Philosophy 405: Knowledge, Truth and Mathematics | Hamilton College
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Intuitionism

I. Paradoxes and Problems

Cantor's transfinite numbers, and their underlying set theory, led to several oddities which created anxiety for mathematicians and philosophers in the early twentieth century.
We have seen the so-called Cantor's paradox, and Burali-Forti's paradox, as well as Russell's paradox.
Many mathematicians located the causes of the paradoxes in transfinite set theory.
As we saw, Hilbert's program was motivated by the desire to maintain set theory, and all its transfinite results, by focusing on meta-mathematics.
Other, more reactionary, mathematicians were dismissive of the infinite.
Leopold Kronecker (1823-1891), for example, is reputed to have said, "God made the natural numbers; all else is the work of man."

The paradoxes of naive set theory were resolved by adopting axiomatic set theory.
Among the axioms of set theory is the axiom of choice, or selection.
The axiom of choice has many forms, most of which appear to be uncontroversial.
For example, the axiom of choice says that given an arbitrary collection of nonempty sets, we may choose an element from each of them.
But the axiom is also equivalent to the claim, first proved by Zermelo in 1904, that every set can be well-ordered.
A brief examination of the conclusion of Zermelo's proof that every set can be well-ordered will be instructive in understanding intuitionism.

A set A is well ordered by a relation <, if for all x, y, and z in A

1. $\sim x < x$
2. $(x < y \bullet y < z) \supset x < z$
3. Ether $x < y$, $x = y$, or $y < x$
4. Every nonempty subset of A has a smallest element.

The natural numbers are well-ordered by their usual sequence.
The real numbers are not well-ordered by their usual sequence, since condition 4 is violated.
There is no least element in the set of all reals, say, between 0 and 1.
Indeed, given Cantor's diagonal argument, which shows that every list of real numbers is in principle incomplete, it is odd to think that the reals could be well-ordered.
Zermelo's proof shows that any set can be well-ordered.
In particular, the real numbers can be well-ordered.
But, Zermelo's proof does not produce the ordering.
Indeed, it seems that no such ordering should be possible within the language of set theory.

One response to Zermelo's proof is to give up on the axiom of choice, from which it follows, despite its obviousness.
As Zermelo's proof that every set can be well-ordered does not provide a method for ordering, the axiom of choice does not provide method for constructing the choice set.

Thus, the axiom of choice and the oddity of pairing the well-ordering of the reals with the impossibility of listing them are related worries about set theory.
Some proofs in set theory are non-constructive existence proofs.
Some existence proofs are weird.

II. Logicism, Formalism, Intuitionism

In the early twentieth century, there was an explosion of research in logic and the philosophy of mathematics.
Three distinct positions were explored, with many variations of those positions.
We have already looked at Frege's logicism, and mentioned Russell's version which was based on set theory with a theory of types used to block the paradoxes of naive set theory.
Frege and Russell defended classical mathematics.
We have also looked at Hilbert's formalism.
Hilbert defended classical mathematics, though taking infinitary, or ideal, elements to be ontologically second-class.
The third position is called intuitionism.
Brouwer was the earliest and most prominent intuitionist.
The debates between Brouwer, and his followers, and Hilbert, and his followers, in the 1920s were intense and productive.
Paolo Mancosu's *From Brouwer to Hilbert* contains papers and correspondence from that period, as well as invaluable introductory essays.

Both Hilbert and Brouwer worried about whether infinitary results in mathematics were consistent with the finitude of mathematicians.
Hilbert takes the utility of infinitary statements to show the need for firmer grounding in mathematics, for demonstrating the ways in which we can have knowledge of completed infinite sequences, say.
He argues that ideal, infinitary, mathematics is not really about external objects, but about the systems themselves.
Hilbert gives up the idea that we are seeking mathematical truth, for ideal elements.
He abandons the claim that there are infinitary objects to serve as the models of transfinite mathematics.
Hilbert retreats to proofs of the consistency of the systems we use.
He does not cede any mathematical results.

Brouwer, in contrast, wants to hold on to the idea that mathematics presents us with a body of verifiable truths.
He is willing to give up some results, especially those which involved infinite quantities.
Hilbert's defense of Cantor's paradise was really directed at the intuitionists, as was his comment that every mathematical problem has a solution.
Denying Hilbert's claim, the intuitionists believe that mathematics is not discovered, that mathematics is not a body of transcendent truths.
Instead, they believe that mathematics is a human construction.
Classical mathematics depends on untenable metaphysics, commitments to an over-stuffed mathematical universe.
Heyting makes this point by considering a dialogue between a classical mathematician and an intuitionist about a mathematical definition.

