Knowledge, Truth, and Mathematics

Philosophy 405
Russell Marcus
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Class 12
Finishing Cantor and Transfinite Set Theory
Frege's Argument Against Kant

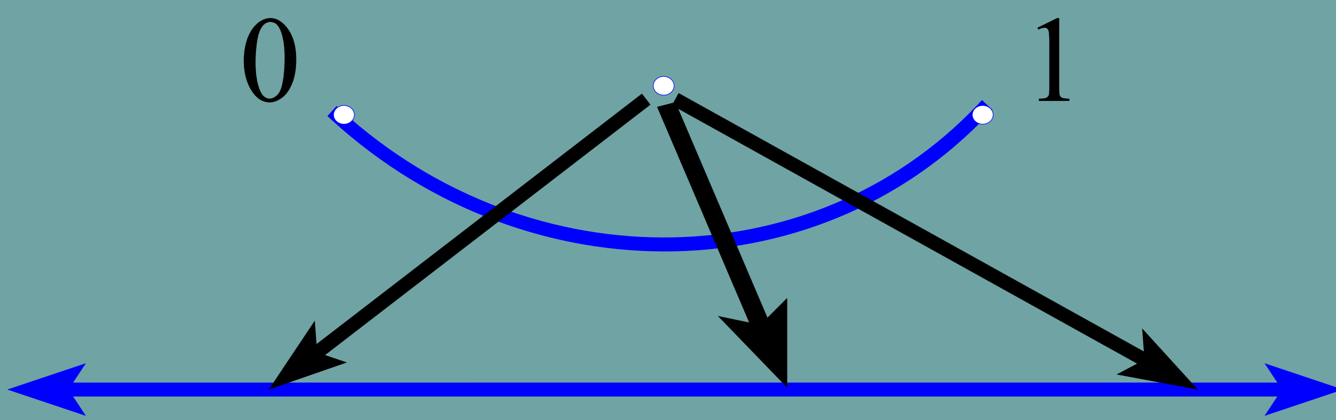
Review

- Aristotle distinguished between actual and potential infinity
- Algebraic representations of functions led to questions about the discrepancy between the picture of a function (graph) and the numbers over which it ranges.
  - There are more points on a line than rational numbers.
  - How many points are on a line?
- Cantor developed set theory to examine the structure of the line.
- Hume's principle: Cardinality is measured by one-one correspondence.
- Leibniz's law for identity:
  - "Things are the same as each other, of which one can be substituted for the other without loss of truth" (Frege, *Grundlagen,* §65).
  - substitutivity *salva veritate*
- Cantor's Theorem (diagonal argument)
  - Cardinal number version: $2^a > a$
  - Set theory version: $\mathbb{C}(\wp(A)) > \mathbb{C}(A)$.

The Natural Numbers and the Real Numbers

- The real numbers, and the real plane, are the size of the set of subsets of the natural numbers.
  - The set of subsets of a set is its power set.
  - We can correlate the power set of the natural numbers with the real numbers, and thus with the points on a line.
- Every subset of the natural numbers can be uniquely correlated with an infinite sequence of zeroes and ones.
  - If the set includes a one, put a one in the first place of the sequence; if not, put a zero in the first place.
  - If it includes a two, put a one in the second place of the sequence; if not, put a zero in the first place.
  - For all n, if the set includes n, put a one in the nth place of the sequence.
  - For all n, if the set does not include n, put a zero in the nth place of the sequence.
- Each infinite sequence of zeroes and ones can be taken as the binary representation of a real number between zero and one, the binary representation of their decimal expansions.
- We can easily provide a mapping between the real numbers (points on a line) between zero and one and all the real numbers (points).

Mapping the Reals to the Points Between Zero and One



If you prefer an analytic proof, take $f(x) = \tan \pi(2x-1)/2$.

Transfinite Cardinals

- We can define a sequence of alephs:
  - $\aleph_0, \aleph_1, \aleph_2, \aleph_3, \aleph_4$...
- Set theorists, by various ingenious methods, including the addition of axioms which do not contradict the standard axioms, generate even larger cardinals.
  - ethereal cardinals, subtle cardinals, almost ineffable cardinals, totally ineffable cardinals, remarkable cardinals, superstrong cardinals, superhuge cardinals
- All of these cardinal numbers are transfinite, and larger than any of the sequence of alephs.
- They require special axioms.

