Hamilton College
Russell Marcus

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Cantor[1]

I. A Brief History of Infinity

Aristotle distinguishes between actual and potential infinity.
A potential infinity is exemplified by the counting numbers.
We can always discover or construct more of them.
But, we can never get to the end of the sequence.
An actual infinity would have to be a complete whole of infinite size.
Bounded figures and individual numbers actually exist, according to Aristotle.
But, there is no actual infinity, since any infinite sequence or construction, like a line, can never be complete.
The potential infinite is real.
There are potentially infinitely many counting numbers.
But the actual infinite is not.
We can not speak sensibly about the counting numbers as a completed whole.

> The infinite does not exist potentially in the sense that it will ever actually have separate existence; its separateness is only in knowledge. For the fact that division never ceases to be possible gives the result that this actuality exists potentially, but not that it exists separately (Aristotle, *Metaphysics* IX.6, 1048b14-17).

Aristotle's distinction between actual and potential infinity applies both to infinitely large quantities and infinitely small ones.
His opposition to the actual infinite comes, in part, as a response to Zeno's paradoxes.
The concept of infinite divisibility seems clearly to lead to contradiction.
Each time Achilles approaches the tortoise, the tortoise moves slightly ahead of Zeno.
If time were really infinitely divisible, and Achilles had to complete an infinite sequence of moves before reaching the tortoise, Achilles could never catch it.
Similarly, if time and space were infinitely divisible, then an arrow would have to fly to its halfway point to reach any endpoint, and thus it must have to complete an unending, actually infinite series of movements in order to get anywhere.

Aristotle's distinction between actual and potential infinity allows him to divide the infinite sequence of rationals, which exists in the mind, from the world, in which no such infinite sequence could exist.
Thus he says that the infinite does not exist separately.
In the middle ages, the actual infinity became aligned with the concept of God, inaccessible to human cognition.

The calculus of Newton and Leibniz put some pressure on Aristotle's distinction.
It requires working with an infinite number of infinitesimals.
Leibniz argues that we can have some knowledge of infinites, as we can have knowledge of necessary

[1] These notes, especially the second section, are based in large part on Mary Tiles' *The Philosophy of Set Theory*, Chapters 4 and 5.

truths, innately.

> Let us take a straight line, and extend it to double its original length. It is clear that the second line, being perfectly similar to the first, can be doubled in its turn to yield a third line which is also similar to the preceding ones; and since the same principle is always applicable, it is impossible that we should ever be brought to a halt; and so the line can be lengthened to infinity. Accordingly, the infinite comes from the thought of likeness, or of the same principle, and it has the same origin as do universal necessary truths. That shows how our ability to carry through the conception of this idea comes from something within us, and could not come from sense-experience; just as necessary truths could not by proved by induction or through the senses (Leibniz, *New Essays on Human Understanding* 158).

Still, Leibniz was careful not to say too much about the infinite.
For Leibniz, matter and space is actually infinitely divided.[2]
But there is no infinite number that could measure the amount of space or matter.
Similarly, the natural numbers form an infinite sequence, but there is no infinite number.
Aristotle's worries about actual infinity remained.

> We do not have the idea of a space which is infinite...and 'nothing is more evident, than the absurdity of the actual idea of an infinite number' (Leibniz, *New Essays on Human Understanding* 159).

Locke, who Leibniz quotes approvingly in the previous passage, claims that we have a negative idea of the infinite, but that we lack any positive idea.
Hume rejects the infinite divisibility of space and time.
For Kant, since mathematical objects must be constructed in the imagination (albeit *a priori*), the actual infinite is impossible to cognize.

> The true transcendental concept of infinitude is this, that the successive synthesis of units required for the enumeration of a quantum can never be completed (Kant, *Critique* A432/B460).

Until Cantor's work, in the mid-nineteenth century, mathematicians still maintained, for the most part, Aristotle's distinction between potential and actual infinity.

> I protest against the use of infinite magnitude as something completed, which is never permissible in mathematics. Infinity is merely a way of speaking, the true meaning being a limit which certain ratios approach indefinitely close, while others are permitted to increase without restriction (Gauss, letter to Schumacher, 1831).

The central claim, which nearly all mathematicians and philosophers held in regard to infinity, in defense against the paradoxes, is that to think of infinity as a completed whole would lead to contradiction.
In contrast, and sparking a mathematical revolution, Cantor shows that we can speak coherently, and without contradiction, about completed infinite sequences.
Indeed, if we are to think clearly about the real numbers, we must.

