is that physical objects do not actually have the mathematical properties they seem to approach.
To be specific, recall LQ.

LQ $G(b \text{ qua } F) \equiv F(b) \bullet (x)(Fx \supset Gx)$

Notice that the first clause in the right-hand side of LQ is always false in the case of perfect geometric objects.
No sensible object, strictly speaking, has perfect geometric shape.
No pizza is a perfect circle, no hat is a perfect triangle.
The problem of applicability was supposed to be solved by taking mathematical objects to be properties of physical objects.
But, the shapes that mathematicians actually study seem not to be the properties that mathematical objects actually have.

Even if we grant there are some physical circles, there are not going to be perfect examples of all the forms that the mathematician explores.
The world is finite, and the mathematician studies infinitely many objects.

> One unfortunate (if not damning) consequence of this account is that a natural number does not exist unless there is a collection of physical objects of that size. Similarly, a geometric object, such as a given polygon, exists only if there is a physical object that has that shape (Shapiro 66).

The restrictions that Aristotle places on the existence of mathematical objects are counter-intuitive.
Lear claims that Aristotle can appeal to a human ability to abstract, to construct a figure in thought.
That claim makes Aristotle too idealistic; mathematical objects are not mental objects.
Aristotle tries to avoid the problem of restriction by taking mathematical objects not to be the limits of physical bodies.

> The mathematician, though he too treats of these [sensible] things, nevertheless does not treat of them as the limits of a natural body; nor does he consider the attributes indicated as the attributes of such bodies (*Physics* II.2: 193b32-4).

The question that remains is whether Aristotle has given a coherent account.
On the one hand, there are no separable objects.
On the other, we take mathematical objects not to be the limits of sensible bodies, but something transcendent.
We will not resolve this tension, here.

In addition to the problem of having to have a physical object of a certain shape in order for there to be a corresponding geometric object, Aristotle limits the world to a potential infinity of objects.
Part of his worry about infinity arises from paradoxes, specifically including Zeno's paradoxes.
Consider Achilles, having to complete an infinite series of motions before he catches the Tortoise.
See David Bostock's discussion, in "Aristotle, Zeno, and the Potential Infinite".

We will return to the problem of restriction when we get to the indispensability argument.
Aristotle's work in mathematics sets a precedent for future empiricists in the philosophy of mathematics, including Mill, in the 19th century, and Quine, in the 20th.