> II. l is the greatest prime such that l - 2 is also a prime, or l = 1 if such a number does not exist...
>
> *Class*: [You claim that] as long as we do not know if there exists a last pair of twin primes, II is not a definition of an integer, but as soon as this problem is solved, it suddenly becomes such a definition. Suppose [in the future] it is proved that an infinity of twin primes exists; from that moment l = 1. Was l = 1 before that date or not?
>
> *Int*: A mathematical assertion affirms the fact that a certain mathematical construction has been effected. It is clear that before the construction was made, it had not been made... It seems to me that in order to clarify the sense of your question you must again refer to metaphysical concepts: to some world of mathematical things existing independently of our knowledge, where "l = 1" is true in some absolute sense. But I repeat that mathematicians ought not to depend upon such notions as these. In fact all mathematicians and even intuitionists are convinced that in some sense mathematics bear upon eternal truths, but when trying to define precisely this sense, one gets entangled in a maze of metaphysical difficulties. The only way to avoid them is to banish them from mathematics (Heyting 67-8).

The intuitionists, following Kant, believe that we construct mathematical objects in intuition.
For the formalists, mathematics is about inscriptions.
For the intuitionists, mathematical objects are mental constructions.

> The question where mathematical exactness does exist, is answered differently by the two sides; the intuitionist says: in the human intellect, the formalist says: on paper (Brouwer 78).

Frege and Hilbert looked to formal systems to secure inference from the misleading powers of subjective intuition.
Brouwer rejects the appeals to formal theories.

> A mathematical construction ought to be so immediate to the mind and its result so clear that it needs no foundation whatsoever. One may very well know whether a reasoning is sound without using any logic; a clear scientific conscience suffices (Heyting 70).

III. Kant, Non-Euclidean Geometry, and the New Intuitionists

While the ancient Greeks had noticed that perfect mathematical objects were not available to sense experience, mathematics through the eighteenth century remained closely tied to our theories of the world.
For Mill, mathematical objects were approximations of physical objects.
For Kant, mathematical objects were constructions in the pure forms of intuition which made experience possible.
The increased abstraction and formalization of mathematics in the nineteenth century distanced mathematics from science and sense experience.
Cantor's transfinites, and the fineness of the continuum, went beyond anything that could be applied in the physical world.
Cantor's paradise was not in this world, but, as Aristotle would say, a separate one.

In addition, the developments of non-Euclidean geometry, as we saw, undermined the Kantian view of

mathematics as based in geometric intuition.
Kant claimed that our knowledge of geometry is synthetic *a priori*, based on our knowledge of the necessary structure of our capacity for representation.

> For Kant...the possibility of disproving arithmetical and geometrical laws experimentally was not only excluded by a firm belief, but it was *entirely unthinkable* (Brouwer 78, emphasis added).

In contrast, consistent non-Euclidean theories showed that geometry transcended our intuition.
Geometry became untethered from experience.

The intuitionists resisted the increased abstraction, and consequent formalization, of mathematics.
While later intuitionists, like Heyting, developed formal logical systems, Brouwer rejected formalization of mathematics precisely for its disconnection from experience.

> We...are interested not in the formal side of mathematics, but exactly in that type of reasoning which appears in metamathematics (Heyting 68)

For Brouwer, mathematics is essentially the construction of human thinkers.

> Intuitionism, as a form of interaction between men, is a social phenomenon (Heyting 75).

Brouwer desired a return to the Kantian view of mathematics.
But Kant's claims that all of mathematics is synthetic *a priori* had become untenable in light of the development of non-Euclidean geometries.
Brouwer's solution, borrowing the inversion of foundations from geometric to set-theoretic from the nineteenth-century logicians, is to claim that our knowledge of arithmetic remains synthetic *a priori*, and that our knowledge of geometry is based in our knowledge of arithmetic.
We can derive our knowledge of mathematics from temporal intuitions.

