Hamilton College
Russell Marcus

Class #17: October 25
Conventionalism

I. A Fourth School

We have discussed the three main positions in the philosophy of mathematics developed during the intense period from 1879-1931: logicism, formalism, and intuitionism.

According to logicism, mathematics is logic in complex disguise.
Mathematical theorems are analytic, since they follow from logical truths by logical transformations.
For the logicists, our knowledge of all of classical mathematics, including transfinite set theory, is justified by its logical status.
Mathematical objects are just logical constructs; mathematical theorems are just logical theorems.
But Frege's project was beset by Russell's paradox.
Russell's fix, the theory of types, showed that mathematics was reducible, in some sense, to set theory.
But, set theory is not logic.
The question whether mathematics is analytic or synthetic, which is central to our understanding of how our knowledge of mathematics is justified, was not solved in the way that Frege promised.

Given the paradoxes and other counter-intuitive results of the developing set theory, the formalists retreated from interpretation, focusing on the formal systems themselves.
To the strictest formalists, mathematics is an empty game, with no content.
The choices of axioms are arbitrary.
Less strictly, Hilbert maintained intuitive interpretations for the real elements of mathematics, the primitive recursive functions.
For the ideal elements, especially the infinitary ones, the practice of mathematics was supposed to be justified by completeness and consistency proofs for the axiomatic theories.
But, as Gödel showed, completeness and consistency proofs of the type that Hilbert sought were impossible to construct, in principle.
In addition, Hilbert's nominalism for ideal statements, his dis-interpretation, leaves open the question of how to explain the utility of mathematics in science.
If mathematics were merely an empty game, its applicability in our best scientific theories would be miraculous.

Hilbert denied the logicists' claim that arithmetic is analytic, appealing to intuition in his argument for the syntheticity of the real elements of mathematics.
The intuitionists returned more fully to the Kantian view that mathematics is synthetic, constructed by human beings.
Unfortunately, taking mathematics to be synthetic almost necessitates a finitism.
The intuitionists are unable to justify knowledge of many classical results, including the general theory of sets.
They rejected non-constructive proofs, and the law of the excluded middle, which were (and are) nearly universally approved and widely used by mathematicians.
Intuitionistic logic is counter-intuitive.
For the intuitionist, '$A \supset B$' means that there is a constructive procedure leading from a proof of A to a proof of B.
In mathematics, we can, perhaps, adopt this logic.

In ordinary language, and in science, the '⊃' doesn't mean that we have constructed proofs from one to the other.
Here is another criticism of intuitionism, from Hilary Putnam.
The intuitionist relies on the consistency of the mind, and its constructions, to secure the health of its mathematics.
But, our minds themselves might be inconsistent.

See the chart at the end of these notes for a summary of the three positions we have studied so far.

There was a fourth position in the philosophy of mathematics in the early twentieth century, not normally grouped among the three others.
This fourth position, or family of positions, had a wide-ranging and profound influence.
We could use the labels conventionalism or deflationism to characterize this family of views.
They were held by the logical positivists, or logical empiricists, working in the wake of Wittgenstein's early work, the *Tractatus Logico-Philosophicus*.

II. Positivism

Logical positivism is sometimes characterized as British empiricism plus logic.
While this characterization is an oversimplification, the core idea is correct.
Like Locke and Hume, the positivists attempted to establish a foundation for knowledge, a systematic justification for our scientific beliefs, that relied on sense experience.
Hume and Locke were content mainly to imagine how all our knowledge could be grounded in sense experience.
The positivists, in contrast, tried actually to trace construction of science from sense data.
Instead of seeing empiricist principles as merely negative, limiting the scope of knowledge, they attempted the positive reconstruction of our knowledge.