Foundations

- The movement from geometry to arithmetic led to further abstraction.
  - Set theory
  - Category theory
- Cantor developed set theory in order to generate his theory of transfinites.
- Frege defined the numbers independently.
- Cantor defined cardinal numbers in terms of ordinal numbers.
- Frege sought independent definitions of the ordinals and cardinals.
- Let’s look at the set-theoretic definitions of ordinal numbers.

ZF

- Substitutivity: $(x)(y)(z)[y=z \supset (y \in x \equiv z \in x)]$
- Pairing: $(x)(y)(\exists z)(u)[u \in z \equiv (u = x \vee u = y)]$
- Null Set: $(\exists x)(y){\sim}x \in y$
- Sum Set: $(x)(\exists y)(z)[z \in y \equiv (\exists v)(z \in v \bullet v \in x)]$
- Power Set: $(x)(\exists y)(z)[z \in y \equiv (u)(u \in z \supset u \in x)]$
- Selection: $(x)(\exists y)(z)[z \in y \equiv (z \in x \bullet \mathscr{F}u)]$, for any formula $\mathscr{F}$ not containing y as a free variable.
- Infinity: $(\exists x)(\varnothing \in x \bullet (y)(y \in x \supset Sy \in x)$

Successor Ordinals and Limit Ordinals

- Ordinal numbers, set-theoretically, are just special kinds of sets, ones which are well ordered.
  - an ordering relation on the set
  - a first element under that order
- For convenience, we standardly pick one example of a well-ordering to represent each particular number.
- To move through the ordinals, we look for the successor of a number.
  - Ordinals generated in this way are called successor ordinals.
- In transfinite set theory, there are limit elements.
  - We collect all the sets we have counted so far into one further set.
  - The union operation
  - If we combine all the sets that correspond to the finite ordinals into a single set, we get another ordinal
  - This limit ordinal will be larger than all of the ordinals in it.
- So, there are two kinds of ordinals: successor ordinals and limit ordinals.

The Ordinal Numbers

$1, 2, 3, \ldots \omega$
$\omega+1, \omega+2, \omega+3 \ldots 2\omega$
$2\omega+1, 2\omega+2, 2\omega+3 \ldots 3\omega$
$4\omega, 5\omega, 6\omega \ldots \omega^2$
$\omega^2, \omega^3, \omega^4 \ldots \omega^\omega$
$\omega^\omega, (\omega^\omega)^\omega, ((\omega^\omega)^\omega)^\omega, \ldots \varepsilon^0$

- Limit ordinals are taken as the completions of an infinite series.
- From Aristotle philosophers and mathematicians denied that there can be any completion of an infinite series.
- Cantor’s diagonal argument shows that there are different levels of infinity.
- We form ordinals to represent the ranks of these different levels of infinity precisely by taking certain series to completion.
- The consistency of Cantor’s theory of transfinites transformed the way we think of infinity.

Natural Numbers in Terms of Ordinals

- Zermelo:
  - 0 = ∅
  - 1 = {∅}
  - 2 = {{∅}}
  - 3 = {{{∅}}}
  - ...
- Von Neumann
  - 0 = ∅
  - 1 = {∅}
  - 2 = {∅, {∅}}
  - 3 = {∅, {∅}, {∅, {∅}}}
  - ...
- Frege has a different way

The Continuum Hypothesis

For Next Wednesday

Logicism

- Frege's grand claim is that mathematics is just logic in complicated disguise.
  - "Arithmetic...becomes simply a development of logic, and every proposition of arithmetic a law of logic, albeit a derivative one" (Frege, *Grundlagen* §87).
- Logicism is the intellectual heir of Leibniz's proposal to reduce all propositions to elementary identities.
- Frege's argument for logicism is that we can define the natural numbers merely by using logic.
  - One step is the reduction of the theory of natural numbers to logic.
  - The other step of Frege's argument is the reduction of all of the rest of mathematics to the theory of natural numbers.

From the Natural Numbers to the Reals

Peano Axioms

P1: 0 is a number

P2: The successor (x’) of every number (x) is a number

P3: 0 is not the successor of any number

P4: If x’=y’ then x=y

P5: If P is a property that may (or may not) hold for any number, and if i. 0 has P; and ii. for any x, if x has P then x’ has P; then all numbers have P.

- P5 is mathematical induction, actually a schema of an infinite number of axioms.