> [Intuitionism] has recovered by abandoning Kant's apriority of space but adhering the more resolutely to the apriority of time. This neo-intuitionism considers the falling apart of moments of life into qualitatively different parts, to be reunited only while remaining separated by time, as the fundamental phenomenon of the human intellect, passing by abstracting from its emotional content into the fundamental phenomenon of mathematical thinking, the intuition of the bare two-oneness (Brouwer 80).

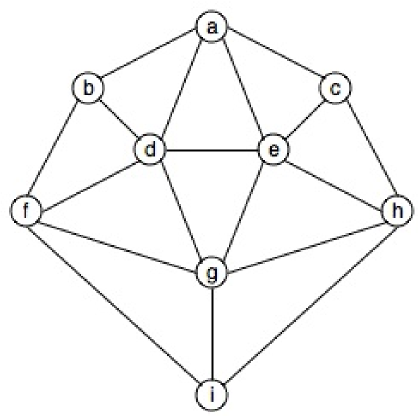
Brouwer's claim is that the intuition of two-oneness underlies all of our mathematical reasoning.
The 'two-oneness' refers to unity in a multiplicity, like Kant's transcendental apperception.
From the two-oneness, we can generate finite ordinals up to ω.
We can generate intuitions of the continuum.

> This basal intuition of mathematics, in which the connected and the separate, the continuous and the discrete are united, gives rise immediately to the intuition of the linear continuum, i.e., of the "between," which is not exhaustible by the interposition of new units and which therefore can never be thought of as a mere collection of units. In this way the apriority of time does not only qualify the properties of arithmetic as synthetic a priori judgments, but it does the same for those of geometry... (Brouwer 80).

Returning thus to intuition as the basis for our knowledge of mathematics, Brouwer diagnoses the problems of Zermelo's proof that every set can be well-ordered, as well as the paradoxes of transfinite set theory.
Mathematicians have relied on non-constructive existence proofs.
Logicists and formalists have embraced such proofs only by dis-interpreting mathematics, by removing the thinking mathematician from mathematical practice.
Brouwer's recommendation is to reject all such reasoning.
Intuitionists should accept only constructive proofs.
Let's look more closely at the difference between constructive and non-constructive proofs.

IV. A Constructive Proof

Definition: A coloring of a graph is an assignment of a color to each node of the graph.
Definition: A graph is 3-colorable if there exists a coloring which uses only three colors, and which does not assign the same color to any two nodes which share a branch.
Definition: A graph is 4-colorable if there exists a coloring which uses only four colors, and which does not assign the same color to any two nodes which share a branch.



Theorem: There are graphs which are 4-colorable but which are not 3-colorable.
Proof: In two stages. Present a graph which is not 3-colorable but which is 4-colorable.

Stage 1: Prove that the graph is not 3-colorable.
Stage 2: Show that the graph is 4-colorable.

Stage 1:
Call the three colors red, green, and blue.
Assign red to a (without loss of generality).
So, b and d must not be red, nor may they be the same color as each other.
So, assign green to b and blue to d, again without loss of generality.
Consider e, which now must be green.
Consider g, which now must be red.
Look at f.
Uh-oh.

Stage 2: Construct a four-coloring, use red, green, blue, yellow.
Left to reader

Note that both stages are clearly constructive.
In stage 1, we construct a counter-example to the claim that the graph is 3-colorable.
In stage 2, we construct a 4-coloring of the graph which satisfies the theorem.

V. A Non-Constructive Proof

Claim: There exist irrational numbers x and y such that x^y is rational.

Proof:

Let $z = \sqrt{2}^{\sqrt{2}}$.
Either z is rational or z is irrational, though we do not know which.
If z is rational then z is our desired number with $x = y = \sqrt{2}$.
If z is irrational, then let $x = z$ and $y = \sqrt{2}$.

$$x^y = {\sqrt{2}^{\sqrt{2}}}^{\sqrt{2}} = \sqrt{2}^{\sqrt{2} \cdot \sqrt{2}} = \sqrt{2}^{2} = 2.$$

On these different assignments of irrational values to x and y, x^y is again rational.
Whether z is rational or irrational, there exist irrational numbers x and y such that x^y is rational.
QED

VI. Intuitionism and the Law of the Excluded Middle

The proof in the previous section is not constructive because we do not know whether z is rational or irrational.
If z is rational, the one set of assignments to x and y works.
If z is irrational, a different set of assignments works.
So, according to classical logic and mathematics, we can be sure of the theorem without being able to construct a particular x and y.
Similarly, we can know that the real numbers can be well ordered without providing an order, and we can be sure of the existence of a choice set even in cases, like transfinite cases, in which we can not provide a method of construction.