The positivists were encouraged to take on their project by the nineteenth- and early-twentieth-century developments in logic, from Frege, Russell, and Wittgenstein.
The positivist's project was developed in and around Vienna between WWI and WWII by philosophers influenced by the *Tractatus*, such as Rudolph Carnap, Otto Neurath, Moritz Schlick, and Herbert Feigl.
Their group came to be known as the Vienna Circle.
There was a similar, though less-influential, Berlin Circle, centered around the physicist Hans Reichenbach.
The young A.J. Ayer visited Vienna from England and wrote about the movement he found there.
His *Language, Truth, and Logic*, from which our selection is taken, became the primary source for positivism for English-speaking philosophers, though most of the positivist's central works eventually were translated into English.
The culmination of the positivist's project was Carnap's 1928 *The Logical Structure of the World* or *Aufbau*, which was not available in English until 1967.
Carnap had been a student of Frege's in Jena, Germany.
In the *Aufbau*, Carnap attempts to develop scientific theory, using the tools of logic, out of sense-data, or sense experiences.

Wittgenstein and the logical positivists were responding in large part to Hegelian idealism, and speculative metaphysics generally, which had taken root in Europe after Kant.

Like Hume, they were intent on ridding philosophy of what they deemed to be pseudo-problems, pseudo-questions, meaningless language, and controversial epistemology.

Focused on science, the positivists derided such concerns as:

> A. The meaning of life
> B. The existence (or non-existence) of God
> C. Whether the world was created, with all its historical remnants and memories, say, five minutes ago
> D. Why there is something rather than nothing
> E. Emergent evolutionary theory, and the elan vital
> F. Freudian psychology
> G. Marxist theories of history

The positivists presented a verificationist theory of meaning, inspired directly by Hume and Locke.
Hume believed that for a term to be meaningful, it had to stand for an idea in one's mind that could be traced back (in some sense) to an initial sense impression.
The verification theory says that for a sentence to be meaningful, it must be verifiable on the basis of observation.
Any sentence which is unverifiable, like any of the examples A-G above, is meaningless.

In particular, questions about the reality of the external world were deemed pseudo-questions.

> The Circle rejected both the thesis of the reality of the external world and the thesis of its irreality as pseudo-statements; the same was the case for both the thesis of the reality of universals (abstract entities, in our present terminology) and the nominalistic thesis that they are not real and that their alleged names are not names of anything... (Carnap 215).

The positivists welcomed scientifically legitimate (i.e. verifiable) reformulations of some traditional philosophical problems, even some of which seemed like metaphysical nonsense.
For example, Newton and Leibniz had debated the question whether space is relational or absolute; see the Leibniz-Clarke correspondence.
The absolute/relational debate persisted through Kant's defense of the absoluteness of space, and it appeared essentially metaphysical.
An early influence on positivism, the scientist and philosopher Ernst Mach, had argued against absolute space on positivist principles.

> No one is competent to predicate things about absolute space and absolute motion; they are pure things of thought, pure mental constructs, that cannot be produced in experience. All our principles of mechanics are...experimental knowledge concerning the relative positions and motions of bodies... No one is warranted in extending these principles beyond the boundaries of experience. In fact, such an extension is meaningless, as no one possesses the requisite knowledge to make use of it. (Mach, *Science of Mechanics*, 280; cited in William Craig, *Time and the Metaphysics of Reality*, p 124)

But the positivists were later able to interpret the question so that it had empirical, scientific meaning.
Einstein's theory of relativity provided evidence for the relativity of space to an inertial frame of reference.

The theory of relativity made testable and verifiable claims which allowed the positivists to transform the old, metaphysical debate into a legitimate, scientific one, decided, for the time, in favor of relational space.
While some metaphysical questions could be re-cast as scientific ones, the positivists believed that many philosophical problems, like the problem of free will, could be dissolved, rather than solved.
The challenge for the positivists was to clarify what it meant to verify a sentence.
This challenge turned out to be more difficult than it seemed, and led, along the way, to the development of the philosophy of science as a significant research area in philosophy.