Models of the Peano Axioms

- The Peano axioms refer to a sequence of numbers, which we can call $\mathbb{N}$ = 0, 1, 2, 3...
- Using logic and set theory, we can define standard arithmetic operations:
  - addition
  - multiplication

Integers

- We can define the integers, Z, in terms of the natural numbers by using subtraction.
  - We can define -3 as the ordered pair <5,8>.
  - But -3 could also be defined as <17,20>.
  - We thus take the negative numbers to be equivalence classes of such ordered pairs.
- The equivalence class for subtraction is defined using addition:
  - <a,b> ~ <c,d> iff a + d = b + c.
- So, we can define Z = ...-3, -2, -1, 0, 1, 2, 3... in terms of $\mathbb{N}$, addition, and the notion of an ordered pair.

Rationals

- The rationals, Q, can be defined in terms of the integers, Z, by using ordered pairs of integers.
- a/b :: <a,b>, where '<a,b> ~ <c,d> iff ad = bc'
- So, the rationals are defined in terms of the natural numbers, ordered pairs, and multiplication.

Reals

- The real numbers, $\mathbb{R}$, are on the number line.
- Rational numbers are also part of the number line, but there are real numbers that are not rational.
- The relation between the real numbers and the rational numbers was unclear in the 19th century.
- Both are dense: between any two there is a third.
- The reals are also continuous.

The Problem of Continuity in the Early 19th Century

- Cantor had not yet produced his set theory, which founded his theory of transfinite numbers.
- There was a growing pressure from analysis to provide a solid underpinning of calculus, and its infinitesimals.
- Niels Abel:
  - "the tremendous obscurity which one unquestionably finds in analysis. It lacks so completely all plan and system that it is peculiar that so many men could have studied it. The worst of it is, it has never been treated stringently. There are very few theorems in advanced analysis which have been demonstrated in a logically tenable manner. Everywhere one finds this miserable way of concluding from the special to the general and it is extremely peculiar that such a procedure has led to so few of the so-called paradoxes" (Abel, quoted in Kline, *Mathematical Thought from Ancient to Modern Times* 947).
- Karl Weierstrass: the arithmetization of analysis.
- a function $f(x)$ is continuous at a if for any $\varepsilon > 0$ (no matter how small) there is a $\delta > 0$ such that
  - for all x such that $|x - a| < \delta$, $|f(x) - f(a)| < \varepsilon$.

Dedekind's Definition of the Reals

- The key concept is of a cut, which has become known as a Dedekind cut.
- The real numbers are identified with separations of the rationals, Q, into two sets, Q_1 and Q_2, such that every member of Q_1 is less than or equal to the real number and every member of Q_2 is greater.
  - Even though π is not a rational, it divides the rationals into two such sets.
  - Not all cuts are produced by rational numbers.
  - So, we can distinguish the continuity of the reals from the discontinuity of the rationals on the basis of these cuts.
- Real numbers are thus defined in terms of sets of rationals, the set of rationals below the cut.
- These sets have no largest member, since for any rational less than ,√2 for example, we can find another one larger.
- But, they do have an upper bound in the reals.

Reducing Mathematics to the Theory of Natural Numbers

- By adding our definition of the real numbers in terms of the rational numbers to our definitions of the rationals in terms of the natural numbers, we have defined the reals in terms of the natural numbers.
- Such definitions do two things.
- First, they make it clear that infinitesimals and real numbers are accessible to finite (or at least denumerable) methods.
- Second, they make it plausible that we can reduce the problem of justifying our knowledge of mathematics to the problem of justifying our knowledge of just natural numbers.
- We have an ontological reduction of the objects of analysis to the objects of number theory: we need not assume the existence of any objects beyond the natural numbers in order to model analysis.
- Despite the ontological reduction that Dedekind cuts and the definitions of real numbers in terms of rationals achieves, there remain questions about whether mathematics is reducible to the theory of natural numbers.
  - “One can only speculate on the extent to which Fregean logicism might accommodate some of the contemporary branches of mathematics, such as complex analysis, topology, and set theory” (Shapiro, *Thinking About Mathematics* 113).

