Knowledge, Truth, and Mathematics

Philosophy 405
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Class 11: Cantor and Transfinite Set Theory

Aristotle on Infinity

- "The infinite does not exist potentially in the sense that it will ever actually have separate existence; its separateness is only in knowledge. For the fact that division never ceases to be possible gives the result that this actuality exists potentially, but not that it exists separately" (Aristotle, *Metaphysics* IX.6, 1048b14-17).
- Distinction between actual and potential infinity
- The potential infinite is real.
  - We can always discover or construct more counting numbers.
  - But, we can never get to the end of the sequence.
- The actual infinite is not real.
  - Actual infinity would be a complete whole of infinite size.
  - Any infinite sequence or construction, like a line, can never be complete.

Zeno's Paradoxes

- The concept of infinite divisibility seems to lead to contradiction.
  - Achilles and the tortoise
  - The arrow
- Aristotle:
  - The infinite sequence of rationals exists in the mind.
  - No infinite sequence could exist in the world
- Thus, the infinite does not exist separately.
- In the middle ages, the actual infinity became aligned with the concept of God, inaccessible to human cognition.

Pressure on Aristotle

- The Calculus of Newton and Leibniz used infinitesimals.
- Leibniz argues that we can have some knowledge of infinites, as we can have knowledge of necessary truths, innately.
  - Let us take a straight line, and extend it to double its original length. It is clear that the second line, being perfectly similar to the first, can be doubled in its turn to yield a third line which is also similar to the preceding ones; and since the same principle is always applicable, it is impossible that we should ever be brought to a halt; and so the line can be lengthened to infinity. Accordingly, the infinite comes from the thought of likeness, or of the same principle, and it has the same origin as do universal necessary truths. That shows how our ability to carry through the conception of this idea comes from something within us, and could not come from sense-experience; just as necessary truths could not by proved by induction or through the senses (Leibniz, *New Essays on Human Understanding* 158).
- Still, Leibniz was careful not to say too much about the infinite.
  - No infinite number that could measure the amount of space or matter.
  - There is no infinite cardinal number.
  - "We do not have the idea of a space which is infinite...and 'nothing is more evident, than the absurdity of the actual idea of an infinite number'" (Leibniz, *New Essays on Human Understanding* 159).

Capitulation to Aristotle in the Moderns

- Locke claims that we have a negative idea of the infinite, but that we lack any positive idea.
- Hume rejects the infinite divisibility of space and time.
- For Kant, since mathematical objects must be constructed in the imagination (albeit *a priori*), the actual infinite is impossible to cognize.
  - "The true transcendental concept of infinitude is this, that the successive synthesis of units required for the enumeration of a quantum can never be completed" (Kant, *Critique* A432/B460).
- Mathematicians, too
  - "I protest against the use of infinite magnitude as something completed, which is never permissible in mathematics. Infinity is merely a way of speaking, the true meaning being a limit which certain ratios approach indefinitely close, while others are permitted to increase without restriction" (Gauss, letter to Schumacher, 1831).

Cantor

- Nearly all mathematicians and philosophers held that to think of infinity as a completed whole would lead to contradiction.
- In contrast, and sparking a mathematical revolution, Cantor shows that we can speak coherently, and without contradiction, about completed infinite sequences.
- Indeed, if we are to think clearly about the real numbers, we must.

The Algebraization of Geometry

an inversion of views about the ultimate nature of mathematics

- From Euclid (and before) geometry was seen as the foundation of arithmetic.
- Descartes inverted that view, taking algebra and arithmetic to be the foundation of geometry.
- Taking arithmetic to be foundational allows for greater abstraction.