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[2] Actually, Leibniz is an idealist who does not believe in the reality of matter or space. But, he writes as if he does, just as the reinterpretive anti-platonist writes about mathematics.

II. Moving Towards Actual Infinity

By the nineteenth century, two centuries of developments in analysis had put substantial mathematical pressure on Aristotle's distinction between the potential and actual infinite.
The algebraization of geometry led to an inversion of views about the ultimate nature of mathematics.
From Euclid (and before) geometry was seen as the foundation of arithmetic.
Even Newton saw the calculus as essentially geometric.
But, Descartes inverted that view, taking algebra and arithmetic to be the foundation of geometry.
By grounding mathematics on arithmetic, mathematics was able to become more abstract, and more formal, and less tied to sense experience.

Consider how one might think about the nature of x^3.
For the Greeks, it is the volume of a cube with side length x.
When they talked about cubic numbers, they were talking about geometric properties.
Now, consider x^5.
If we take the (Euclidean) geometric view, x^5 is a five-dimensional cube, a sort of mind-blowing creature.
If we take the (Cartesian) analytic view, x^5 is just another curve.
We can plot $f(x)=x^5$ on a standard Cartesian plane.
It is nothing more than a more rapidly growing curve in two-dimensions.
Thus, analysis, and the algebraization of geometry opened up mathematics to a wider, more general treatment of functions.
Any function, indeed any equation of two variables, can be graphed.
The graph of a function is complete, in the sense that it defines a range for any given domain, including irrationals.
We thus see a shape, or curve, as containing all magnitudes.

At the same time as the study of mathematics became more general and more abstract, mathematicians moved from seeing geometry itself as the study of closed figures to seeing it as the study of curves, more generally.
If we could graph every function, and find an algebraic representation of every curve, the question of whether algebra or geometry is fundamental would be moot.
In the early days of analysis, it was assumed that there is a graph for every algebraic function, but not a function for every curve, so the Euclidean view continued to dominate.
In the mid-eighteenth century, Euler identified functions with their graphs, while D'Alembert, identified functions with their algebraic expressions.
But, in the eighteenth and nineteenth centuries, mathematicians began to discover algebraic functions which both behaved nicely (e.g. could be integrated) but which were too pathological to graph.

Consider Bernoulli's equation for the motion of the vibrating string.

$$y = \alpha \sin \pi x/a + \beta \sin 2\pi x/a + \gamma \sin 3\pi x/a + \delta \sin 4\pi x/a + \ldots$$

This single equation is the result of a superposition of an infinity of curves.
As we add terms, the function becomes increasingly fecund, and the graph becomes increasingly unable to represent it.
Still, it is a perfectly well-defined algebraic formula.

In the nineteenth century, Riemann and Weierstrass explored an everywhere-continuous but nowhere-differentiable function, now known as the Weierstrass function.[3]

$$f_a(x) = \sum_{k=1}^{\infty} \frac{\sin(\pi k^a x)}{\pi k^a}$$, originally defined for a=2

The Weierstrass function is like a fractal, in its fineness.
Again, the geometric view of mathematics was seen as more limiting, and less effective.
Some algebraic functions are geometrically ineffable.
Mathematicians thus moved beyond graphical representations to treat algebraic functions more broadly.

> Functions had been conceived in inseparable association with their graphs - the 'paths' traced by points moving in accordance with an algebraically expressed law. But when that law dictates a 'motion' which involves infinitely frequent oscillations, or infinitely frequent jumps, it is a path which can no longer be geometrically traced either in the mind's eye or on paper. But if the law can be written and by this means rationally investigated, the graph of the function must be presumed, in some sense, to exist and to be a totality of points over which our only hold is now algebraic (Tiles, *The Philosophy of Set Theory* 82).

The graph has lost its utility.
Functions exist beyond our ability to picture them.

Still, it is unclear how deep the discrepancy is between the picture of a function and the numbers over which it ranges.
We know that there are more points on a line than rational numbers, for example, since there are incommensurable numbers.
Now, it looks like there are even more numbers, or more structure to the numbers, than there are geometric points or regions.
But, mathematicians lacked the tools to express the discrepancy.
Cantor's work on transfinite numbers was an attempt to explore the fine structure of the numbers, and to see the relations among natural numbers, real numbers, and points on a line.
His work allows us to distinguish among different levels of infinity, and to reject Aristotle's claim that the only infinite we can understand is potential.