Ontological and Methodological Reductions

- Contemporary reductionists (or foundationalists) in mathematics tend to focus their efforts on reducing mathematics to set theory, or to the more-abstract category theory.
- It seems fairly obvious that set theory can serve as a reductionist base for mathematics, ontologically.
- But, there remain questions about what we might call a methodological reduction.
- We could not prove sophisticated theorems in specialized branches of mathematics in set-theoretic terms.
- Even fairly simple number-theoretic inferences become impossibly complex in set theory.
- So, even if we think that there is no need to posit objects other than the natural numbers to serve as models for our mathematical theorems, we do need to posit methods and techniques beyond those of number theory.
- This methodological claim seems to undermine Frege's allegation that we need no special mathematical laws.
  - "The present work will make clear that even an inference like that from *n* to *n* + 1, which on the face of it is peculiar to mathematics, is based on the general laws of logic, and that there is no need of special laws for aggregative thought" (Frege, *Grundlagen* iv).
- Our concern here is more ontological than methodological (this is, in fact, a seminar in metaphysics) and I will put this concern about methodological reduction to the side.

Frege On Mill

- Frege approved of Mill's attempts to avoid psychologistic explanations of mathematics.
- He deplored Mill's claim that mathematical theorems were, strictly speaking, false.
- Frege attacked Mill's definition of number, and his claim that mathematical propositions are mere inductions from sense experience.

Mill and Frege on Number

- Mill said that all numbers were numbers of something, that numbers were properties of physical objects: five apples, ten giraffes.
  - "Ten must mean ten bodies, or ten sounds, or ten beatings of the pulse" (Mill, *A System of Logic* 189).
- Against Mill's definition of number, Frege complains that we see only apples, and not numbers of apples.
- Numbers apply to concepts, not to objects.
  - "While looking at one and the same external phenomenon, I can say with equal truth both "It is a copse" and "It is five trees," or both "Here are four companies" and "Here are 500 men." Now what changes here from one judgement to the other is neither any individual object, nor the whole, the agglomeration of them, but rather my terminology. But that is itself only a sign that one concept has been substituted for another" (Frege, *Grundlagen* §46).
- We can rephrase sentences like 'There are ten bodies', which are paradigmatic for Mill, as 'The number of bodies is ten', in which the number is clearly an object, rather than a predicate.
  - $(\exists x)(Nx \bullet x = 10)$
- Even to apply the number zero to anything is to undermine Mill's account.
  - Zero is a number, but it is not the number of any thing.
- Mill on infinite numbers is also lame.

Frege on Mill’s Inductions

- Mill claimed that mathematical propositions are the results of enumerative inductions.
- Frege notes that induction can not support universal and modal claims.
- Induction can not support knowledge of mathematical objects which do not appear in nature.
- Most importantly, all induction presumes probabilistic reasoning, but, probabilistic reasoning itself relies on arithmetic.
- “Induction [then, properly understood,] must base itself on the theory of probability, since it can never render a proposition more than probably. But how probability theory could possibly be developed without presupposing arithmetical laws is beyond comprehension” (Frege, *Grundlagen* §10).

Frege and Kant on Analyticity

- Frege agrees with Kant that geometry is synthetic *a priori*.
- But Frege denies Kant's claims that arithmetic is synthetic.
  - "Kant obviously - as a result, no doubt, of defining them too narrowly - underestimated the value of analytic judgements, though it seems that he did have some inkling of the wider sense in which I have used the term" (Frege, *Grundlagen* §88).
- Kant analyzes judgments as linking subject concepts with object concepts.
- Frege first worries about when the sentence contains an individual object as the subject.
  - Matt is a cat.
  - The statement's analyticity conditions depend not on whether the subject concept is contained in the predicate concept, but whether the object to which 'Matt' refers necessarily has the property of being a cat.
  - 'Matt' is not a concept.

Fregean Logic and Analyticity

- Second, Frege worries about Kant's analysis of existential statements, e.g. 'There are electrons'.
  - not subject-predicate form
- Predicates are not closed concepts, but functions.
  - In 'Matt is a cat', the predicate is not catness, but '...is a cat'.
  - 'Clinton is between Syracuse and Albany' does not predicate a property of being between Syracuse and Albany of an object, Clinton.
    - It is a three-place function, betweenness.
    - The same predicate holds among New York, Boston and Philadelphia.
    - On the Kantian notion of a predicate, being between Boston and Philadelphia was completely distinct from being between Albany and Syracuse.