The core idea of the principle of verification is that all our justifiable claims are traceable to a core set of claims which refer only to things or events that we can experience.
The positivists claimed that all of science (and philosophy) can be founded on the basis of observation statements in conjunction with the logical and mathematical principles used to regiment and derive those observations.
Claims that are not observable may be derived from the axiomatic observations, or introduced by definition.
But, all and only meaningful statements will be analytic, observable, or derivable (using logic) from observable axioms.

III. Necessity or Nonsense?

The *Tractatus* was intended, and was hailed by Russell, as the culmination of the enterprise of logical analysis begun by Frege.
According to the picture theory in the *Tractatus*, both the world and our language consist of independent atomic elements, which are combined according to strictly logical principles.
The structure both of language and of the world is governed by strict logical rules, like those depicted in truth tables (which originated in the *Tractatus*, §4.31).
The world is a collection of independent states of affairs.
If I am standing to the right of you, we have, let's say, two atomic facts (my standing and your standing) and a logical relation (to the right of) between those facts.
Language consists of atomic statements of those facts, connected (into more complex statements) by logical principles.
Language provides a picture of the world, and mirrors the world by providing logical structure which is isomorphic to the structure of the world.

One of the most important advances in Frege's logic was its ability to characterize the most general logical properties, including logical truth.
All logical truths are tautologies, complex statements which are true no matter the truth values of their component variables.
We might characterize these statements as necessary truths.
For Descartes, the certainty of logic and mathematics had provided essential support to his claim that our minds have substantial content built into their structures.
From the claim that logic and mathematics are innate, it is reasonable to ask whether there are other innate ideas, including the idea of God.

Wittgenstein and the positivists believe that characterizing logical truths as necessary imbues them with too much importance.

In contrast, he calls them nonsense.
The only statements that can picture the world are those that have sense, that can be either true or false, that can picture accurately or not.

> Propositions show what they say; tautologies and contradictions show that they say nothing. A tautology has no truth conditions, since it is unconditionally true; and a contradiction is true on no condition. Tautologies and contradictions are not pictures of reality. They do not represent any possible situations. For the former admit *all* possible situations, and the latter *none* (Wittgenstein, *Tractatus* §4.461-4.462).

Logical truths (and mathematical truths, given the logicist reduction which loomed large over Wittgenstein's early work) are unknowable because they are too thin to be objects of knowledge.
They don't picture any fact.

Ayer maintains Wittgenstein's deflationary view of logic and mathematics.

> It is to be noticed that the proposition "Either some ants are parasitic or none are" provides no information whatsoever about the behavior of ants, or, indeed, about any matter of fact. And this applies to all analytic propositions. They none of them provide any information about any matter of fact. In other words, they are entirely devoid of factual content (Ayer 79).

Ayer agrees with the rationalists and the logicists that mathematics consists of necessary truths.
They are necessary in the sense that no experience could refute them, no experience could lead us to give them up.
If we found five pairs of socks yielding only nine socks, we would give up our claim to having five pairs.
We would look for a missing sock.
But, we would not give up the claim that five times two is ten.

> One would say that I was wrong in supposing that there were five pairs of objects to start with, or that one of the objects had been taken away while I was counting, or that two of them had coalesced, or that I had counted wrongly. One would adopt as an explanation whatever empirical hypothesis fitted in best with the accredited facts. The one explanation which would in no circumstances be adopted is that ten is not always the product of two and five (Ayer 75-6).

In contrast to Mill's implausible view that mathematical assertions are the results of enumerative inductions and that our claims that they are necessary are merely psychological claims about our inability to conceive of their falsity, Ayer insists that we would never cede a mathematical claim.
But, there is also no way to verify, empirically, the existence of numbers, or circles.
Thus, Ayer's claim is that mathematical theorems tell us about the ways in which we use language, not about the way the world is.

