Hamilton College
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Class #1: August 30
Mathematics and Philosophy

I. Introduction

All of philosophy is philosophy of mathematics.

> If we taught philosophy today in a way that reflected its history, the current curriculum would be overwhelmed with the philosophy of mathematics. Think of these great philosophers and how important mathematics is to their thought: Plato, Descartes, Leibniz, Kant, Frege, Russell, Wittgenstein, Quine, Putnam, and so many others. And interest in the nature of mathematics is not confined to the so-called analytic stream of philosophy; it also looms large in the work of Husserl and Lonergin, central figures in, respectively, the continental and Thomistic philosophical traditions. Anyone sincerely interested in philosophy must be interested in the nature of mathematics... As for those who persist in thinking otherwise - let them burn in hell (Brown, xi-xii).

Some people are scared of mathematics, and so obscure its central role in philosophy.
Brown's assertion is too strong, but the core idea is right.
A satisfactory understanding of philosophy can not be gotten without serious contemplation of mathematics.

Many philosophers have contributed to mathematics.
Plato's students were implored to excel in mathematics.
A sign over the door to his Academy said, "Let no one enter who is ignorant of geometry."
Aristotle wrote, "Mathematics has come to be the whole of philosophy for modern thinkers" (*Metaphysics* I.9: 992a32).
Descartes founded analytic geometry.
Leibniz developed the calculus.
Frege and Russell made advances in the foundations of mathematics proper.
Quine, Kripke, Field and many others contribute to set theory and the foundations of mathematics.

In the other direction, mathematicians from Euclid forward have contributed to philosophy.
Euclid's method has had a long and profound influence on the methods of philosophy.
Cantor's work on transfinite numbers transformed the philosopher's concept of infinity, which had played a central role in philosophical debate about God and the origins of the universe for millennia.
Other philosophical topics like necessity and contingency have received mathematical treatment which has changed the way philosophers argue about these concepts.
Some mathematicians, like Hilbert, Gödel, von Neumann, and Tarski, are central philosophical figures.

Some philosophers pay close attention to their accounts of mathematics.
Plato used the abstractness of mathematics to motivate the reality of the forms.
Descartes cleaved thought from sensation by considering how mathematical beliefs were not ultimately sensory.
Kant's transcendental idealism begins with the question of what the structure of our reasoning must be in order to yield mathematical certainty.
Wittgenstein's *Remarks on the Foundations of Mathematics* contain core elements of his philosophical

positions, specifically his skepticism about rule-following.
Some contemporary mathematical nominalists, those who deny the existence of mathematical objects, take the difficulty of explaining the status of mathematical claims to be the most difficult problem they face.

Historically, some philosophers think that philosophical results can affect how we do mathematics.
Berkeley tried to debunk the calculus; check out *The Analyst* in the Ewald collection.
But, the more dominant contemporary view is that mathematics is independent of philosophy.

> Philosophy may in no way interfere with the actual use of language; it can in the end only describe it. For it cannot give it any foundation either. It leaves everything as it is. It also leaves mathematics as it is, and no mathematical discovery can advance it (Wittgenstein, *Philosophical Investigations*, §124).
>
> There is no mathematical substitute for philosophy (Kripke, "Is There a Problem About Substitutional Quantification").

This course will blend historical and contemporary approaches to the philosophy of mathematics.
Philosophy of mathematics is taught rarely.
When it is taught, it tends to focus either exclusively on twentieth century and contemporary readings or on a small period, like the fecund time between 1879 and 1931, when the debate among logicists, intuitionists and formalists was prominent.
The first half of this course will be a broad survey of historical approaches to the philosophy of mathematics, from the Pre-Socratic philosophers through the early twentieth century.
Then, starting with Quine's work, we will narrow our focus to the topic on which I do my main research, the indispensability argument.
We will spend the second half of the course working through some recent work on the indispensability argument.
The course will culminate in some new work that I am preparing.

II. The Syllabus

The course website is:

> http://www.thatmarcusfamily.org/philosophy/Course_Websites/Math_F10/Course_Home.html

I use Blackboard mainly just for grades.

The **readings** for class come in two levels: primary and secondary
The primary readings are required for each class, as is any seminar paper for that day.
I will post reading guides, lists of questions, for each primary reading.
The reading guides will be the basis for the final exam.
The secondary readings are suggested, but if you are writing your seminar paper for a class with secondary readings, you should take them as required.
The two books I've ordered are mainly secondary sources, and good ones.
Brown is more fun, while Shapiro is more traditional.
Some of you are going to have to work a bit harder with the history.