Frege, Kant, and Definitions

- Third, and most important, Frege criticizes Kant's notion of a concept, as a list waiting to be unpacked.
- The elements of a concept are linked, not merely appended.
- We can unpack them, tracing them back to their justificatory grounds.
- Definitions should not be mere lists of properties, but chains of inference.
  - "The more fruitful type of definition is a matter of drawing boundary lines that were not previously given at all. What we shall be able to infer from it, cannot be inspected in advance; here, we are not simply taking out of the box again what we have just put into it. The conclusions we draw from it extend our knowledge, and ought therefore, on Kant's view, to be regarded as synthetic; and yet they can be proved by purely logical means, and are thus analytic. The truth is that they are contained in the definitions, but as plants are contained in their seeds, not as beams are contained in a house. Often we need several definitions for the proof of some proposition, which consequently is not contained in any one of them alone, yet does follow purely logically from all of them together" (Frege, *Grundlagen* §88).

Against Kantian Construction

- For Kant, mathematical theorems had to be constructed, synthetically, in intuition.
- Frege laments our lack of ability to construct some mathematical objects.
- "Nought and one are objects which cannot be given to us in sensation. And even those who hold that the smaller numbers are intuitable, must as least concede that they cannot be given in intuition any of the numbers greater than" (*Frege, Grundlagen* §89).

Frege’s Goal

- Frege’s goal is to show that mathematics is analytic by providing a logical system in which one can define and then derive all of mathematics.
- Instead of taking mathematical statements to be true on the basis of our intuitive apprehension of them, Frege argues that we can take them to be true because they are derivable from logical truths using a secure logic.
  - “The fundamental propositions of arithmetic should be proved, if in any way possible, with the utmost rigour; for only if every gap in the chain of deductions is eliminated with the greatest care can we say with certainty upon what primitive truths the proof depends; and only when these are known shall we be able to answer our original questions” (*Frege, Grundlagen* §4).
- Frege was pursuing Leibniz’s distinction between the origins of our beliefs and their justifications.
  - “It can...be asked, on the one hand, by what path a proposition was gradually reached, and on the other hand, in what way it is now finally to be most firmly established. The former question possibly needs to be answered differently for different people; the latter is more definite, and its answer is connected with the inner nature of the proposition concerned. The firmest proof is obviously the purely logical, which, prescinding from the particularity of things, is based solely on the laws on which all knowledge rests. Accordingly, we divide all truths that require justification into two kinds, those whose proof can be given purely logically and those whose proof must be grounded on empirical facts” (Frege, Preface to *Begriffsschrift*, III).

Three Books, Three Principles, One Project

- Almost all of Frege’s work traces back to his logicist project.
- The *Begriffsschrift* (1879) formulates his logical language.
  - Standard contemporary predicate logic is mainly just a notational variant of Frege’s logic.
- The *Grundlagen* (1884) presents a philosophical defense of the logicist project.
- The *Grundgesetze*, in two (of three planned) volumes (1893 and 1903), did some of the technical work promised in the *Grundlagen*.
- After the discovery of Russell’s paradox, Frege despaired of completing the work.

Frege’s Three Principles

- always to separate sharply the psychological from the logical, the subjective from the objective;
- never to ask for the meaning of a word in isolation, but only in the context of a proposition;
- never to lose sight of the distinction between concept and object (Frege, *Grundlagen,* x).
  - against Mill’s definition of number

Against Psychology

- Frege uses the first principle against both Mill and Kant.
  - "Now these distinctions between a priori and a posteriori, synthetic and analytic, concern, as I see it, not the content of the judgement but the justification for making the judgement" (Frege, *Grundlagen* §3).
- In Mill's case, Frege argues that classifying mathematics as empirical because it requires sense experience errs by missing the distinction between the origin of our beliefs and their justifications.
  - "We are making a psychological statement, which concerns solely the content of the proposition; the question of its truth is not touched. In this sense, all of Münchhausen's tales are empirical too; for certainly all sorts of observations must have been made before they could be invented" (Frege, *Grundlagen* §8).
- In Kant's case, Frege argues that making intuition an essential component of mathematical knowledge violates the first principle.
  - "Number is no whit more an object of psychology than, let us say, the North Sea is...If we say, "The North Sea is 10,000 square miles in extent" then neither by "North Sea" nor by "10,000" do we refer to any state of or process in our minds: on the contrary, we assert something quite objective, which is independent of our ideas and everything of the sort" (Frege, *Grundlagen* §26).