In contrast, mathematical objects are used in science, which the positivists esteemed most highly.
Carnap's "Empiricism, Semantics, and Ontology" was published in 1950, after the end of the Vienna Circle.
Carnap, like Ayer, maintains the positivist's view that logic and mathematics are analytic, and thus devoid of empirical content.
He does not believe that our uses of mathematical terms in science commit us to the existence of abstract entities.

They are artifacts of the conventions of language.
We choose languages, or linguistic frameworks, on the basis of pragmatic considerations.
Once we have chosen a language, certain truths follow within the language.
But, the question of the correct language is merely conventional.
Mathematical theorems are necessary, once we have adopted mathematical language.
But the choice of whether to adopt mathematical language is merely conventional, and does not reflect any transcendent necessity.

IV. Internal and External Questions

Carnap's article focuses on the status of both semantic and mathematical abstract entities.
In mathematics, the ontological question concerns the existence of mathematical objects, like numbers and shapes.
In semantics, the central ontological question concerns the existence of propositions.
Propositions are meanings of sentences.
If I say, "Snow is white," and Pierre says, "La niege et blanc," we are asserting different sentences, but the same proposition.
The proposition that snow is white is shared by me and Pierre.
It is an abstract object, independent of any particular mind.
Indeed, it is independent of any particular language.
We often indicate propositions using a that-clause: that snow is white; that grass is green; that two plus two is four.
But, our use of a particular language to express a proposition is no indication that a proposition is a linguistic object.
The same proposition may be expressed in many different languages, just as we have different ways of representing the same number: 2, II, two, ⋅⋅, {φ, {φ}}.
Similarly, a property (or universal) might be regarded as an abstract object.

To dissolve metaphysical questions such as whether there really exist propositions, numbers, and circles, Carnap distinguishes between external and internal questions.
External questions regard whether or not to adopt a linguistic framework.
A linguistic framework consists of a general term, like 'number' or 'proposition', and variables which range over those objects.
Carnap provides examples of linguistic frameworks.

F1. Thing language
F2. Mathematical languages
F3. The language of propositions
F4. Property language
F5. Systems of space-time points

Carnap depicts the choice of whether to adopt a linguistic framework as a practical decision.
We can talk about things, concrete objects.
Equivalently, we can talk about our sense data.
One way of speaking might be more useful than another, depending on the context.
But, there is no fact of the matter about whether things or sense data actually exist.

> The decision of accepting the thing language, although itself not of a cognitive nature, will nevertheless usually be influenced by theoretical knowledge, just like any other deliberate decision concerning the acceptance of linguistic or other rules. The purposes for which the language is intended to be used, for instance, the purpose of communicating factual knowledge, will determine which factors are relevant for the decision. The efficiency, fruitfulness, and simplicity of the use of the thing language may be among the decisive factors. And the questions concerning these qualities are indeed of a theoretical nature. But these questions cannot be identified with the question of realism (Carnap 208).

Whether to adopt mathematical language or not, then, is, like every external question, just a practical question about the utility of the language.

> It can only be judged as being more or less expedient, fruitful, conducive to the aim for which the language is intended (Carnap 214).

In contrast to external questions, internal questions arise once a framework has been adopted.
Within the framework, there is a very easy argument for the existence of abstract objects.

EA EA1. There are two prime number between 4 and 8.
EA2. So, there are (at least) two prime numbers.
EAC. So, there are numbers.

Internally, the existence of numbers is analytically true.

> Nobody who meant the question "Are there numbers?" in the internal sense would either assert or even seriously consider a negative answer (Carnap 209).

We can ask whether there really are mathematical objects.
But, that question, in itself, is ambiguous.
If it is internal, it is obviously the case.
If it is meant as an external question, it is nonsensical.
Philosophers who ask whether there are numbers, as an external question, are posing an ill-formed question.

> Unfortunately, these philosophers have so far not given a formulation of their question in terms of the common scientific language. Therefore our judgment must be that they have not succeeded in giving to the external question and to the possible answers any cognitive content. Unless and until they supply a clear cognitive interpretation, we are justified in our suspicion that their question is a pseudo-question... (Carnap 209).

