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# Some Sets of Mathematical Axioms

**Propositional Logic**, following Mendelson, *Introduction to Mathematical Logic*

The symbols are ~, ⊃, (, ), and the statement letters $A_i$, for all positive integers i.

All statement letters are wffs.

If α and β are wffs, so are ~α and (α ⊃ β)

If α, β, and γ are wffs, then the following are axioms:

A1: (α ⊃ (β ⊃ α))

A2: ((α ⊃ (β ⊃ γ)) ⊃ ((α ⊃ β) ⊃ (α ⊃ γ)))

A3: ((~β ⊃ ~α) ⊃ ((~ β ⊃ α) ⊃ β))

β is a direct consequence of α and (α ⊃ β)

**Zermelo-Fraenkel Set Theory**, again following Mendelson, but with adjustments

ZF may be written in the language of first-order logic, with one special predicate letter, ∈.

Substitutivity: (x)(y)(z)[y=z ⊃ (y∈x ≡ z∈x)]

Pairing: (x)(y)(∃z)(u)[u∈z ≡ (u = x ∨ u = y)]

Null Set: (∃x)(y)~x∈y

Note: the null set axiom ensures the existence of an empty set, so we can introduce a constant, ∅, such that (x)~x∈∅.

Sum Set: (x)(∃y)(z)[z∈y ≡ (∃v)(z∈v • v∈x)]

Power Set: (x)(∃y)(z)[z∈y ≡ (u)(u∈z ⊃ u∈x)]

Selection: (x)(∃y)(z)[z∈y ≡ (z∈x • $\mathscr{F}$u)], for any formula $\mathscr{F}$ not containing y as a free variable.

Infinity: (∃x)(∅∈x • (y)(y∈x ⊃ Sy∈x)

Note: 'Sy' stands for y∪{y}, the definitions for the components of which are standard.

**Peano Arithmetic**, again, following Mendelson with adjustments

P1: 0 is a number

P2: The successor (x') of every number (x) is a number

P3: 0 is not the successor of any number

P4: If x'=y' then x=y

P5: If P is a property that may (or may not) hold for any number, and if

i. 0 has P; and

ii. for any x, if x has P then x' has P;

then all numbers have P.

Note: P5 is also called mathematical induction, and is actually a schema of an infinite number of axioms.

**Birkhoff's Postulates for Geometry**, following James Smart, *Modern Geometries*

*Postulate I: Postulate of Line Measure*. The points A, B,... of any line can be put into a 1:1 correspondence with the real numbers x so that $|x_B-x_A| = d(A,B)$ for all points A and B.

*Postulate II: Point-Line Postulate*. One and only one straight line l contains two given distinct points P and Q.

*Postulate III: Postulate of Angle Measure*. The half-lines l, m... through any point O can be put into 1:1 correspondence with the real numbers a(mod 2π) so that if A≠0 and B≠0 are points on l and m, respectively, the difference $a_m - a_l$ (mod 2π) is angle△AOB. Further, if the point B on m varies continuously in a line r not containing the vertex O, the number $a_m$ varies continuously also.

*Postulate IV: Postulate of Similarity*. If in two triangles △ABC and △A'B'C', and for some constant k>0, d(A', B') = kd(A, B), d(A', C')=kd(A, C) and △B'A'C'=±△BAC, then d(B', C')=kd(B,C), △C'B'A'=±△CBA, and △A'C'B'=±△ACB.