The Objectivity of Numbers

- Frege is not claiming that numbers are concrete, just that they are objects.
- Part of the objectivity of numbers comes from the truth value of expressions which contain numbers.
  - ‘snow is white’ is true because there is snow, and it has the property of being white.
  - ‘five is prime’ is true because there is a five, and it has the property of being prime.
- Numbers are not spatio-temporal objects, but they are objects nonetheless.
  - “Not every objective object has a place” (Frege, *Grundlagen* §61).
- The objectivity of mathematics consists in part in its independence from us.
- It also depends on the logicist project more generally.
  - Since mathematical theorems are really just logical truths, they can be seen as applicable most generally, most objectively.
  - Every mathematical theorem will be justified by a purely objective, secure derivation.
  - the *Begriffsschrift* and “The Old Euclidean standards of rigour” (Frege, *Grundlagen* §1).

Proofs

- In the *Begriffsschrift* and the *Grundlagen* Frege expresses his view that there is just one, most estimable proof per proposition.
- Such proofs will be completely gap-free, if long.
- All appeals to intuition must be eliminated, and every step must be guided purely syntactically.
  - "The demand is not to be denied: every jump must be barred from our deductions" (Frege, *Grundlagen* §91).
- To prove that an arithmetic statement is analytic, its derivation from basic logical principles must be possible.
- Only such proofs will determine whether a proposition is analytic or synthetic.
- Analytic statements will be provable from mere logic.
- Synthetic proofs will rely on assumptions given in intuition.
  - "The problem [of determining whether a statement is analytic or synthetic] becomes, in fact, that of finding the proof of the proposition, and of following it up right back to the primitive truths" (Frege, *Grundlagen* §3).
- The success of Frege's work, over all three books, depends on whether he can define the numbers, and the axioms governing them, in purely logical terms.

Frege’s Context Principle

- He defines numbers within sentences, rather than as individual objects.
- Violating the context principle is the source of the Berkeley problem.
  - If we look for meanings of individual number terms, and any other individual terms, we are tempted to think of them as particular ideas in our minds.
  - Lockean conceptualism, Berkeleyan nihilism, or Kantian intuitionism follows.
  - “If the second principle is not observed, one is almost forced to take as the meanings of words mental pictures or acts of the individual mind, and so to offend against the first principle as well” (Frege, *Grundlagen* x).
- Frege’s central point here is the argument against any of the other reductive views about numbers.
  - Kant reduced numbers to intuition;
  - Locke reduced them to abstract ideas, which are psychological;
  - Mill reduced them to empirical objects.
- Frege’s logicism rejects all of these views: mathematical objects are logical.

Numbers as Sets

- Too-short answer: Numbers are certain kinds of sets, sets of sets.
- Frege takes sets to be logical objects, extensions of predicates.
- The extension of a concept is the set of things which fall under that concept, or which have that property.
- For Fregean logic, a predicate is a mathematical function.
- The extension of a predicate is the range of that function.
- The extension of a concept is just the set of things which have the property assigned by the concept.
- To define ‘number’, Frege relies directly on extensions.
  - The number which belongs to the concept *F* is the extension of the concept “equal to the concept *F*” (Frege, *Grundlagen* §68).

Numbers as Objects and Predicates

- Frege's definition tells us when a number belongs to a concept.
- But numbers are objects themselves, not merely properties of concepts.
- Recall Frege's argument against Mill's definition of number, that he takes them as properties of objects when they are really objects themselves.
- Frege must provide a definition of the number terms without appealing merely to when they hold of concepts.
- Numbers are second-order extensions, extensions of extensions.
- In particular, numbers are extensions of all extensions of a particular size.

Zero

- 0 belongs to a concept if nothing falls under the concept.
- Thus, Frege can define zero by appealing to a concept with no extension.
  - 0 is the Number which belongs to the concept “not identical with itself” (Frege, *Grundlagen* §74).
- Again in Russell’s terms, 0 is the set of all sets which are not identical to themselves (i.e. the number of x such that $x \neq x$).

Succession

- The definitions of the rest of the numbers can be generated inductively, using the successor definition.
  - "There exists a concept *F*, and an object falling under it *x* such that the Number which belongs to the concept *F* is *n* and the Number which belongs to the concept 'falling under *F* but not identical with *x*' is *m*" is to mean the same as "*n* follows in the series of natural numbers directly after *m*" (Frege, *Grundlagen* §76.)
- 1 applies to a concept if that concept
  - a) applies to at least one thing, and
  - b) if it applies to two things, they are the same thing.
  - (∃!x)Fx :: (∃x)[Fx • (y)(Fy ⊃ y=x)]
- 1 then may be defined as the number which belongs to the concept 'identical to 0', since there is only one concept 0.
- More succinctly, 1 is the set of all 1-membered sets; 2 is the set of all 2-membered sets.