I'm not presuming knowledge of the history of philosophy, but you will have an easier time if you already know some of the terminology, if you've already read Kant or Aristotle.
In addition to the primary and secondary readings, there are further suggestions on the course bibliography.

R**eading prècises** are 100-150-word summaries, or distillations, of some portion of an assigned reading.
These are just ways of getting you writing a very little bit about the material for class before class.
You may write about an entire reading or readings, or focus on a small portion of one reading.
Alternatively, you can write a list of 6-8 detailed questions on the reading.

Seminar papers are due, to the class, by noon on Sunday or Tuesday.
They should be expository.
They may be critical as well.
The person presenting the seminar paper will basically run the class, though I will kibitz.
We will sign up for the first two seminar papers on Wednesday.
You are likely to write your term paper on a topic you have begun researching as a seminar paper.

The **term paper** is due in three stages:
A one-paragraph abstract of you paper is due on **Wednesday, October 13**.
A full draft of your term paper is due on **Monday, November 15**.
The final draft is due on **Monday, December 6**.
The Course Bibliography has many references.

The **final exam** will be on **Wednesday, December 15**, from 9am to noon.
Preparatory questions will be posted on the course website.

III. What is a proof?

This example comes from the Kline reading for Wednesday, on Pythagoras.
Greek mathematics was essentially geometric.
Euclid's *Elements* contained number theory, but it was derived from geometric relations.
The Pythagoreans, for example, were fascinated by figurate numbers: triangular numbers, square numbers, pentagonal numbers, and so on.
When they thought about square numbers, the Pythagoreans thought of the numbers which, when taken as discrete dots, could be arranged as squares.

The triangular numbers, were especially interesting to the Pythagoreans: 1, 3, 6, 10, 15, 21, 28, 36...
The formula for calculating the nth triangular number is: (n/2)(n+1)
The sum of two consecutive triangular numbers is a square number.
This is easily shown algebraically:

$$
\begin{aligned}
&(n/2)(n+1) + ((n+1)/2)((n+1)+1) &&= \\
&(n^2 + n)/2 + (n+1)(n+2)/2 &&= \\
&(n^2 + n)/2 + (n^2 + 3n + 2)/2 &&= \\
&(2n^2 + 4n + 2)/2 &&= \\
&n^2 + 2n + 1 &&= \\
&(n+1)^2
\end{aligned}
$$

Kline says: “That the Pythagoreans could prove this general conclusion, however, is doubtful” (30).
Is Kline’s claim correct?

In Kline’s defense, the Pythagoreans did not have algebra, which was not developed until the 9th century, by the Arabs.
Still, the Pythagoreans did have the picture in Kline, p 30, Figure 3.2.
Is the picture not a proof?
Consider Wittgenstein’s proof of commutativity, from *Remarks on the Foundations of Mathematics*, IV.17.
The picture is convincing.

Pictures have severe limitations as sources of proof.
It is difficult to draw intuitively useful pictures of odd spaces.
More importantly, as Brown points out, some pictures are misleading.
Compare the sums of two infinite series:

1, 1/4, 1/9, 1/16...
1, ½, 1/3, 1/4...

The graphs of the two series look essentially the same.
But the first series sums to a finite number, $\pi^2/6 \approx 1.64$
The sum of the second series is infinite; see Brown for the proof.
The picture was misleading.

Wittgenstein’s suggestion, in defense of picture proofs, and against Kline’s claim that the Pythagoreans did not have a proof that the sum of two consecutive triangular numbers is a square number, is that we overvalue the algebraic proof.
Wittgenstein worried that we do not know how to extend, or project, results to new cases.
Further, the picture proof gives you more, in some ways, than the algebraic one.
Most people get the ‘aha’ moment more substantially from the picture than from the algebra.
What does that mean?

IV. The Nature of Mathematics

We start with the question whether a picture can be a proof as an introduction to a general characterization of mathematics.
Brown characterizes the “mathematical image,” a set of characteristics that most people apply to mathematics.

1. Mathematical results are certain.
2. Mathematics is objective.
3. Proofs are essential.
4. Diagrams are psychologically useful, but prove nothing.
5. Diagrams can even be misleading.
6. Mathematics is wedded to classical logic.
7. Mathematics is independent of sense experience.
8. The history of mathematics is cumulative.

9. Computer proofs are merely long and complicated regular proofs.
10. Some mathematical problems are unsolvable in principle (Brown, 7).

We will spend some time exploring many of these characteristics in depth.
For now, note that Brown omits the term *a priori*, at #7.

