Philosophy 109, Modern Logic, Queens College
Russell Marcus, Instructor
email: philosophy@thatmarcusfamily.org
website: http://philosophy.thatmarcusfamily.org
Office phone: (718) 997-5287

Relational Predicates and Overlapping Quantifiers, §8.6

I. **Introducing Relational Predicates**

Consider the argument:
Bob is taller than Charles. Andrew is taller than Bob. For any x, y and z, If x is taller than y and y is taller than z, then x is taller than z. So, Andrew is taller than Charles.

The conclusion should follow logically, but how do we translate the predicates?
If we only have monadic (1-place) predicates, like the ones we have so far considered, we have to translate the two first sentences with two different predicates:
Bob is taller than Charles: Tb
Andrew is taller than Bob: Ya

We really want a predicate that takes two objects. This is called a dyadic predicate. For examples:
Txy: x is taller than y
Kxy: x knows y
Bxy: x believes y
Dxy: x does y

We can have three-place predicates too, called triadic predicates:
Gxyz: x gives y to z
Kxyz: x kisses y in z
Bxyz: x is between y and z

Also, we can have four-place and higher level predicates. All predicates which take more than one object are called relational.

II. **Exercises A**. Translate each sentence into predicate logic.

1. John loves Mary
2. Tokyo isn't smaller than New York.
3. Marco was introduced to Erika by Paco.
4. America took California from Mexico.

III. **Quantifiers with relational predicates**

Return to the original problem:
Bob is taller than Charles: Tbc
Andrew is taller than Bob: Tab
But what about the general statement?
We need to put quantifiers on the relations.

Putting on a single quantifier:
Joe is bigger than some thing : (∃x)Bjx
Something is bigger than Joe: (∃x)Bxj
Joe is bigger than everything: (x)Bjx
Everything is bigger than Joe: (x)Bxj

We can dispense with constants altogether, introducing overlapping quantifiers.
Consider: ‘Everything loves something’:
(x)(∃y)Lxy
Note the different quantifier letters: overlapping quantifiers must use different variables.
Also, the order of quantifiers matters: ‘(∃x)(y)Lxy’ means that something loves everything, which is different.

Consider these more complex examples:
1. Something taught Plato. (Txy: x taught y)
(∃x)Txp
2. Someone taught Plato.
(∃x)(Px · Txp)
3. Plato taught everyone.
(x)(Px ⊃ Tpx)
4. Everyone knows something. (Kxy: x knows y)
(x)[Px ⊃ (∃y)Kxy]
5. Everyone is wiser than someone. (Wxy: x is wiser than y)
(x)[Px ⊃ (∃y)(Py · Wxy)]
6. Someone is wiser than everyone.
(∃x)[Px · (y)(Py ⊃ Wxy)]
7. Some financier is richer than everyone. (Fx, Rxy: x is richer than y)
(∃x)[Fx · (y)(Py ⊃ Rxy)]
8. No deity is weaker than some human. (Dx, Hx, Wxy: x is weaker than y)
~(∃x)[Dx · (∃y)(Hy · Wxy)] or (x)[Dx ⊃ (y)(Hy ⊃ ~Wxy)]
9. Honest candidates are always defeated by dishonest candidates. (Hx, Cx, Dxy: x defeats y)
(x){(Cx · Hx) ⊃ (∃y)[(Cy · ~Hx) · Dyx]}
10. No mouse is mightier than himself. (Mx, Mxy: x is mightier than y)
(x)(Mx ⊃ ~Mxx)
11. Everyone buys something from some store. (Px, Sx, Bxyz: x buys y from z)
(x)[Px ⊃ (∃y)(∃z)(Sz · Bxyz)]
12. There is a store from which everyone buys something.
(∃x){Sx · (y)[Py ⊃ (∃z)Byzx]}
13. No store has everyone for a customer.
~(∃x){Sx · (y)[Py ⊃ (∃z)Byzx]} or (x){Sx ⊃ (∃y)[Py · (z)~Byzx]}

IV. **Order of quantifiers, and scope**

Mostly we keep narrow scope.
This means we don’t introduce a quantifier until it’s needed.
On occasion, not here, but later, we will just put all quantifiers in front, using broad scope.
We must be careful.

Sometimes the order of the quantifiers and the scope doesn’t matter:
‘Everyone loves everyone’ can be written as any of the following:
(x)[Px ⊃ (y)(Py ⊃ Lxy)]
(x)(y)[(Px · Py) ⊃ Lxy]
(y)(x)[(Px · Py) ⊃ Lxy]
Technically, this latter is everyone is loved by everyone. But these are logically equivalent.

Similarly, someone loves someone can be written as any of the following:
(∃x)[Px · (∃y)(Py · Lxy)]
(∃x)(∃y)[(Px · Py) · Lxy]
(∃y)(∃x)[(Px · Py) · Lxy]
Again, the latter is someone is loved by someone. Again, these are equivalent.

But when you mix universals with existentials, you have to be careful, since reversing the order of the quantifiers changes the meaning of the proposition.

None of the following examples are equivalent:

1. $(x)(\exists y)[(Px \cdot Py) \cdot Lxy]$ - Everyone loves someone.
2. $(\exists y)(x)[(Px \cdot Py) \supset Lxy]$ - Someone is loved by everyone.
3. $(\exists x)(y)[(Px \cdot Py) \supset Lxy]$ - Someone loves everyone.
4. $(y)(\exists x)[(Px \cdot Py) \cdot Lxy]$ - Everyone is loved by someone.

Take your time with the above. Make sure you understand them.