Using Graphs for Functions

- Consider how one might think about the nature of x^3.
- For the Greeks, it is the volume of a cube with side length x.
- Now, consider x^5.
  - On the (Euclidean) geometric view, x^5 is a five-dimensional cube.
  - On the (Cartesian) analytic view, x^5 is just another curve on a standard Cartesian plane.
  - It is nothing more than a more rapidly growing curve in two-dimensions.
- Thus, analysis, and the algebraization of geometry, opened up mathematics to a wider, more general treatment of functions.
- Any function, indeed any equation of two variables, can be graphed.
- The graph of a function is complete.
  - It defines a range for any given domain, including irrationals.
- We thus see a shape, or curve, as containing all magnitudes.

The Limitations of Graphs

- Mathematicians started seeing geometry as the study of curves, more generally.
- If we could graph every function, and find an algebraic representation of every curve, the question of whether algebra or geometry is fundamental would be moot.
- In the early days of analysis, it was assumed that there is a graph for every algebraic function, but not a function for every curve.
  - The Euclidean view continued to dominate.
- Mid-eighteenth century views
  - Euler identified functions with their graphs.
  - D'Alembert identified functions with their algebraic expressions.
- In the eighteenth and nineteenth centuries, mathematicians began to discover algebraic functions which both behaved nicely (e.g. could be integrated) but which were too pathological to graph.

Pathological Functions

- Bernoulli's equation for the motion of the vibrating string.
  - $y = \alpha \sin \pi x/a + \beta \sin 2\pi x/a + \gamma \sin 3\pi x/a + \delta \sin 4\pi x/a + \ldots$
  - As we add terms, the function becomes increasingly fecund, and the graph becomes increasingly unable to represent it.
  - Still, it is a perfectly well-defined algebraic formula.
- Riemann and Weierstrass explored an everywhere-continuous but nowhere-differentiable function, now known as the Weierstrass function.

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$$f_a(x) = \sum_{k=1}^{\infty} \frac{\sin(\pi k^a x)}{\pi k^a}$$

Algebra Wins!

- Some algebraic functions are geometrically ineffable.
- “Functions had been conceived in inseparable association with their graphs - the ‘paths’ traced by points moving in accordance with an algebraically expressed law. But when that law dictates a ‘motion’ which involves infinitely frequent oscillations, or infinitely frequent jumps, it is a path which can no longer be geometrically traced either in the mind’s eye or on paper. But if the law can be written and by this means rationally investigated, the graph of the function must be presumed, in some sense, to exist and to be a totality of points over which our only hold is now algebraic” (Tiles*, The Philosophy of Set Theory* 82).
- The graph has lost its utility.
- Functions exist beyond our ability to picture them.

How Deep is the Discrepancy?

between the picture of a function (graph)
and the numbers over which it ranges

- We know that there are more points on a line than rational numbers, for example, since there are incommensurable numbers.
- Now, it looks like there are even more numbers, or more structure to the numbers, than there are geometric points or regions.
- But, mathematicians lacked the tools to express the discrepancy.
- Cantor's work on transfinite numbers was an attempt to explore the fine structure of the numbers, and to see the relations among natural numbers, real numbers, and points on a line.
- His work allows us to distinguish among different levels of infinity, and to reject Aristotle's claim that the only infinite we can understand is potential.

The Infinite Hotel

- The hotel is fully booked.
- A new guest arrives.
  - Shift every current guest from Room n to Room n+1.
  - For any finite number of guests, m, shift all current guests from Room n to Room n+m.
- An infinite bus with an infinite number of guests arrives.
  - Shift every current guest from Room n to Room 2n.
  - All the even-numbered rooms are filled, but the odd-numbered rooms are vacant.
- An infinite number of infinite busloads of guests arrives.
  - Shift all current guests from Room n to Room 2^n.
  - Lots of empty rooms
    - – Place the people on the first bus in room numbers 3^n
    - – the people in the second bus go to rooms 5^n
    - – the people in the third bus go to rooms 7^n
    - – etc.
  - There will be lots of empty rooms left over!
- Are there any sets of guests that the infinite hotel could not accommodate?
  - What is the fine structure of the numbers?
  - Are there different sizes of infinity?

