Philosophy 405: Knowledge, Truth and Mathematics | Hamilton College
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Gödel's Platonism

I. The Continuum Hypothesis and Its Independence

The continuum problem is an interesting technical question in set theory.
Our interest in that technical question is subordinate to the philosophical questions about our knowledge of mathematics that considering the question raises.
Gödel's claim is that we have a capacity, called mathematical intuition, that can support our beliefs about an objective set-theoretic universe.
Intuition is analogous to sense perception, and does not require immediate apprehension of the mathematical world.

The question of the size of the continuum was raised by Cantor.
The period between Cantor's original development of set theory and Gödel's 1964 version of his paper on the continuum hypothesis was extraordinarily fecund.
Set theory and modern logic were born, and grew to maturity in less than a century.
Cantor raised the continuum hypothesis, and thought several times he had solved it.
Yet, it persisted as an open question.
Even today, it remains a subject for debate.
Let's first get a bit clearer about the problem.

Feferman, in his editor's introduction to Gödel's paper, distinguishes three versions of the continuum hypothesis.
The weak continuum hypothesis says that every uncountable set of reals is the same cardinality of the set of all reals.
In other words, there is no cardinality less than the full size of the real numbers but greater than that of the natural numbers.
The continuum hypothesis says that $2^{\aleph_0} = \aleph_1$, that size of the power set of the natural numbers, which is provably equivalent to the size of the real numbers, is the next transfinite cardinal after $\aleph_0$.
The generalized continuum hypothesis says that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$, for all cardinals α.
In other words, the power set operation is the way to move through the infinite cardinalities.
On the generalized continuum hypothesis, the power set operation is the successor function for transfinite cardinalities; it doesn't skip any.

The 1964 version of Gödel's paper came out just before Paul Cohen showed that the size of the continuum is independent of the other, standard axioms of set theory.
Cohen showed that the continuum hypothesis is undecidable by the standard axioms using a new model-theoretic method called forcing.
That the size of the continuum is independent (undecidable) means that one can add an axiom to the standard axioms asserting that the continuum is any size greater than the size of the natural numbers, and not derive a contradiction.
So $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_0} = \aleph_2$ and $2^{\aleph_0} = \aleph_3$ ... are all consistent with the other axioms.
Gödel anticipated Cohen's result, as you can tell by reading the paper.
He had earlier thought that the continuum hypothesis was true.

Then, he decided that it was probably false, and he had tried to prove its independence.
But he had seen proofs of only smaller solutions.
Gödel includes a postscript in which he lauds Cohen's independence proof.

> Cohen's work...no doubt is the greatest advance in the foundations of set theory since its axiomatization... (Gödel 270).

Our interest is only in passing on the specifics of the continuum hypothesis, though those results are intrinsically compelling.
The philosophical interest of Gödel's paper comes mainly from §3 and, more importantly, note 4 of the postscript.
Gödel raises questions about what the independence of the continuum hypothesis entails for set theory, and about the nature of mathematics, including the relationship between axiomatic systems and mathematical truth.
Most importantly, he posits a faculty of mathematical intuition for learning about an objective mathematical universe.

II. Intuitionism, Paradox and the Continuum Hypothesis

One response to the independence of the continuum hypothesis is to call it meaningless.
Gödel points out that the intuitionists would take this option.
The intuitionists, as we have discussed briefly, and as we will examine in more depth next week, were committed finitists.
For them, all discussion of the infinite is illegitimate.

The claim that the continuum hypothesis is meaningless, if motivated by finitism, is not a response to the independence of the hypothesis *per se*.
It is a more general rejection of infinitistic reasoning.