III. The Infinite Hotel

To get in the mood for infinite numbers, let's consider the infinite hotel.
The infinite hotel has infinitely many rooms.
Imagine that the hotel is fully booked.
A new guest arrives.
We can add the new guest, by shifting every current guest from Room n to Room n+1.
Then, Room 1 will be available for the arriving guest.

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[3] According to Wikipedia, Weierstrass's original function was slightly different.

We can perform the same procedure repeatedly, adding one guest at a time.
We can generalize the procedure to add any finite number of guests, m, by shifting all current guests from Room n to Room n+m.

Next, an infinite bus with an infinite number of guests arrives.
If we tried to shift all guests from Room n to Room n+(the number of guests on the bus), we would have to complete an actual infinity.
We would have no first room for the current guests, no place to send them.
But, we can accommodate them by using a new procedure.
We shift every current guest from Room n to Room 2n.
Now, all the even-numbered rooms are filled, but the odd-numbered rooms are vacant.
We can put the infinite number of new guests in the odd-numbered rooms.

Next, an infinite number of infinite busloads of guests arrives.
We can still accommodate them.
Shift all current guests from Room n to Room 2^n.
Now, all the rooms that are powers of two are filled, leaving lots of empty rooms.
We can place the people on the first bus in room numbers 3^n (for n people on the bus), the people in the second bus in rooms 5^n, the people in the third bus to rooms 7^n, and so on for each (prime number)^n.
Since there are an infinite number of prime numbers, there will be an infinite number of infinite such sequences.
And, there will be lots of empty rooms left over!

A natural question to arise is whether there are any sets of guests that the infinite hotel could not accommodate.
This question is precisely a question about the fine structure of the numbers, and about whether there are different sizes of infinity.

IV. Two Notions of Cardinal Number

The splitting headache which may arise from thinking about infinite numbers may correspond to a split among different ways to think about numbers.
Numbers have at least two different functions: measuring the size of a set; and ordering, or ranking, a series.
When we use numbers to measure size, we use the property of the numbers called cardinality.
When we use them to measure rank, we use the property called ordinality.
It has become useful to consider the numbers in their different uses as different objects altogether.
Thus we have cardinal numbers and ordinal numbers.
Ordinal numbers (first, second, third...) measure rank.
We use cardinal numbers (1, 2, 3...) to measure size.
We also use one-one correspondence to characterize cardinal numbers.

With finite numbers, these two approaches converge.
The size of a group is the same as the correspondence between the objects in the group and some initial segment of the natural numbers.
If we have five hedgehogs, we can line them up and give them each a number from one to five.
The view that we can define numbers in terms of one-one correspondence became known, in the

twentieth century, as Hume's principle, due to a passage from the *Treatise*.

> We are possessed of a precise standard by which we can judge of the equality and proportion of numbers and, according as they correspond or not to that standard, we determine their relations without any possibility of error. When two numbers are so combined as that the one has always a unit answering to every unit of the other, we pronounce them equal... (Hume, *Treatise* §I.III.1, p 8).

With transfinite numbers, as with the infinite hotel, the two concepts diverge.
For example, the size of the integers seems to be bigger than the size of the even numbers.
As Tiles points out, we have intuitions that the size of a whole is greater than the size of its proper part.
And, the even numbers are a proper part of the integers.
But, the even numbers (E) and the integers (N) can be put into one-one correspondence with each other.

E: 2, 4, 6, 8...
↕ ↕ ↕ ↕
I: 1, 2, 3, 4...