Cardinals and Ordinals

- Numbers have at least two different functions:
  - measuring the size of a set
  - ordering, or ranking, a series
- When we use numbers to measure size, we use cardinality.
- When we use them to measure rank, we use ordinality.
- It has become useful to consider the numbers in their different uses as different objects altogether.
  - Ordinal numbers (first, second, third...) measure rank.
  - Cardinal numbers (1, 2, 3...) measure size.
  - We use one-one correspondence to characterize cardinal numbers.

Size and One-One Correspondence

- With finite numbers, the size of a group is the same as the correspondence between the objects in the group and some initial segment of the natural numbers.
- Hume's principle
  - "We are possessed of a precise standard by which we can judge of the equality and proportion of numbers and, according as they correspond or not to that standard, we determine their relations without any possibility of error. When two numbers are so combined as that the one has always a unit answering to every unit of the other, we pronounce them equal..." (Hume, *Treatise* §I.III.1, p 8).
- With transfinite numbers, two concepts of size diverge.
  - The size of the integers seems to be bigger than the size of the even numbers

    The size of a whole is greater than the size of its proper part.

    The even numbers are a proper part of the integers.
  - The even numbers (E) and the integers (N) can be put into one-one correspondence with each other.

    E: 2, 4, 6, 8...

    ↕ ↕ ↕ ↕

    I: 1, 2, 3, 4...

Two Concepts of Size

- Two sets have the same *size*$_h$ if they can be put in one-one correspondence with each other.
- Two sets have the same *size*$_w$ if it is not possible to put either in one-one correspondence with a proper part of itself.
- So, N and E have the same size$_h$ but different size$_w$s.
- "N and E will have the same 'number' of elements even though there are infinitely many numbers in N which are not in E, so that in this sense N is 'bigger than' E. This suggests that the elements of an infinite set are without number not just because the notion of number, as a measure of size, can get no grip here. All infinite sets *seem to* come out as being of the same 'size' if one-one correspondence is taken as indicating the sameness of size for sets" (Tiles, *The Philosophy of Set Theory* 97; emphasis added).
- Despite appearances, not all infinite numbers have the same size$_h$.

Lists and Infinite Sizes

- Cantor relies on $size_h$ to generate different kinds of infinite, or transfinite, numbers.
- When we list the members of something, we are putting them into one-one correspondence with the natural numbers.
  - We can list the even numbers.
  - We can list the prime numbers.
  - We can even list the rational numbers.

	1	2	3	4	5	6	…
1	1/1	1/2	1/3	1/4	1/5	1/6	…
2	2/1	2/2	2/3	2/4	2/5	2/6	…
3	3/1	3/2	3/3	3/4	3/5	3/6	…
4	4/1	4/2	4/3	4/4	4/5	4/6	…
5	5/1	5/2	5/3	5/4	5/5	5/6	…
6	6/1	6/2	6/3	6/4	6/5	6/6	…
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- When we list the members of something, we are putting them into one-one correspondence with the natural numbers.
  - We can list the even numbers.
  - We can list the prime numbers.
  - We can even list the rational numbers.
  - All of these sets have the $size_h$, despite having different $size_w$s.
- If there were some kinds of sets whose members could not be put into a list, then that set would be strictly larger than the set of natural numbers, both in $size_h$ and $size_w$.
- We could show that there are different sizes of infinity, whatever way we measure size.

The Diagonal Argument

- Cantor shows that we can not list the real numbers.
  The real numbers may be represented as their decimal expansions.
- Imagine that we have a list of all the real numbers.
- Let's represent that list abstractly, using a concatenation of variables.