The question of whether to adopt a framework admits of degrees.
A framework may be more or less useful depending on context.
But, whether a question is internal or external does not admit of degrees.
Either we have a method of verification or we do not.
If there is a method to verify an answer, then the question has content, and can not be merely external.
If there is no method, then we have a pseudo-question.

V. Surprise!

The positivists present a method for eliminating speculative metaphysics as meaningless.
But statements of mathematics are both impossible to verify empirically and essential to the construction of empirical science.
Thus, as Ayer notes, the empiricist is in a bind.

> The truths of mathematics appear to everyone to be necessary and certain. But if empiricism is correct no proposition which has a factual content can be necessary or certain. Accordingly the empiricist must deal with the truths of logic and mathematics in one of the two following ways: he must say either that they are not necessary truths, in which case he must account for the universal conviction that they are; or he must say that they have no factual content, and then he must explain how a proposition which is empty of all factual content can be true and useful and surprising (Ayer 72-3).

The empiricist must either deny the necessity of mathematics, as Mill does, or explain how claims which are not derived from (or verifiable by) experience, have content.
Against Mill, Ayer argues that we decide to hold onto mathematical claims, as when we refuse to give up 'two times five is ten' when we find that five pairs of objects are nine.
The necessity of mathematics is a decision, like the decision of whether to adopt a linguistic framework.

> The principles of logic and mathematics are true universally simply because we never allow them to be anything else. And the reason for this is that we cannot abandon them without contradicting ourselves, without sinning against the rules which govern the use of language... (Ayer 77).

Ayer's deflationary view of mathematics, that the principles of mathematics are empty, analytic tautologies, thus aligns with Carnap's conventional view, that our mathematical practices reflect a decision to use mathematical language.
As Ayer deflates the old view of necessity, Carnap wants to replace the old notion of apriority, for sureness, with a decision.
They are both replacing necessity with linguistic convention.

Ayer's remaining challenge is to explain how mathematical sentences, mere empty tautologies, can nevertheless appear surprising.
Logical truths are supposed to be obvious.
The advantage of classifying logical truths as analytic, conceptual truths is that we can justify our mathematical beliefs by their obviousness.
But mathematical sentences often appear to be full of interesting, unexpected content.
Ayer appeals to psychological factors to explain how some tautologies can be surprising.
Tautologies tell us about how we use language.
We have limited intellects, and some language use is complicated.

> The power of logic and mathematics to surprise us depends, like their usefulness, on the limitations of our reason. A being whose intellect was infinitely powerful would take no interest in logic and mathematics. For he would be able to see at a glance everything that his definitions implied, and, accordingly, could never learn anything from logical inference which he was not fully conscious of already (Ayer 85-6).

Since we are of limited intelligence, we have to learn the logical truths.
We even appeal to intuition, at times, in our uses of diagrams in geometry, say.
But such appeals are merely accidental and psychological, and tell us nothing about how we justify our mathematical beliefs.

> The fact that most of us need the help of an example to make us aware of those consequences does not show that the relation between them and the axioms is not a purely logical relation. It shows merely that our intellects are unequal to the task of carrying out very abstract processes of reasoning without the assistance of intuition. In other words, it has no bearing on the nature of geometrical propositions, but is simply an empirical fact about ourselves (Ayer 83).

Following Leibniz's distinction between the ways in which we learn propositions and the ways in which we justify them, Ayer claims that Kant and the intuitionists erred in appealing to the psychological fact of our uses of intuition as evidence for their ultimate nature.

VI. Double-Talk and the Guilty Conscience

Carnap and Ayer present tempting views.
On the one hand, they account for our entrenched beliefs in the security of mathematics.
As an internal question, the existence of mathematics is answered uncontroversially.
Mathematical theorems are analytic truths, thin but secure.
On the other hand, they are able to avoid the problems that face traditional platonists.
They need not posit innate ideas, or abstract objects, since the truths of mathematics are just logical truths.
As external claims, platonism and anti-platonism are both regarded as nonsensical.