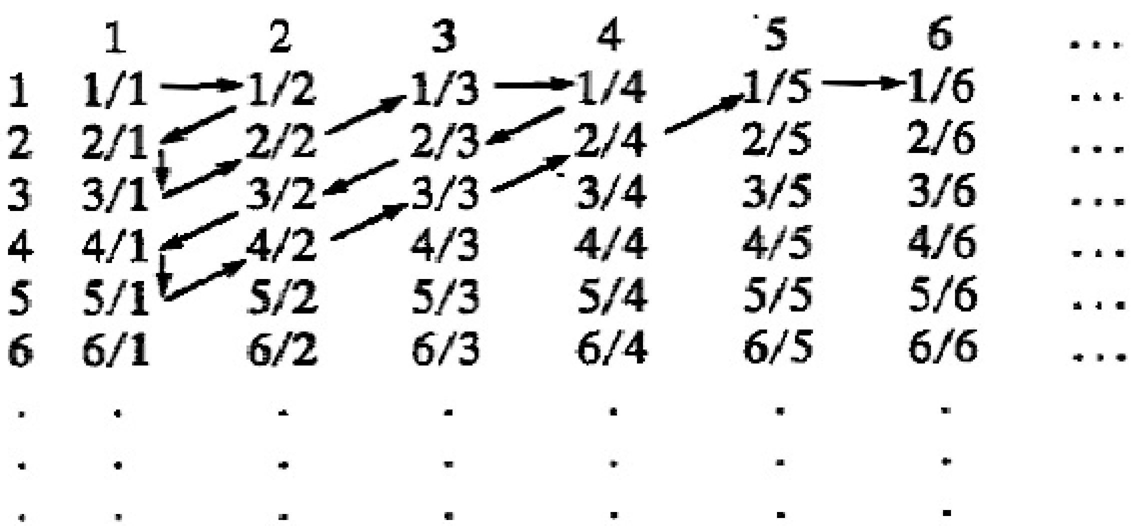
Let's give names to these different concepts of size.
Two sets have the same $\textit{size}_h$ (for Hume) if they can be put in one-one correspondence with each other.
Two sets have the same $\textit{size}_w$ (for the whole is greater than the sum of its parts) if it is not possible to put either in one-one correspondence with a proper part of itself.
So, N and E have the same size_h but different size_ws.

> N and E will have the same 'number' of elements even though there are infinitely many numbers in N which are not in E, so that in this sense N is 'bigger than' E. This suggests that the elements of an infinite set are without number not just because the notion of number, as a measure of size, can get no grip here. All infinite sets *seem to* come out as being of the same 'size' if one-one correspondence is taken as indicating the sameness of size for sets (Tiles, *The Philosophy of Set Theory* 97; emphasis added).

In contrast to the way it seems, though, it is not the case that all infinite numbers have the same size_h.

V. Cantor's Diagonal Argument

Cantor relies on size_h to generate different kinds of infinite, or transfinite, numbers.
When we list the members of something, we are putting them into one-one correspondence with the natural numbers.
We can list the even numbers.
We can list the prime numbers.
We can even list the rational numbers.



Just follow the arrows to construct the complete list.
All of these sets have the size_h, despite having different size_ws.
If there were some kinds of sets whose members could not be put into a list, then that set would be strictly larger than the set of natural numbers, both in size_h and size_w.
We could show that there are different sizes of infinity, whatever way we measure size.

Cantor shows that we can not make certain lists.
In terms of the infinite hotel, he shows that there are sets of guests we could not accommodate.
In particular, Cantor developed an argument which proves that we can not list the real numbers.
The real numbers may be represented as their decimal expansions, many of which are non-repeating and non-terminating.
Imagine that we have a list of all the real numbers, like the list we can generate of the rationals.
Let's represent that list abstractly, using a concatenation of variables.

L $\quad a_1\, a_2\, a_3\, a_4\, a_5\, a_6\, a_7 \ldots$
$b_1\, b_2\, b_3\, b_4\, b_5\, b_6\, b_7 \ldots$
$c_1\, c_2\, c_3\, c_4\, c_5\, c_6\, c_7 \ldots$
$d_1\, d_2\, d_3\, d_4\, d_5\, d_6\, d_7 \ldots$
...

By hypothesis, L contains all real numbers.
We can, by Cantor's technique, demonstrate a number which does not appear in L.

Consider the following number

$N = a_1\ b_2\ c_3\ d_4\ e_5\ f_6\ g_7...$

N could be in L.
But, we can change each digit in N to create a new number N*.
To construct N*, add one to each digit of N other than nine, and replace all nines in N with zeroes.
N* is certainly not in L.
For, N* is different from the first number in L in its first digit, different from the second number in L in its second digit, and so on, for all numbers on the list.
We could add N* to L, to make a new list, L*.
But the same procedure allows us to form a new number that's not on L*.
However complete we make our list, we can always find a number that's not in it.
Thus, all possible lists of real numbers are necessarily incomplete.
We are in principle prevented from establishing a one-one correspondence between the natural numbers and the real numbers.
There are strictly more real numbers than natural numbers, on both a one-once correspondence notion of size ($size_h$) and a whole-is-greater-than-the-sum-of-its-parts notion of size ($size_w$).

