Knowledge, Truth, and Mathematics

Philosophy 405
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Class 18: Mathematical Truth

Benacerraf's Two Conditions
On Philosophical Accounts of Mathematics

- Semantics
  - ‣ Mathematical semantics should be consistent with our broader views on semantics.
  - ‣ Semantics: theories of meaning, truth and reference
  - ‣ Includes specifications of truth conditions for the sentences of a language.
  - ‣ Tarski's theory of truth

- Epistemology
  - ‣ Mathematical epistemology should be consistent with our broader views about epistemology.
  - ‣ Causal theory of knowledge
  - ‣ But, the causal theory is not essential to the Benacerraf problem.

2500 Years of Epistemology in Ten Minutes

JTB

- X knows that p iff X believes that p and X is justified in believing that p and p is true

- *Theaetetus* 200-201

- In 1962, Edmund Gettier published counter-examples to the JTB definition.

Gettier Counter-Example #1

- Smith believes that the man who will get a job has ten coins in his pocket because he believes (on good evidence) that Jones will get the job and he also believes that Jones has ten coins in his pocket (because he counted them).

- Smith does not know how many coins he has in his own pocket.

- It turns out, though, that Smith himself gets the job, and that Smith has ten coins in his pocket.

- Smith's belief is true, and justified.

- But it is not knowledge.

Gettier Counter-Example #2

- Smith believes that either Jones owns a Ford or Brown is in Barcelona, since he believes that Jones owns a Ford and he knows the logical rule of addition.

- He does not know where Brown is, but he has good evidence that Jones owns a Ford.

- It turns out, though, that Jones is just leasing his Ford, and Brown is, in fact, in Barcelona.

- Again, Smith has a justified true belief, but not knowledge.

Wackiness Ensues

- The Gettier cases show that people can have a JTB without knowledge.
- The JTB definition is insufficient for knowledge.
- Suddenly, epistemologists did not know what they were chasing.
- Additional conditions were proposed to fix the JTB account.

CTK

- CTK adds a fourth condition on JTB: the justification has to include appropriate causal connections between the knower and the proposition known.

- CTK and Counter-Example #1
  - ‣ Smith does not have an appropriate causal connection to the object of his knowledge, which is the coins in the pocket of Smith himself, rather than those in Jones's pocket.
  - ‣ Smith does not know that the man who will get the job has ten coins in his pocket.

- By the mid-1970s, it became clear that CTK was itself flawed.
  - ‣ Causation is tricky.
  - ‣ Tachyons?
 - ‣ Backwards causation

Fake Barn Country

from Alvin Goldman

- In fake barn country, there are thousands of barn facades.
  - The facades look like real barns, but they are not real.
- There are also one or two real barns, among the thousands of fake ones.
- If you are, unknowingly, driving through fake barn country, and happen to see one of the rare real barns, you might believe that you have seen a barn.
- You would have a JTB that you have seen a barn.
- You would be appropriately causally connected to a barn.
- You do not know that you have seen a barn.

Theories of Truth: Inflationary and Deflationary (Or, 2500 Years of Truth in Fifteen Minutes)

Semantic Theories
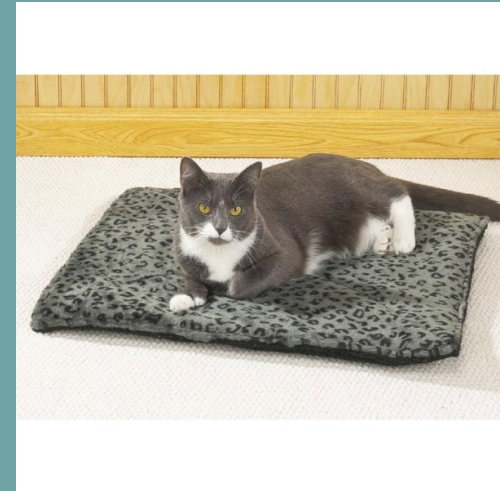
- Three parts
  - a theory of truth
  - a theory of reference
  - a theory of meaning

- A term, like 'cat', has some meaning, and refers to certain objects.

- A sentence, like 'the cat is on the mat', has some meaning, and some truth conditions.

- We are mainly concerned here with the theories of truth and reference.
  - 'The cat is on the mat' seems to require, for its truth, that the term 'cat' refer to some specific cat, and the predicate 'is on the mat' refer to some sort of property or relation of being on the mat.

Correspondence Truth

To say of what is that it is not, or of what is not that it is, is false, while to say of what is that it is, and of what is not that it is not, is true (Aristotle, *Metaphysics* 1011b25).