  L $a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7$...
  $b_1\ b_2\ b_3\ b_4\ b_5\ b_6\ b_7$...
  $c_1\ c_2\ c_3\ c_4\ c_5\ c_6\ c_7$...
  $d_1\ d_2\ d_3\ d_4\ d_5\ d_6\ d_7$...
  ...
- Consider N = $a_1\ b_2\ c_3\ d_4\ e_5\ f_6\ g_7$...
- Create N*:
  - add one to each digit of N other than nine
  - replace all nines in N with zeroes
- N* is certainly not in L.
  N* is different from the first number in L in its first digit
  different from the second number in L in its second digit
  and so on.
- All possible lists of real numbers are necessarily incomplete.
- There are strictly more real numbers than natural numbers, on both a one-once correspondence notion of size ($size_h$) and a whole-is-greater-than-the-sum-of-its-parts notion of size ($size_w$).

Properties of Cardinal Numbers

- Some properties of finite numbers do not extend to infinite numbers.
- For transfinite numbers:

  $a + 1 = a$

  $2a = a$

  $a \bullet a = a$

- For all cardinal numbers a, b, and c, whether finite or transfinite, the following properties hold:

  1. $a+b=b+a$
  2. $ab=ba$
  3. $a + (b + c) = (a + b) + c$
  4. $a \bullet (b \bullet c) = (a \bullet b) \bullet c$
  5. $a \bullet (b + c) = ab + ac$
  6. $a^{(b+c)} = a^b \bullet a^c$
  7. $(ab)^c = a^c \bullet b^c$
  8. $(a^b)^c = a^{bc}$

The Big One

- For infinite n, sets with n members are the same size as sets with n+1 members, or with 2n members, or with n^2 members.
- With infinite numbers, it is not always clear that what we think of as a larger set is in fact larger.
- We might conclude that sets with n members are the same size as sets with 2^n members.
- This conclusion would be erroneous.

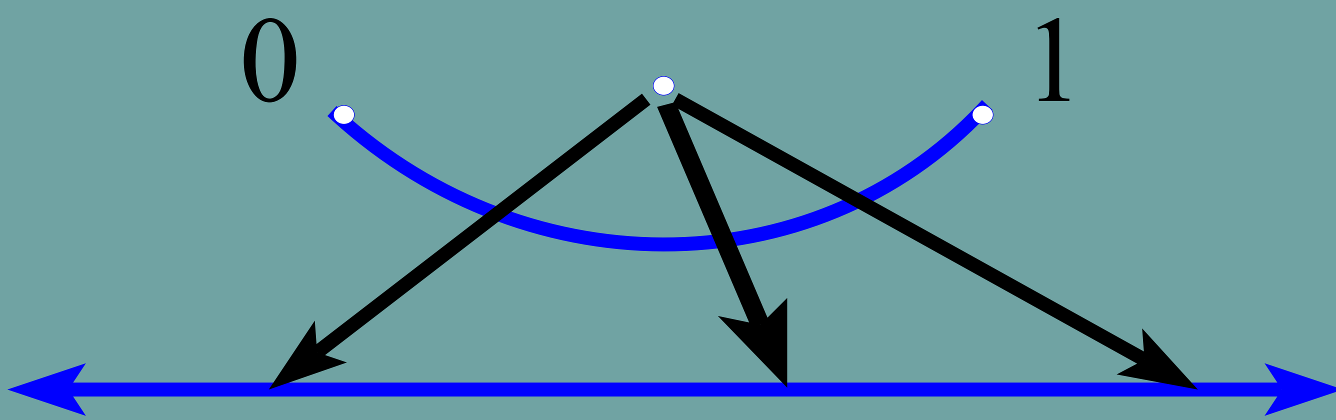
  9. $2^a > a$

  $\mathbb{C}(\wp(A)) > \mathbb{C}(A)$.
- This fact was once called Cantor's Paradox.
- Now it's called Cantor's Theorem.
- The proof of the theorem is a set-theoretic version of the diagonalization argument.
- See the class notes.