Heyting provides constructive and non-constructive definitions of a number; we saw the non-constructive one above.

I. *k* is the greatest prime such that *k* - 1 is also a prime, or *k* = 1 if such a number does not exist
II. *l* is the greatest prime such that *l* - 2 is also a prime, or *l* = 1 if such a number does not exist (Heyting 67).

I is intuitionistically legitimate, since we can calculate $k = 3$.
II is not intuitionistically legitimate, since we have no proof of the twin prime conjecture on which the definition of l depends.
The classical mathematician can argue that II defines a number, even though we do not know the value of that number.

CL CL1. The sequence of twin primes is either finite or infinite.
CL2. If it is finite, then x is the larger element of the largest pair.
CL3. If it is infinite, then x is 1.
CLC: x is some integer.

Classical logic, the logic underlying classical mathematics, includes the law of the excluded middle.

LEM $P \lor \sim P$

LEM is used in CL at step 1, and it is used in the con-constructive proof in §V, above.
The intuitionist denies the legitimacy of both uses, as a consequence of her rejection of LEM.
In both arguments, neither of two options has been demonstrated.
We have not shown that the sequence of twin primes is finite.
We have not shown that the sequence of twin primes is infinite.
We have not shown that z is rational or irrational.
We can not conclude, says that intuitionist, that the sequence of twin primes must be either one or the other.
We can not disjoin the options, even if they are exhaustive.
For more illustrations of the differences between intuitionist and classical logic, and between constructive and non-constructive definitions and proofs, see Brown 125 et seq.

The classical logician charges that by rejecting the law of the excluded middle, the intuitionists change the meanings of logical terms.
The intuitionist accepts this charge.

> In the study of mental mathematical constructions, "to exist" must be synonymous with "to be constructed" (Heyting 67).

For the intuitionist, to assert 'P' is to assert that P has been proven, not that it is true.
'$P \lor Q$' means that either P or Q have been proven.
'$P \supset Q$' means that there is a construction which, when added to the construction of P, yields Q.
Perhaps most interestingly, '$\sim P$' means not that P is false, but that there exists a constructive disproof of P.

For the intuitionist, mathematical statements do not have transcendental truth values.
They are made true or false by our proofs, by our constructions.
Statements which do not produce a number, or a construction of some sort, may be neither proven nor refuted.
These statements, like step 1 in CL, are to be taken as neither true nor false.

Thus, if we are to formalize intuitionist logics, we need a third truth value, beyond truth and falsity.
We need a new logic to treat infinitary, and other non-constructive, mathematics.

> For the description of some kinds of objects another logic may be more adequate than the customary one has sometimes been discussed. But it was Brouwer who first discovered an object which actually requires a different form of logic, namely the mental mathematical construction...The reason is that in mathematics from the very beginning we deal with the infinite, whereas ordinary logic is made for reasoning about finite collections (Heyting 66).

Heyting tried to formalize intuitionist logic, but Brouwer did not approve.
Formalism makes mathematics about the rules of a formal system.
Brouwer insists that mathematics is informal, about mental constructs, not formal systems.

For some classical mathematicians, intuitionists are "damned relativists" (Heyting 70), due to their reliance on subjective procedures of mental construction.
For others, intuitionist mathematics picks out an interesting sub-class of mathematical results.
For the intuitionist, the classical mathematicians goes beyond the bounds of truth, and engages in speculative metaphysics.

VII. Onward

We have looked at the three big schools of philosophy of mathematics in the early twentieth century: logicism, formalism, intuitionism.
Proponents of all three were responding largely to the nineteenth-century advances in set theory and logic.
All three positions are heirs, to some degree, of earlier views.
The logicists were mainly platonists, like Descartes or Leibniz without the commitment to innate ideas.
The formalists, like Hume, denied the existence of (many) mathematical objects while maintaining the truth of many mathematical claims.
The intuitionists inherited from Locke and Kant an appeal to our psychological capacities in order to explain our knowledge of mathematics.

We have not yet looked at a twentieth-century descendent of Berkeley's nihilism.
But, the conventionalism associated with Wittgenstein and logical positivism might, though it is a bit of a stretch, be compared to the Berkeleyan view.
We proceed to look at the positivist's view of mathematics.