- A correspondence theory says that truth is a correspondence between words and worlds.

- A proposition is true if the world is the way that the proposition claims that it is.

- One worry
  - ▸ We have no extra-linguistic way to apprehend reality.
  - ▸ We have no access to the world as it is in itself.

Other Options

- Coherence theory: the truth of a sentence consists in its consistency with other beliefs we hold.
  - ‣ But different people hold different beliefs.
  - ‣ Coherence theories collapse into relativism.
- Deflationary (or minimalist or redundancy) theories: there is no essence to truth.
  - ‣ There is no single reduction of truth to a specific property, like correspondence or consistency.
  - ‣ Truth is a device for simplifying long conjunctions.
- Coherence Theorists and Correspondence Theorists are Inflationists.

The T-schema

- A minimal condition for truth is the T-schema.
  - ‣ T-schema: p is true iff x

- Instances of the T-schema:
  - ‣ 'The cat is on the mat' is true iff the cat is on the mat.
  - ‣ '2+2=4' is true iff 2+2=4
  - ‣ 'Barack Obama is president' is true iff the husband of Michelle Obama and father of Sasha Obama and Malia Obama is head of the executive branch of the United States of America.
  - ‣ 'El gato está en el alfombrilla' is true iff the cat is on the mat.

- Inflationists and deflationists disagree about whether the T-schema is all there is to know about truth.
  - ‣ The inflationist believes that there are explanations of the concept of truth inherent in the truth conditions on the right side of the T-schema.
  - ‣ The deflationist believes that the T-schema is all there is to know about truth, and that there is no single kind of explanation of why all sentences are true.

The Central Problem with Correspondence Truth

The Liar

- L: L is false

- Our natural language contains the words 'true' and 'false', as predicates.

- If we include those predicates in our formal language, we can construct the liar sentence.

- If we can construct the liar sentence, we can formulate an explicit contradiction.

- Contradictions explode; everything would be derivable.

- But, we know that not every sentence is true.

- So, we can not include a truth and falsity predicates in our formal language.

Russell on Self-Reference

- The liar sentence, like Russell's paradox, is self-referential.

- Russell developed the theory of types in such a way as to prevent impredicative definitions, definitions which refer to themselves.

- He relied on a vicious circle principle to eliminate such definitions.
  - "Whatever involves *all* of a collection must not be one of that collection"; or, conversely: "If, provided a certain collection had a total, it would have members only definable in terms of that total, then the said collection has no total" (Whitehead and Russell, *Principia Mathematica*, Chapter II, p 37).

Tarski to the Rescue

- Tarski's theory of truth similarly proscribes self-reference.
- Segregates object language from metalanguage
- Banishes semantic terms from the object language
  - ‣ never allowing 'true' to apply to sentences which contain semantic terms, like 'false'
- Allows semantic terms in the meta-language
  - ‣ they apply only to sentences of the object language
- We can construct theories of truth for the object language in the meta-language.
  - ‣ Instances of the T-schema are sentences of the meta-language which we can use to characterize truth for the object language.

Tarski and the Correspondence Theory

- It is not enough just to list the true and false sentences of a language.
- We need to supplement Tarski's T-schema with an account of why we choose certain sentences to be true.
  - For 'the cat on the mat' to be true, 'the cat' must refer to a specific cat, 'the mat' must refer to a specific mat, and 'is on' must refer to the relation of being on, and the cat must be on the mat.
  - In order to know that 'the cat on the mat' is true, we have to know all of that.
  - We can not merely present the appropriate instance of the T-schema.
- Tarski's construction thus reduces 'truth' to other semantic notions, like reference.
- Benacerraf believes that the use of Tarski's theory in a broader semantic account is the only viable theory of truth.
  - "I take it that we have only one such account: Tarski's, and that its essential feature is to define truth in terms of reference (or satisfaction) on the basis of a particular kind of syntactico-semantic analysis of the language, and thus that any putative analysis of mathematical truth must be an analysis of a concept which is a truth concept at least in Tarski's sense" (Benacerraf 667).

Benacerraf's Problem

- It seems impossible to match our epistemic capacities with standard semantics.

- The platonist mathematics that underlines standard interpretations of mathematical language seems incompatible with our epistemic capacities.

Standard Semantics

- B1. There are at least three large cities older than New York.

- B2. There are at least three perfect numbers greater than 17.

- B1'. $(\exists x)(\exists y)(\exists z)(Lx \bullet Ly \bullet Lz \bullet Oxn \bullet Oyn \bullet Ozn)$

- B2'. $(\exists x)(\exists y)(\exists z)(Px \bullet Py \bullet Pz \bullet Gxn \bullet Gyn \bullet Gzn)$

- B3. There are at least three FGs that bear R to a.