> This negative attitude toward Cantor's set theory, and toward classical mathematics, of which it is a natural generalization, is by no means a necessary outcome of a closer examination of their foundations, but only the result of a certain philosophical conception of the nature of mathematics, which admits mathematical objects only to the extent to which they are interpretable as our own constructions or, at least, can be completely given in mathematical intuition. For someone who considers mathematical objects to exist independently of our constructions and of our having an intuition of them individually, and who requires only that the general mathematical concepts must be sufficiently clear for us to be able to recognize their soundness and the truth of the axioms concerning them, there exists, I believe, a satisfactory foundation of Cantor's set theory in its whole original extent and meaning, namely, axiomatics of set theory... (Gödel 258)

The intuitionists were motivated by the paradoxes of set theory.
Cantor's naive set theory is inconsistent, and some solution to the paradoxes had to be developed.
The standard resolution, as we have seen, is to present set theory axiomatically.

Gödel discusses the construction of axiomatic set theory using numbers as ur-elements.
That is, he is not presenting pure set theory.

But, the use of ur-elements is not the central issue.
The key point here is to use a bottom-up definition of set, like that in ZF, rather than a top-down one, like Cantor's, or Frege's.
In a bottom-up definition, we start with well-formed sets, and form new sets, applying constructive axioms to form new sets, like the pair set axiom, to the sets we already have.
The result is called the constructive set-theoretic universe.

Thus, the set-theoretic paradoxes can be resolved without damage to Cantor's work on transfinites.
We do not know that axiomatic set theories are consistent.
But, we do know that they do not lead to Russell-style paradoxes.
Thus, the continuum hypothesis can be formulated in a (presumably) consistent set theory.
We need not become finitists in order to block the paradoxes.
The continuum hypothesis remains a viable mathematical question.

III. Platonism and Independence

Another way to argue that the continuum hypothesis has lost its meaning is to look at the precedent set by the parallel postulate.
Mathematicians tried to prove that the parallel postulate could be derived from the other Euclidean axioms.
It turned out that it and the two forms of its negation were all consistent with the other axioms.
That is, the parallel postulate turned out to be independent of the other axioms.
The question of which version is right became meaningless, since each version describes a distinct, consistent space.

Gödel argues that the situation is different in the case of the continuum hypothesis.
He believes that there is one right answer to the size of the continuum.
The undecidability of the continuum hypothesis by the standard axioms of set theory shows that we need better, more constructive axioms, in order to settle the matter.

> The set-theoretical concepts and theorems describe some well-determined reality, in which Cantor's conjecture must be either true or false. Hence its undecidability from the axioms being assumed today can only mean that these axioms do not contain a complete description of that reality (Gödel 260).

Gödel believes that if the current axioms do not settle the truth value of the continuum hypothesis, then we can strengthen the axioms.
But, we have to figure out in what way we should strengthen them.

Gödel argues that a better analogy than the parallel postulate would be the axiom which asserts the existence of inaccessible cardinals.
This axiom is also independent of the other axioms of set theory.
There are lots of such axioms, now.
(Gödel discusses the axioms of von Neumann-Bernays set theory, which is now usually called NBG after von Neumann, Bernays and Gödel himself, but the same point holds for ZF.)
The independence of such axioms does not entail that there is no basis for judging whether to accept the axiom.

There are mathematical consequences for accepting (or rejecting) inaccessible cardinals, just as there are mathematical consequences for accepting (or rejecting) the continuum hypothesis.
The theory which denies the existence of inaccessible cardinals is a weaker theory; the one that accepts them is a greater extension of set theory.

> A closely related fact is that the assertion (but not the negation) of the axiom [which asserts the existence of inaccessible cardinals] implies new theorems about integers (the individual instances of which can be verified by computation)...[O]nly the assertion yields a "fruitful" extension, while the negation is sterile outside its own very limited domain (Gödel 267).

In each case, of the continuum hypothesis and of the axioms for inaccessible numbers, we have to consider the consequences of accepting or rejecting further axioms.
We have to see the connections among those axioms or their negations and other theorems.