Bad News for Gottlob

- Russell's paradox showed that Frege's project, as he originally conceived it, was unsuccessful.
- Russell sent word of the paradox to Frege just as the second volume of the *Grundgesetze* was being published.
- Frege added an attempt to avoid the paradox, but it was, in the end, unsuccessful.
- Russell worked out a more thorough, if not fully intuitive, way to avoid the paradox, and used it in his *Principia Mathematica*.
- The paradox also applies to Cantor's earlier set theory, though not in a way that undermines the generation of the transfinite numbers.
- The source of the paradox is an unrestricted axiom of comprehension.

Naive Set Theory

- Frege's logic and Cantor's set theory can both be called naive set theory, for their use of an axiom of comprehension (or abstraction).
- The axiom of comprehension says that any property determines a set.
- For Frege, the relevant version is that every predicate has an extension.
- Frege adds this comprehension claim to his treatment in the *Grundgesetze*, as Axiom 5.
  - $\{x|Fx\} = \{x|Gx\} \equiv (\forall x)(Fx \equiv Gx)$
- Axiom 5 leads to Proposition 91.
  - $Fy \equiv y \in \{x|Fx\}$
- Axiom 5 says that the extensions of two concepts are equal if and only if the same objects fall under the two concepts.
- Proposition 91 says that a predicate F holds of a term iff the object to which the term refers is an element of the set of Fs.

Deriving Russell's paradox

- Take F to be 'is not an element of itself'.
- So, y is not element of itself is expressed:
  - $y \notin y$(which is short for '$\sim y \in y$')
- Take y to be the set of all sets that are not elements of themselves:
  - $\{x | x \notin x\}$
- Substitute that expression for y in the above expression, to get Fy, the left side of proposition 91:
  - $\{x | x \notin x\} \notin \{x | x \notin x\}$
- On the right of proposition 91, we get
  - $\{x | x \notin x\} \in \{x | x \notin x\}$
- Put them together:
  - $\{x | x \notin x\} \notin \{x | x \notin x\} \equiv \{x | x \notin x\} \in \{x | x \notin x\}$
- Which is of the form:
  - $\sim P \equiv P$

Burali-Forti

- The axiom of comprehension is also responsible for the Burali-Forti paradox.
- Consider the set of all ordinals.
- By the axiom of comprehension, any collection, any property, determines a set.
- So, there should be a set of all ordinals.
- By definition, the set of all ordinals will itself be an ordinal larger than itself.
- So, any set of ordinals will be necessarily incomplete.
- Some collections are just too large to form sets.

The Paradoxes as Reductios

- The Burali-Forti paradox shows that there is no set to which every ordinal number belongs.
- Russell's paradox shows that there can not be a set of all sets, a universal set.
- In current axiomatic set theory, we avoid we avoid the paradoxes of unrestricted comprehension by building sets iteratively.
- We start with a few basic axioms, and build up the rest from those, as we have seen with ZF.
- We iterate, or list, the sets from the beginning.

After the Paradox

- Russell introduced a theory of types.
  - According to the theory of types, a set can not be a member of itself.
  - ZF provides a similar solution in that there is no way to generate the problematic sets.
- Frege was able to argue that mathematics reduced to logic, because he claimed just the basic insight that every property determined a set.
- But both Russell's solution and ZF substitute a substantive set theory for Frege's original logical insight.
- Set theory does not appear to be a logical theory, but a mathematical theory.
- Thus, given the paradoxes, Frege was able to show that mathematics is reducible to mathematics, but not to logic.

Is Mathematics Analytic?

- Frege's method of showing that a statement is analytic is to trace it back to its fundamental assumptions.
- We can trace mathematical theorems back to set theory, working the translations I described at the beginning of these notes in reverse.
- So, the question becomes whether the axioms of set theory are analytic.
- For a long time, Frege's program was considered a failure.
- In recent years, renewed interest in Frege's work has led some philosophers to re-think this conclusion.
- It turns out that more technical results can be salvaged from Frege's system than had been thought.
- Interest in what came to be known as neo-logicism was high over the last twenty years, but seems to be waning now.