- The truth of an existential sentence depends on whether there are objects in the domain of quantification that can substitute for the variables in the sentence so that the properties ascribed to those objects hold.
  - ‣ To be is to be the value of a variable.

Uniform Analysis

- Two sentences with the same grammatical structure are to be analyzed in the same way.

- Structure should not depend on content.
  - "One consequence of...the standard view is that logical relations are subject to uniform treatment: they are invariant with subject matter. Indeed, they help define the concept of "subject matter." The same rules of inference may be used and their use accounted for by the same theory which provides us with our ordinary account of inference, thus avoiding a double standard" (Benacerraf 670).

- We need cities to satisfy B1.

- We require mathematical objects to satisfy B2.

- The standard view is thus platonist.
  - 'There is a number between 4 and 6.'
  - 'There is a chair between the desk and the door.'

Epistemic Access

- We need some account of how we can know about the things we think exist.

- Such an account seems to be absent in the case of platonistic entities.

- Benacerraf claims that the best theories of knowledge and reference are causal.
  - My connection to any object to which a term I know refers must be based in some causal link between me and the object.

- We lack causal connection to the mathematical objects required to interpret sentences like B2.

- "If, for example, numbers are the kinds of entities they are normally taken to be, then the connection between the truth conditions for the statements of number theory and any relevant events connected with the people who are supposed to have mathematical knowledge cannot be made out. It will be impossible to account for how anyone knows any properly number-theoretical propositions" (Benacerraf 673).

Field's Reformulation

The way to understand Benacerraf's challenge, I think, is not as a challenge to our ability to *justify* our mathematical beliefs, but as a challenge to our ability to *explain the reliability* of these beliefs... Benacerraf's challenge...is to provide an account of the mechanisms that explain how our beliefs about these remote entities can so well reflect the facts about them. The idea is that *if it appears in principle impossible to explain this*, then that tends to *undermine* the belief in mathematical entities, *despite* whatever reason we might have for believing in them (Field 25-6).

The Gödel Solution

- Mathematical intuition

- Benacerraf finds Gödel's view troubling.
  - "What troubles me is that without an account of *how* the axioms "force themselves upon us as being true," the analogy with sense perception and physical science is without much content. For what is missing is *precisely* what my second principle demands: an account of the link between our cognitive faculties and the objects known" (Benacerraf 674).

- Field dismisses Gödel's view.
  - "Someone *could* try to explain the reliability of these initially plausible mathematical judgments by saying that we have a special faculty of mathematical intuition that allows us direct access to the mathematical realm. I take it though that this is a desperate move..." (Field 28)
  - Remote Nepalese villages
  - The Gödel-platonist owes us an explanation of the reliability of mathematical beliefs.

The Combinatorial Solution

- Combintorialists construe mathematical truth as depending on the manipulation (or combination) of objects other than traditional mathematical objects.
    - Hilbert's program: inscriptions
    - formalism: inscriptions
    - intuitionism: mental constructs
    - conventionalism: nothing at all
- Give up standard semantics
    - '5' does not refer to a five.

Benacerraf Against the Combinatorialists

- A theory of truth demands a theory of reference.
  - ▸ We have not only to partition the set of statements of a theory into two classes.
  - ▸ We also have to know what those two classes are, and why we put some terms in one class and some in the other.
- If we appeal to the provability of mathematical theorems, and to the manipulation of non-mathematical objects, we still don't know why the axioms are true.
- For a semantics which tells us why proofs yield truths, we must have a standard interpretation of the terms.
  - ▸ "Although it may be a truth condition of certain number-theoretic propositions that they be derivable from certain axioms according to certain rules, *that* this is a truth condition must also follow from the account of *truth* if the condition referred to is to help connect truth and knowledge, if it is by their proofs that we know mathematical truths" (Benacerraf 673).
- The combinatorialists provide access to the objects of mathematics only by changing the subject.
  - ▸ They save their epistemology by abandoning their theory of truth.
  - ▸ "Motivated by epistemological considerations, they come up with truth conditions whose satisfaction or nonsatisfaction mere mortals can ascertain; but the price they pay is their inability to connect these so-called "truth conditions" with the truth of the propositions for which they are conditions" (Benacerraf 678)

Ways Out

- Develop a Gödel-style epistemology?

- Pursue the conventionalist account for mathematics, though not for logic?

- Fictionalism?

- Quine's indispensability argument