> It is very suspicious that, as against the numerous plausible propositions which imply the negation of the continuum hypothesis, not one plausible proposition is known which would imply the continuum hypothesis. I believe that adding up all that has been said one has good reason for suspecting that the role of the continuum problem in set theory will be to lead to the discovery of new axioms which will make it possible to disprove Cantor's conjecture (Gödel 264).

Gödel predicts, but does not assert, that we will eventually see that the generalized continuum hypothesis is false.

> The generalized continuum hypothesis, too, can be shown to be sterile for number theory and to be true in a model constructible in the original system, whereas for some other assumption about the power of $2^{\aleph_0}$ this perhaps is not so. On the other hand, neither one of those asymmetries applies to Euclid's fifth postulate. To be more precise, both it and its negation are extensions in the weak sense (Gödel 267).

That the current axioms do not decide the answer of the size of the continuum is no reason, Gödel claims, to think that it has no size.
Now, we need to think about the deeper question of how one can determine what that size is.

IV. Success

Gödel uses criteria of success and fruitfulness to argue that the continuum is likely to have one, and only one, acceptable size, just as the axioms for inaccessible numbers.
We look to the consequences of accepting or rejecting any one size of the continuum to decide whether it is that size.
If the consequences for other established areas of mathematics are salutary, we have good reasons to adopt a new axiom settling the size.

Gödel mentions, as one kind of fruitfulness, the addition as a new axiom a statement which facilitates the derivation of theorems that are already proven.
If the new axiom makes the derivations easier, that might be a good reason to adopt the axioms.

> Success here means fruitfulness in consequences, in particular in "verifiable" consequences, i.e. consequences demonstrable without the new axiom, whose proofs with the help of the new axiom, however, are considerably simpler and easier to discover, and make it possible to contract into one proof many different proofs (Gödel 261).

Another kind of fruitfulness is when a new axiom allows us to derive theorems that we have not yet been able to derive.
Gödel is hoping for axioms that would decide the continuum hypothesis, while at the same time yielding independent reasons for us to accept it.
Thus, he looked carefully at entailments from the continuum hypothesis.
The fact that there were plausible propositions which implied the negation of the continuum hypothesis moved Gödel toward rejecting it.

Contrast Gödel's emphasis on the fruitfulness of an axiom to the standard view that we prove the theorems, but take the axioms as assumptions.
The game formalist takes the axioms as empty, or meaningless.
But the axioms do not lack content.
Frege took the axioms to be logical truths.
But we saw that their derivations required set theory.
Most philosophers and mathematicians agree that set theory is not merely logic.
Gödel is offering to use arguments about success to argue, not merely for the utility of assuming them, but for their truth.

For example, Gödel uses the criteria of success and fruitfulness in arguing for the truth of the axiom of choice.
The axiom of choice is consistent with the other axioms of set theory, and its negation is also consistent.
It is thus another independent claim.
The axiom of choice has both seemed obviously true to set theorists, and seemed obviously false.
On the true side, consider the version that says that, given a set of sets, we can construct (or there is) another set which contains one member from each of the member of the original set.
For a simple example, consider the set:

A: {{2, 4}, {1, 5}, {7, Hillary Clinton}}

The axiom of choice says the existence of A ensures that there is a set:

B: {2, 1, Hillary Clinton}

Who could argue?

> This axiom, from almost every possible point of view, is as well-founded today as the other axioms of set theory which are usually assumed...The axiom of choice is just as evident as the other set-theoretical axioms..." (Gödel 255 (fn 2)).

On the false side, the axiom of choice entails the theorem, first proved by Zermelo in 1904, that every set can be well-ordered.
In particular, the set of all real numbers can be well-ordered if the axiom of choice is assumed.
And, that seems wrong.

The other axioms do not settle the truth of the axiom of choice.
So, we are faced with a choice of whether to take it as an axiom, whether we should think that it is true.
Some of our intuitions tell us that it is.
Some of our intuitions tell us that it is not.
It looks like we will, on pain of consistency, have to give up some intuitions.
Our method for deciding which to give up will depend on which choice provides the most satisfying overall picture of mathematics.