The Natural Numbers and the Real Numbers

- Let's call the size of the natural numbers $\aleph_0$.
- Every subset of the natural numbers can be uniquely correlated with an infinite sequence of zeroes and ones.
  - If the set includes a one, put a one in the first place of the sequence; if not, put a zero in the first place.
  - If it includes a two, put a one in the second place of the sequence; if not, put a zero in the first place.
  - For all n, if the set includes n, put a one in the nth place of the sequence.
  - For all n, if the set does not include n, put a zero in the nth place of the sequence.
- Each infinite sequence of zeroes and ones can be taken as the binary representation of a real number between zero and one, the binary representation of their decimal expansions.
- We can easily provide a mapping between the real numbers (points on a line) between zero and one and all the real numbers (points).

Mapping the Reals to the Points Between Zero and One



If you prefer an analytic proof, take $f(x) = \tan \pi(2x-1)/2$.

Real Numbers and Subsets of Natural Numbers

- We can thus correlate the subsets of the natural numbers with the real numbers, and thus with the points on a line.
- Then the real numbers, and the real plane, are the size of the set of subsets of the natural numbers.
- The set of subsets of a set is called its power set.

Exponentiation and Transfinite Numbers

- We can generate larger and larger cardinals by exponentiation.
- We thus define a sequence of alephs:
  - $\aleph_0$, $\aleph_1$, $\aleph_2$, $\aleph_3$, $\aleph_4$…
- Further, set theorists, by various ingenious methods, including the addition of axioms which do not contradict the standard axioms, generate even larger cardinals.
  - ethereal cardinals, subtle cardinals, almost ineffable cardinals, totally ineffable cardinals, remarkable cardinals, superstrong cardinals, superhuge cardinals
- All of these cardinal numbers are transfinite, and larger than any of the sequence of alephs.

Foundations

- The movement from geometry to arithmetic led to further abstraction.
  - Set theory
  - Category theory
- Cantor developed set theory in order to generate his theory of transfinites.
- Frege defined the numbers independently.
- Cantor defined cardinal numbers in terms of ordinal numbers.
- Frege sought independent definitions of the ordinals and cardinals.
- Let’s look at the set-theoretic definitions of ordinal numbers.

ZF

- Substitutivity: $(x)(y)(z)[y=z \supset (y \in x \equiv z \in x)]$
- Pairing: $(x)(y)(\exists z)(u)[u \in z \equiv (u = x \vee u = y)]$
- Null Set: $(\exists x)(y){\sim}x \in y$
- Sum Set: $(x)(\exists y)(z)[z \in y \equiv (\exists v)(z \in v \bullet v \in x)]$
- Power Set: $(x)(\exists y)(z)[z \in y \equiv (u)(u \in z \supset u \in x)]$
- Selection: $(x)(\exists y)(z)[z \in y \equiv (z \in x \bullet \mathscr{F}u)]$, for any formula $\mathscr{F}$ not containing y as a free variable.
- Infinity: $(\exists x)(\varnothing \in x \bullet (y)(y \in x \supset Sy \in x)$

Successor Ordinals and Limit Ordinals

- Ordinal numbers, set-theoretically, are just special kinds of sets, ones which are well ordered.
  - an ordering relation on the set
  - a first element under that order
- For convenience, we standardly pick one example of a well-ordering to represent each particular number.
- To move through the ordinals, we look for the successor of a number.
  - Ordinals generated in this way are called successor ordinals.
- In transfinite set theory, there are limit elements.
  - We collect all the sets we have counted so far into one further set.
  - The union operation
  - If we combine all the sets that correspond to the finite ordinals into a single set, we get another ordinal
  - This limit ordinal will be larger than all of the ordinals in it.
- So, there are two kinds of ordinals: successor ordinals and limit ordinals.