The preceding proof is called a diagonal argument, due to its method of producing N*, along the diagonal of the list.
Tiles presents a diagonal argument much more neatly by using the binary representation of each real number, pp 109-10.
For finite numbers, the concept of a whole being larger than its proper part is consistent with the concept of size as measured by one-one correspondence.
For infinite numbers, the two concepts diverge.
Mathematicians now almost universally think of size as one-one correspondence, as $size_h$.
Thus, I will cease to distinguish $size_h$ from $size_w$, and just use 'size' to refer to $size_h$.

Mathematicians' use of 'size' differs from that of ordinary people, once we get to transfinite numbers.
Indeed, it is standard to define the term 'transfinite set' as a set which can be put into one-one correspondence with a proper subset of itself.

VI. Cardinal Arithmetic and Cantor's Theorem

Let's look a bit more at both cardinal and ordinal numbers, and how transfinite numbers can be used for both measuring size and ranking.
I'll start with the cardinals.
We are all familiar with many properties of cardinal numbers.
For all cardinal numbers a, b, and c, whether finite or transfinite, the following properties hold:

1. $a+b=b+a$
2. $ab=ba$
3. $a + (b + c) = (a + b) + c$
4. $a \bullet (b \bullet c) = (a \bullet b) \bullet c$
5. $a \bullet (b + c) = ab + ac$
6. $a^{(b+c)} = a^b \bullet a^c$

7. $(ab)^c = a^c \bullet b^c$
8. $(a^b)^c = a^{bc}$

But some properties of finite cardinal numbers do not hold for transfinite numbers.
Notice that a + 1 = a, when a is transfinite.
And 2a = a holds as well.
Even a • a = a.
We can show these three facts by considering a bijective mapping from one set to the other, as we did in the discussion of the infinite hotel.

Consider one final important property which holds both of finite and transfinite numbers.

9. $2^a > a$

In its most fundamental, set-theoretic terms, this ninth claim is that $\mathbb{C}(\wp(A)) > \mathbb{C}(A)$.
'$\mathbb{C}$(A)' refers to the cardinality of A.
$\mathbb{C}$ is the measure of the size of a set.
For finite sets, $\mathbb{C}$(A) is just the number of elements of A.
'$\wp$(A)' refers to the power set of a, the set of all subsets of a set a.
Here are two examples of finite sets and their power sets.

A={a, b} $\wp$(A)={{a}, {b}, {a, b}, ∅}
A={2, 4, 6} $\wp$(A)={{2}, {4}, {6}, {2, 4}, {2, 6}, {4, 6}, {2, 4, 6}, ∅}

In general the power set of a set with n elements will have 2^n elements, which is why the number-theoretic claim that $2^n > a$ is the arithmetic correlate of the set-theoretic claim that $\mathbb{C}(\wp(A)) > \mathbb{C}(A)$.
For infinite n, sets with n members are the same size as sets with n+1 members, or with 2n members, or with n^2 members.
With infinite numbers, it is not always clear that what we think of as a larger set is in fact larger.
We might conclude that sets with n members are the same size as sets with 2^n members.
This conclusion would be erroneous.
$\mathbb{C}(\wp(A)) > \mathbb{C}(A)$.

The claim that $\mathbb{C}(\wp(A)) > \mathbb{C}(A)$ was called Cantor's paradox, but is now called Cantor's theorem.
The proof of the theorem is a set-theoretic version of the diagonalization argument.
I am going to present the proof here, though understanding it requires some familiarity with set theory.
Most basically, a set is a collection of objects, a plurality considered as a unit.
We can define sets either by listing their elements, or by stating a rule for inclusion in the set.
A is defined in the first way; B is defined in the second way.

A = {Alvin, Simon, Theodore}
B = {x | x is one of the three most popular singing chipmunks}

An element, ∈, of a set is just one of its members.

Alvin ∈ A
Theodore ∈ A

The subset S of a set A includes only members of A.
If S omits at least one member of A, it is called a proper subset.

C = {Alvin, Simon}

C is a subset of A: $C \subset A$.
C is a proper subset of A: $C \subseteq A$.

To prove Cantor's theorem, we need two more set-theoretic definitions.
A function is called *one-one* if it every element of the domain maps to a different element of the range, that is: $f(a)=f(b) \Rightarrow a=b$
A *function maps a set A onto another set B* if the range of the function is the entire set B, i.e. if no elements of B are left out of the mapping.