V. Platonism and Intuition

The picture we have been examining, on which we have to decide whether to accept the axiom of choice, or large cardinal axioms, or the continuum hypothesis, makes such questions seem conventional, or pragmatic.
But, mathematical claims are, on almost all views, neither conventional nor pragmatic.
When we say that 2 + 2 = 4, we are not indicating a convention or a decision.
We are making an objective claim.
Everyone we have read so far, including Locke and Mill, with the sole exception of the nihilist Berkeley, believes that mathematical claims are objective, rather than conventional.
There are other dissenters, like Wittgenstein.
We will read Carnap on conventionalism.
But, one need not be a platonist, like Frege or Gödel, to believe in the objectivity of mathematical claims.

The point here is that Gödel's claim that we have to decide what axioms to adopt on the basis of considerations like success and fruitfulness is not an indication that he believes that the choice is merely pragmatic.
Gödel believes that success and fruitfulness are reliable indicators of the objective truth of mathematical claims.

If all mathematical claims were to be judged by intra-theoretic criteria like fruitfulness, then Gödel's claim that mathematical truths are objective would be less plausible.
Using fruitfulness and success alone as criteria for mathematical truth leads to a kind of coherentist circle within mathematics.
For at least some theorems, we would like a more direct defense, some foundational claims.

Even for Gödel, fruitfulness is not the only criterion for mathematical truth.
In addition to intra-theoretic considerations, Gödel believes that we have mathematical intuition, analogous to sense perception.

> Despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. I don't see any reason why we should have less confidence in this kind of perception, i.e. in mathematical intuition, than in sense perception, which induces us to build up physical theories and to expect that future sense perceptions will agree with them, and, moreover, to believe that a question not decidable now has meaning and may be decided in the future. The set-theoretical paradoxes are hardly any more troublesome for mathematics than deceptions of the senses are for physics (Gödel 268).

With both mathematical intuition and sense perception, we are forced to believe in the objects that we sense and the way that mathematical axioms force themselves on us.
In both cases, we are liable to error.
We make errors of illusion or hallucination with the senses.
We can make *a priori* errors like those which led to the axiom of comprehension in set theory.

Kant also appealed to a kind of intuition.
For Kant, 'intuition' referred to our psychological capacity for representation.
For Gödel, 'intuition' refers to a capacity for acquiring mathematical beliefs.
Gödel agrees with Kant that intuition involves conceptualizing some matter given to us prior to our thought.
We do not immediately apprehend the set-theoretic universe.
But, Gödel also claims an objectivity for our mathematical beliefs that goes beyond Kant's subjective psychological constructions.

> Evidently the "given" underlying mathematics is closely related to the abstract elements contained in our empirical ideas. It by no means follows, however, that the data of this second kind, because they cannot be associated with actions of certain things upon our sense organs, are something purely subjective, as Kant asserted (Gödel 268).

Despite his claims that we have intuitive knowledge of an objective mathematical reality, based on our (at least metaphorical) apprehension of a transcendent set-theoretic universe, and that we have criteria for evaluating mathematical theorems beyond testing whether they are consistent with our accepted axioms, Gödel does not believe that we can judge whether Cantor's continuum hypothesis is true or false.
The jury is still out.

> This criterion, however, though it may become decisive in the future, cannot yet be applied to the specifically set-theoretical axioms...because very little is known about their consequences in other fields...On the basis of what is known today, however, it is not possible to make the truth of any set-theoretical axiom reasonably probable in this manner (Gödel 269).

Our interest in Gödel's work was not based on a particular claim about the continuum hypothesis.
We were interested in his claims about mathematical intuition, and about the role of success and fruitfulness, in determining the nature of an objective set-theoretic universe.