Carnap sees his position as responding to the conundrum of the person who uses mathematics in science, but worries about metaphysics.

> A physicist who is suspicious of abstract entities may perhaps try to declare a certain part of the language of physics as uninterpreted and uninterpretable, that part which refers to real numbers as space-time coordinates or as values of physical magnitudes, to functions, limits, etc. More probably he will just speak about all these things like anybody else but with an uneasy conscience, like a man who in his everyday life does with qualms many things which are not in accord with the high moral principles he professes on Sundays (Carnap 205).

Carnap is acknowledging a worry about double-talk.
If we claim there are numbers (in science) at the same time as thinking that there are no numbers (in philosophy), we seem to be contradicting ourselves.
In scientific contexts, we use real numbers as space-time coordinates, and for measurement.
It seems illicit to deny in one context what we affirm in another.

One response to the double-talk worry is to admit that one is a platonist.
Such an admission upsets the empiricist, since it admits mathematical objects which can not be sensed.
Carnap's internal/external distinction is an attempt to legitimate these two different ways of speaking.
When we are doing science, and using mathematics, we are admitting mathematical objects as an internal matter.

When we step out of the language of science, we can deny that we mean anything by such talk.

> Some contemporary nominalists label the admission of variables of abstract types as "Platonism". This is, to say the least, an extremely misleading terminology. It leads to the absurd consequence, that the position of everybody who accepts the language of physics with its real number variables (as a language of communication, not merely as a calculus) would be called Platonistic, even if he is a strict empiricist who rejects Platonic metaphysics (Carnap 215).

Carnap urges that platonism and nominalism, from an external perspective, are equally incoherent.
He considers Gilbert Ryle's 'Fido'-Fido criticism of Platonism.
According to Ryle, the platonist makes the error of thinking that every meaningful term has to have an object as a referent.
The platonist argues that since 'two' is meaningful, as we can see by its utility in science, there must be abstract mathematical objects.
But, according to Ryle, some meaningful terms do not mean anything.
The platonist is guilty of illicit reification, or hypostatization.
Carnap argues that Ryle, a nominalist, is engaged in the same kind of metaphysical speculation that the platonist is.

> Ryle says that the 'Fido'-Fido principle is "a grotesque theory". Grotesque or not, Ryle is wrong in calling it a theory. It is rather the practical decision to accept certain frameworks (Carnap 218).

Both Carnap and Ryle deny the inference from our uses of mathematics to the existence of mathematical objects.
Ryle just denies the inference, preferring nominalism.
Carnap provides an explanation for blocking the inference: the very question itself is ill-formed.
There is no evidence that can decide between the platonist and the nominalist.
So, the question of whether mathematical objects exist is a mere pseudo-question.

> I cannot think of any possible evidence that would be regarded as relevant by both philosophers, and therefore, if actually found, would decide the controversy or at least make one of the opposite these more probably than the other...Therefore I feel compelled to regard the external question as a pseudo-question, until both parties to the controversy offer a common interpretation of the question as a cognitive question; this would involve an indication of possible evidence regarded as relevant by both sides (Carnap 219).

VII. Positivism and the Analytic/Synthetic Distinction

Carnap's distinction between internal and external questions relies on the empiricist criterion for meaning.
Internal questions are meaningful, since we have a way of verifying them.
External questions are meaningless, since we lack a way of verifying them.
So, it is critical for the logical positivist to defend the verifiability theory of meaning.

The verifiability theory of meaning derives from the concern, of both Wittgenstein and the positivists, to clear the world of metaphysics.

But, the doctrine of meaninglessness seems to be both metaphysical, and problematic.
The verifiability theory is harder to defend than Carnap and the positivists thought it was.
One serious problem with the verifiability theory of meaning is its apparent circularity.
The theory claims that a proposition is meaningless unless it is verifiable.
But, to know whether the statement is verifiable, we need to know what it means.
For example, few of us know whether 1 is verifiable.