To prove Cantor's theorem, we want to show that the cardinality of the power set of a set is strictly larger than the cardinality of the set itself (i.e. $\mathbb{C}(\wp(A)) > \mathbb{C}(A)$).
It suffices to show that there is no function which maps A one-one and onto its power set.

Proof of Cantor's Theorem

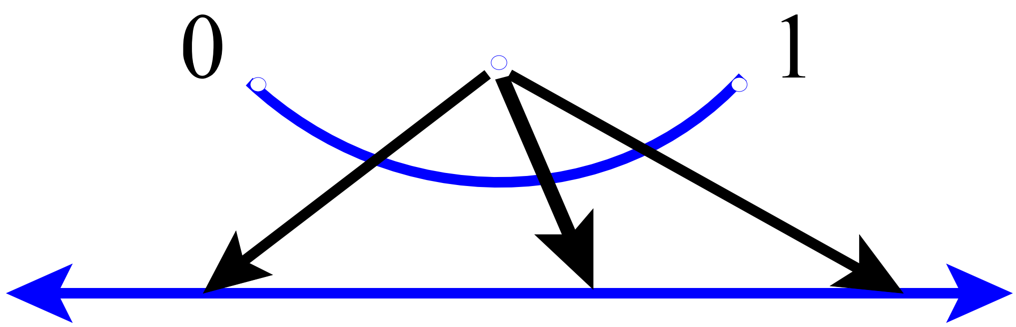
Assume that there is a function f: $A \Rightarrow \wp(A)$
Consider the set $B = \{x \mid x \in A \bullet x \notin f(x)\}$
B is a subset of A, since it consists only of members of A.
So, B is an element of $\wp(A)$, by definition of the power set.
That means that B itself is in the range of f.
Since, by assumption, f is one-one and onto, there must be an element of A, b, such that f(b) is B itself.
Is $b \in B$?
If it is, then there is a contradiction, since B is defined only to include sets which are not members of their images.
If it is not, then there is a contradiction, since B should include all elements which are not members of their images.
Either way, we have a contradiction.
So, our assumption fails, and there must be no such function f: $A \Rightarrow \mathcal{P}(A)$.
$\mathcal{P}(A)$ is strictly larger than A.
$\mathbb{C}(\wp(A)) > \mathbb{C}(A)$).

QED

VII. The Natural Numbers and the Real Numbers

With Cantor, let's call the size of the natural numbers $\aleph_0$.
Every subset of the natural numbers can be uniquely correlated with an infinite sequence of zeroes and ones.
If the set includes a one, put a one in the first place of the sequence; if not, put a zero in the first place.
If it includes a two, put a one in the second place of the sequence; if not, put a zero in the first place.
For all n, if the set includes n, put a one in the nth place of the sequence.

For all n, if the set does not include n, put a zero in the nth place of the sequence.
Each infinite sequence of zeroes and ones can be taken as the binary representation of a real number between zero and one, the binary representation of their decimal expansions.
We can easily provide a mapping between the real numbers (points on a line) between zero and one and all the real numbers (points).
Here is a geometric demonstration.



For each point on the curved line between zero and one, we can find a point on the infinite line, and vice-versa.
If you prefer an analytic proof, take $f(x) = \tan \pi(2x-1)/2$.
We can thus correlate the subsets of the natural numbers with the real numbers, and thus with the points on a line.
Then the real numbers, and the real plane, are the size of the power set of the natural numbers, $2^{\aleph_0}$.

VIII. The Power Set and Transfinite Numbers

We can generate larger and larger cardinals by taking the power set of any cardinal number.
We thus define a sequence of alephs:

$\aleph_0, \aleph_1, \aleph_2, \aleph_3, \aleph_4...$

Further, set theorists, by various ingenious methods, including the addition of axioms which do not contradict the standard axioms, generate even larger cardinals.
Cardinal counting gets pretty wild.
There are ethereal cardinals, subtle cardinals, almost ineffable cardinals, totally ineffable cardinals, remarkable cardinals, superstrong cardinals, and superhuge cardinals, among many others.
All of these cardinal numbers are transfinite, and larger than any of the sequence of alephs.