The question of whether we have *a priori* knowledge is widely debated.
Shapiro quotes Blackburn's *Dictionary of Philosophy* on the characterization of the a priori:

> A proposition is known *a priori* if the knowledge is not based on any 'experience of the specific course of events of the actual world' (Shapiro, 22).

Brown presents the discovery that the square root of two is irrational as an illustration of an a priori process.
Here is a good discussion of the difference between *a priori* and empirical knowledge.
The debates over the *a priori* are subtle and complex.
But, the question of whether there is *a priori* knowledge seems easily answered in mathematics.
Consider Brown's example of the irrationality of the square root of two.
We could never discover that the square root of two is irrational by experience.
Since the rationals are dense, we could not get to the result by measuring.
We can always find a rational which will fulfill our measurement needs.
Consider a circle of radius 2.
We know, *a priori*, that its circumference is 4π.
But we could never measure 4π empirically.
By measure, we can only get to about 12.5.
But, we have a clean proof of the irrationality of $\sqrt{2}$, which was discovered by the Pythagoreans.

That $\sqrt{2}$ is irrational:

> Suppose that $\sqrt{2}$ is rational.
> Then, it's expressible as a/b, where a and b are integers.
> We can suppose a/b to be in lowest terms, which means that a and b have no common divisors.
> $a^2 = 2b^2$
> So, a^2 is even.
> Thus a is even, since only even numbers have even squares.
> So, a = 2c, for some c.
> $a^2 = 4c^2 = 2b^2$
> So, $b^2 = 2c^2$.
> Which means that b is also even.
> So a and b have been shown even, which contradicts our assumption that a/b is in lowest terms.
> *Tilt*

We do use our senses to perceive proofs, of course.
But we should not confuse names for objects with the objects themselves.
Sense experience is necessary for *a priori* knowledge, but not sufficient.

The Pythagoreans were supposed to have drowned the person who discovered the proof that the square root of two is irrational.
For, it undermined their belief that the world was essentially made of whole numbers.
We will discuss the Pythagoreans more on Wednesday.

For now, note that the method of this proof is called *reductio ad absurdum*, or proof by contradiction.
In a *reductio* proof, we assume the opposite of what we want to demonstrate, and show that it leads to a contradiction.
In order for proof by contradiction to be a legitimate method, we must assume bivalence, that either a statement or its negation is true.
Bivalence is an essential claim of classical logic.
If we eliminate the possibility of the negation of a statement being true, then we can safely claim that the original statement is true.
Some philosophers reject bivalence, and its object-language correlate called the law of the excluded middle:

Law of the excluded middle: $P \vee \sim P$

In particular, we will look at intuitionists in mathematics, who claim that only constructive, and not *reductio*, proofs are legitimate methods.

There has been a long confusion of apriority with necessity, which is coming to an end.
Though, Shapiro still calls them "twin notions" (23).
One problem with the traditional notion of apriority, on which anything believed *a priori* must be true, can be seen by considering Kant's claim that Euclidean space is the result of the a priori application of our concepts on the noumenal world.
Since space turns out to be non-Euclidean, according to special relativity, what seemed *a priori* turned out to be false.
If we assume that any proposition which was believed on *a priori* principles must be true, then if the statement turns out false, it must never have been held *a priori*.
Kant's entire metaphysical system depended on the application of *a priori* concepts to the noumenal world.
When space turned out to be non-Euclidean, Kant's system fell apart.

If we adopt a fallibilistic *a priori*, on which we can be wrong about a proposition, even if we hold it *a priori*, then a discovery that a proposition is false need not impugn the methods we used to acquire that belief.
That is, we can believe a proposition independently of experience, and still be wrong about that belief.
A wonderful example of this is found in Cantor and Frege, who held contradictory axioms of comprehension in their set theories.
Had Kant held a fallibilistic *a priori*, he might have been able to salvage some of his approach, though not the necessity of mathematics.
The fallibilist can, alternatively, hold that statements believed on the basis of a priori reasoning are necessarily true, *if true*.
(And if they are false, they are necessarily false.)
We will return to these topics.

The twentieth century dominance of philosophy by philosophy of language has confused the matter worse by admixing analyticity, as an explanation of apriority.
Analyticity is a semantic notion, about meanings of terms.
For example, 'Bachelors are unmarried' and 'We walk with those with whom we stroll' are analytic.
Apriority is an epistemic notion, about belief and knowledge.
Necessity is a metaphysical notion, about the nature of the universe, broadly conceived.

Certainty is an epistemic notion, masquerading as a metaphysical notion.
I can be certain about something non-necessary, like that I am here now.
I can be uncertain about something necessary, like whether Goldbach's conjecture is true.
Brown is liable at #1 for confusing certainty and necessity.