1. Kichwa chake kikubwa.

If we know that 1 means that the meaning of life is 42, we can claim that it is not verifiable.
If we know that 1 is Swahili for 2, then we can claim that it is verifiable.

2. His head is big.

If we know what a proposition means before we verify it, then verificationism is not doing any semantic work.
There seems to be a difference between real nonsense (gibberish) and metaphysical claims.
Metaphysical claims can be grammatical, and composed of terms which otherwise might refer.
They can combine with other claims in consistent ways.
Some terms which are supposed by the positivists to be meaningless do have content.
If purported external claims like 'There are numbers' are not meaningless, if they have content, then Carnap's proposal does not succeed.

So I am skeptical that Carnap's internal/external distinction solves the problem of the empiricist's guilty conscience regarding uses of mathematics.
We can not wish away the commitments to mathematics that come with the acceptance of the linguistic framework that includes mathematical objects.
Quine will return to the double-talk criticism, as we will see in his attack on the analytic/synthetic distinction.
Quine argues that there is no good way to distinguish between analytic and synthetic statements.
If 'There are numbers' is taken as an internal question, it is supposed to be analytic, and devoid of any experiential evidential basis.
But, if there is no analytic/synthetic distinction, then, Carnap's distinction collapses, too.

Quine attacks conventionalism for logic.
He argues that for logic to be conventional, in Carnap's sense, we would have to adopt a framework including it.
But, the adoption of a framework is itself guided by logical laws.
So, some logic has to be presupposed.
Similar claims might be made for mathematics.

I think that Quine is wrong about the analytic/synthetic distinction, as we will discuss.
But, Carnap is also wrong that external questions are meaningless.

The positivist's project was a general account of our knowledge of science which begins with foundational elements (e.g. sense data) and uses the tools of modern, Fregean logic to construct the most sophisticated scientific theory.
Like Hume's claim that we should commit to the flames all books which rely on claims which are not

capable of being traced back to initial impressions, the positivists derided as pseudo-statements all claim that could not be verified.
It is thus central to the positivist's project that some statements be taken as basic truths.

> There is a class of empirical propositions of which it is permissible to say that they can be verified conclusively. It is characteristic of these propositions, which I have elsewhere called "basic propositions," that they refer solely to the content of a single experience, and what may be said to verify them conclusively is the occurrence of the experience to which they uniquely refer... Propositions of this kind are "incorrigible,"...[in that] it is impossible to be mistaken about them except in a verbal sense (Ayer, *Language, Truth and Logic*, 10).

Among these basic, incorrigible principles are the logical and mathematical principles, which are analytic.
These analytic claims are atomic facts.

> Every logical proposition is valid in its own right. Its validity does not depend on its being incorporated in a system (Ayer 81).

Quine's attack on positivism takes this claim, that there are atomic facts independent of a broader conceptual framework, as its main target.

Philosopher	**Is Mathematics Analytic or Synthetic?**	**Math Knowledge**	**Are Mathematical Truths Necessary?**	**What are Mathematical Objects?**	**The Infinite**
Frege's Logicism	Arithmetic is analytic; geometry is synthetic.	All mathematical knowledge is *a priori* logic.	Yes	Platonic entities, but logical constructions	Accepts the actual infinite
Brouwer's Intuitionism	Synthetic	Synthetic *a priori*, constructed out of our intuition of time	Yes; disproving mathematical truths is unthinkable	Mental constructions	Accepts only a potential infinite; nothing beyond ω
Hilbert's Formalism	Analytic for ideal elements; synthetic for real elements	Proofs are known *a priori* (logic); elementary truths are known by intuition.	Yes; and all mathematical truths are provable	Finite mathematics is about stroke symbol sequences; Infinite mathematics is about nothing	Accepts the infinite as an ideal, meaningless element, useful for deriving further finitary mathematical results