IX. Ordinals and Counting

I have been sliding back and forth between discussions of number theory and discussions of set theory, as if there is no difference.
The movement away from thinking of geometry as the foundational mathematical theory to thinking of arithmetic and algebra as foundational led to further abstraction.
Now, most mathematicians mostly think of set theory, which is more abstract than either arithmetic or geometry, as the fundamental mathematical theory.
Recent work on foundations has led to an even more abstract, and fundamental theory called category theory.
Cantor developed set theory in order to generate his theory of transfinites.
Cantor's work on set theory was followed by Frege's work defining the numbers.
Cantor defined cardinal numbers in terms of ordinal numbers, making the ordinals more fundamental.
Frege sought independent definitions of the ordinals and cardinals.
Despite their differences, both Cantor and Frege actually used inconsistent set theories, called naive set theory.
We will discuss Frege's approach to defining the numbers, and the inconsistency of Cantor/Frege set theory, which was discovered by Bertrand Russell in his formulation of the eponymous paradox, in our next class.
For now, let's look briefly at the ordinal numbers, and at how we can define arithmetic by using the more general set theory.

Let's start by reminding ourselves of an axiomatization of set theory, ZF.

Substitutivity:	$(x)(y)(z)[y=z \supset (y\in x \equiv z\in x)]$
Pairing:	$(x)(y)(\exists z)(u)[u\in z \equiv (u = x \vee u = y)]$
Null Set:	$(\exists x)(y){\sim}x\in y$
Sum Set:	$(x)(\exists y)(z)[z\in y \equiv (\exists v)(z\in v \bullet v\in x)]$
Power Set:	$(x)(\exists y)(z)[z\in y \equiv (u)(u\in z \supset u\in x)]$
Selection:	$(x)(\exists y)(z)[z\in y \equiv (z\in x \bullet \mathscr{F}u)]$, for any formula $\mathscr{F}$ not containing y as a free variable.
Infinity:	$(\exists x)(\varnothing\in x \bullet (y)(y\in x \supset Sy\in x)$

Ordinal numbers, set-theoretically, are just special kinds of sets, ones which are well ordered.
A set is well-ordered if we can find an ordering relation on the set, and the set has a first element under that order.
For convenience, we standardly pick a particular ordinal to represent each particular number.
We choose one example of a well-ordering for each number, and use it as the definition of that number.

In a moment, we'll start counting through the ordinal numbers, which (recall) are used to measure rank: first, second, third, etc.
To move through the ordinals, most often, we will just look for the successor of a number, for the set which stands for the next ordinal number.
Ordinals generated in this way are called successor ordinals.
In transfinite set theory, there are also sets which are called limit elements.
We get to them not by finding a successor of a set, but by collecting all the sets we have counted so far into one further set.
This operation of collecting several sets into one is called union.

If we combine all the sets that correspond to the finite ordinals into a single set, we get another well-ordered set.
This new set will be another ordinal: there will be a well-ordering on it, and it will have a minimal element.
This limit ordinal will be larger than all of the ordinals in it.
So, there are two kinds of ordinals: successor ordinals and limit ordinals.

Remember, that the following counting list is of ordinals, so by '1', I mean the first ordinal, rather than the cardinal '1'.
The limit ordinals are the ones found after the ellipses on each line.

The Ordinal Numbers:

$1, 2, 3, \ldots \omega$
$\omega+1, \omega+2, \omega+3 \ldots 2\omega$
$2\omega+1, 2\omega+2, 2\omega+3 \ldots 3\omega$
$4\omega, 5\omega, 6\omega \ldots \omega^2$
$\omega^2, \omega^3, \omega^4 \ldots \omega^\omega$
$\omega^\omega, (\omega^\omega)^\omega, ((\omega^\omega)^\omega)^\omega, \ldots \varepsilon^0$

Notice that limit ordinals are taken as the completions of an infinite series.
From Aristotle (and likely before), through the nineteenth century, philosophers and mathematicians denied that there can be any completion of an infinite series.
Cantor's diagonal argument shows that there are different levels of infinity.
We form ordinals to represent the ranks of these different levels of infinity precisely by taking certain series to completion.
The consistency of Cantor's theory of transfinites transformed the way we think of infinity.

X. Defining the Natural Numbers

In order to link number theory with set theory, Cantor identifies the natural numbers with certain sets.
As I mentioned earlier, we define the natural numbers as certain sets among the transfinitely many.
There are different ways to do this, and different ways to define ordinals and cardinals.
The following definitions are standard, if conventional.