The Ordinal Numbers

$1, 2, 3, \ldots \omega$
$\omega+1, \omega+2, \omega+3 \ldots 2\omega$
$2\omega+1, 2\omega+2, 2\omega+3 \ldots 3\omega$
$4\omega, 5\omega, 6\omega \ldots \omega^2$
$\omega^2, \omega^3, \omega^4 \ldots \omega^\omega$
$\omega^\omega, (\omega^\omega)^\omega, ((\omega^\omega)^\omega)^\omega, \ldots \varepsilon^0$

- Limit ordinals are taken as the completions of an infinite series.
- From Aristotle philosophers and mathematicians denied that there can be any completion of an infinite series.
- Cantor's diagonal argument shows that there are different levels of infinity.
- We form ordinals to represent the ranks of these different levels of infinity precisely by taking certain series to completion.
- The consistency of Cantor's theory of transfinites transformed the way we think of infinity.

Natural Numbers in Terms of Ordinals

- Zermelo:
  - $0 = \varnothing$
  - $1 = \{\varnothing\}$
  - $2 = \{\{\varnothing\}\}$
  - $3 = \{\{\{\varnothing\}\}\}$
  - ...
- Von Neumann
  - $0 = \varnothing$
  - $1 = \{\varnothing\}$
  - $2 = \{\varnothing, \{\varnothing\}\}$
  - $3 = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}$
  - ...
- Frege has a different way

The Continuum Hypothesis

Difficult Questions

- Is Fermat's theorem true?
- Is Goldbach's conjecture true?
- Is the parallel postulate true?

- Some questions, like Fermat's conjecture, are clearly answered affirmatively.
- We expect the same kind of answer for Goldbach's Conjecture.
- The parallel postulate is more interesting.
  - It can fail, but it can also hold.
  - The question is ill-formed.
  - There are different kinds of spaces, and they are each defined by a different answer to the parallel postulate.

The Continuum Hypothesis

- Cantor provided a method for generating larger and larger transfinite numbers.
- He shows that the cardinal number of the reals is equal to .
- He also shows that $2^\wedge\aleph_0$ is greater than $\aleph_0$.
- Cantor's theorem does not show, however, that it is the next greater transfinite number.
- The continuum hypothesis is that $\aleph_1 = 2^\wedge\aleph_0$.
- More abstractly, the generalized continuum hypothesis is that $\aleph_{n+1} = 2^\wedge\aleph_n$.
- Cantor believed that the continuum hypothesis was true, but he could not prove it.

Against CH

- Certain operations which generate larger cardinal numbers, like exponentiation, skip numbers in between.
  - Only succession actually gives the next number.
- We do not even know that the sizes of transfinite cardinal numbers can be ordered linearly.
- “It has been assumed that cardinalities, or cardinal numbers, can be arranged in a single linear order. But just making that assumption does not tell us anything about the nature of the cardinal number ‘sequence’, about how to establish where any given cardinality lies in it, or even whether it is correct to talk about there being a *next* cardinal number after $\aleph_0$. Our assumption does not rule out the possibility that infinite cardinalities might, like the rational numbers, be densely ordered. If that were the case, there would always be another cardinal number between any two given cardinalities and given any cardinal number there would be no ‘next ‘one” (Tiles, *The Philosophy of Set Theory* 103).

The Independence of CH

- In 1940, Kurt Gödel showed that the continuum hypothesis is consistent with the standard axioms of set theory.
- In 1963, Paul Cohen showed that its negation is consistent with set theory.
- Thus, the continuum hypothesis is independent of the standard axioms.
- We can consistently consider the continuum to be of all different sizes
  - $\aleph_1$, $\aleph_2$, $\aleph_3$, etc.
- Further, none of the large cardinal axioms proposed settle the question.

A Solution?

- We could settle the question of the size of the continuum by adopting stronger axioms for set theory.
- Some mathematicians believe that the continuum hypothesis, even the generalized version, is so intuitively true that we should just adopt it, or an equivalent, as part of set theory.
- Gödel
- Alternatively, we could take the question to be ill-formed, like the question of whether the parallel postulate is true.
- Perhaps there are different set theories, with different sizes of the continuum.