Two standard definitions of ordinals derive from work of Zermelo and von Neumann, in the early twentieth century.
The null set axiom ensures the existence of an empty set, so we can introduce a constant, ∅, such that (x)~x∈∅.
Both Zermelo and von Neumann take ∅ to stand for zero.
After that, their definitions diverge.

Zermelo:

$0 = \varnothing$
$1 = \{\varnothing\}$
$2 = \{\{\varnothing\}\}$
$3 = \{\{\{\varnothing\}\}\}$
...

Von Neumann

$0 = \emptyset$

$1 = \{\emptyset\}$

$2 = \{\emptyset, \{\emptyset\}\}$

$3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

...

Von Neumann's definitions are less elegant than Zermelo's.
But they are more efficient, and have become standard.
Frege has a different way of picking out the natural numbers from the sets, one which depends on his inconsistent set theory, but which has, and continues, to attract a lot of attention.

XI. The Continuum Hypothesis

Finally, before we get to Frege's logicist definition of number in our next class, we should spend a moment on Cantor's continuum hypothesis.
Certain questions in the history of mathematics have proven difficult to answer.
Here are three:

Is Fermat's theorem true?
Is Goldbach's conjecture true?
Is the parallel postulate true?

Some questions, like Fermat's conjecture, are clearly answered affirmatively.
We expect the same kind of answer for Goldbach, even though we lack one at the moment.
The parallel postulate is more interesting.
It can fail, but it can also hold.
Thus, we have decided that the question is ill-formed.
There is no one true answer.
There are different kinds of spaces, and they are each defined by a different answer to the parallel postulate.

Cantor provided a method for generating larger and larger transfinite numbers.
He shows that the cardinal number of the reals is equal to $2^{\aleph_0}$.
He also shows that $2^{\aleph_0}$ is greater than $\aleph_0$.
Cantor's theorem does not show, however, that it is the next greater transfinite number.
Let's take $\aleph_1$ to be the name we give to the next transfinite cardinal after $\aleph_0$.
The continuum hypothesis is that $\aleph_1 = 2^{\aleph_0}$.
More abstractly, the generalized continuum hypothesis is that $\aleph_{n+1} = 2^{\aleph_n}$.
Cantor believed that the continuum hypothesis was true, but he could not prove it.
Indeed, at the 1900 Paris Congress, David Hilbert cited the continuum hypothesis as one of the ten most important unsolved problems in mathematics.

As Tiles points out, certain operations which generate larger cardinal numbers, like exponentiation, skip numbers in between.
Only succession actually gives the next number.
When we multiply a finite cardinal by two, or seventeen, or add six, or raise to the π power, we generate cardinals that are larger, but not merely one larger.
Indeed, we do not even know that the sizes of transfinite cardinal numbers can be ordered linearly.

> It has been assumed that cardinalities, or cardinal numbers, can be arranged in a single linear order. But just making that assumption does not tell us anything about the nature of the cardinal number 'sequence', about how to establish where any given cardinality lies in it, or even whether it is correct to talk about there being a *next* cardinal number after $\aleph_0$. Our assumption does not rule out the possibility that infinite cardinalities might, like the rational numbers, be densely ordered. If that were the case, there would always be another cardinal number between any two given cardinalities and given any cardinal number there would be no 'next 'one (Tiles, *The Philosophy of Set Theory* 103).

Mathematicians really are split on whether the continuum hypothesis is true.
Though, I suppose opinion has generally turned against it.
Two results in the twentieth century further entrenched the problem.
In 1940, Kurt Gödel showed that the continuum hypothesis is consistent with the standard axioms of set theory.
In 1963, Paul Cohen showed that its negation is consistent with set theory.
Thus, the continuum hypothesis is independent of the standard axioms.
We can consistently consider the continuum to be of all different sizes: $\aleph_1$, $\aleph_2$, $\aleph_3$, etc.
Further, none of the large cardinal axioms proposed settle the question.

We could settle the question of the size of the continuum by adopting stronger axioms for set theory.
Some mathematicians believe that the continuum hypothesis, even the generalized version, is so intuitively true that we should just adopt it, or an equivalent, as part of set theory.
As we will see, Gödel favored this approach.
Alternatively, we could take the question to be ill-formed, like the question of whether the parallel postulate is true.
Perhaps there are different set theories, with different sizes of the continuum.