Note that the first word in each translation above corresponds to the leading quantifier.

Also, note the main connective is determined by the innermost quantifier.

If the innermost quantifier is existential, the main connective is a conjunction.

If the innermost quantifiers is universal, the main connective is a conditional.

This may be clearer if we take the quantifiers inside.

1. $(x)[Px \supset (\exists y)(Py \cdot Lxy)]$
2. $(\exists y)[Py \cdot (x)(Px \supset Lxy)]$

Moral: Keep your scopes narrow to avoid confusion.

V. **Exercises B**. Translate each of the following sentences into predicate logic.

1. Everyone loves something. (Px, Lxy)
2. No one knows everything. (Px, Kxy)
3. No one knows everyone.
4. Every woman is stronger than some man. (Wx, Mx, Sxy: x is stronger than y)
5. No cat is smarter than any horse. (Cx, Hx, Sxy: x is smarter than y)
6. Dead men tell no tales. (Dx, Mx, Tx, Txy: x tells y)
7. There is a city between New York and Washington. (Cx, Bxyz: y is between x and z)
8. Everyone gives something to someone. (Px, Gxyz: y gives x to z)
9. A dead lion is more dangerous than a live dog. (Ax: x is alive, Lx, Dx, Dxy: x is more dangerous than y)
10. A lawyer who pleads his own case has a fool for a client. (Lx, Fx, Pxy: x pleads y's case; Cxy: y is a client of x)

VI. **Deductions using Relational Predicates and Overlapping Quantifiers**

Consider again the original problem:

Prove: Bob is taller than Charles. Andrew is taller than Bob. For any x, y and z, If x is taller than y and y is taller than z, then x is taller than z. So, Andrew is taller than Charles.

1. Tbc
2. Tab
3. $(x)(y)(z)[(Txy \cdot Tyz) \supset Txz]$ / Tac

Use the same rules of inference, one at a time. (there's one exception, to UG, which we will note shortly.)

4. $(y)(z)[(Tay \cdot Tyz) \supset Taz]$	3, UI
5. $(z)[(Tab \cdot Tbz) \supset Taz]$	4, UI
6. $(Tab \cdot Tbc) \supset Tac$	5, UI
7. $(Tab \cdot Tbc)$	2, 1, Conj
8. Tac	6, 7, MP

QED

An example, taking quantifiers off in the middle of the proof:

1. (∃x)[Hx · (y)(Hy ⊃ Lyx)]		/ (∃x)(Hx · Lxx)
2. Ha · (y)(Hy ⊃ Lya)	1, EI	
3. Ha	2, Simp	
4. (y)(Hy ⊃ Lya)	2, Com, Simp	
5. Ha ⊃ Laa	4, UI	
6. Laa	5, 3, MP	
7. Ha · Laa	3, 6, Conj	
8. (∃x)(Hx · Lxx)	7, EG	

QED

The restriction on UG:
Consider the following derivation

1. (x)(∃y)Lxy	Everything loves something.	
2. (∃y)Lxy	1, UI	
3. Lxa	2, EI	
4. (x)Lxa	3, UG	But incorrect!
5. (∃y)(x)Lxy	4, EG	There's something that everything loves.

You shouldn't be able to derive step 5 from step 1.
The problem is in step 4
You may never UG on a variable when there's a constant present, and the variable was free when the constant was introduced.
I.e. In line 4, because 'x' was free in line 3 when 'a' was introduced

The justification of the restriction:
In line 2, we were picking some random object x.
Then, at line 3, we introduced 'a' as the name of what x loves .
Since everything loves something, there must be some thing 'a' loved by whatever x we pick.
But now we can't say that every 'x' loves 'a'. 'x' has become as particular an object as 'a' is.

Another warning: When quantifying, using (UG) or (EG), watch for accidental binding.
Consider : (Pa · Qa) ⊃ (Fx ∨ Gx)
If you try to quantify over the 'a' using EG with the variable 'x', you accidentally bind the latter two terms:
(∃x)[(Px · Qx) ⊃ (Fx ∨ Gx)]
Instead, use a 'y':
(∃y)[(Py · Qy) ⊃ (Fx ∨ Gx)]
Now the latter terms remain free.

An example with several quantifiers:

1. (∃x)(y)[(∃z)Ayz ⊃ Ayx]		
2. (y)(∃z)Ayz	/ (∃x)(y)Ayx	
3. (y)[(∃z)Ayz ⊃ Aya]	1, EI	EI before you UI, replace the 'x'.
4. (∃z)Ayz ⊃ Aya	3, UI	Keep the same variable - easier to keep track!
5. (∃z)Ayz	2, UI	You may instantiate to the same variable - it's UI.
6. Aya	4, 5, MP	
7. (y)Aya	6, UG	'y' was not free when 'a' was introduced in line3!
8. (∃x)(y)Ayx	7, EG	

QED

An example using conditional proof:

1. (x)[Ax ⊃ (y)Bxy]	
2. (x)[Ax ⊃ (∃y)Dxy]	/(x)(∃y)[Ax ⊃ (Bxy · Dxy)]
│3. Ax	ACP
│4. Ax ⊃ (∃y)Dxy	2, UI
│5. (∃y)Dxy	4, 3 MP
│6. Dxa	5, EI
│7. Ax ⊃ (y)By	1, UI
│8. (y)Bxy	7, 3, MP
│9. Bxa	8, UI
│10. Bxa · Dxa	9, 8, Conj
11. Ax ⊃ (Bxa · Dxa)	3-10, CP
12. (∃y)[Ax ⊃ (Bxy · Dxy)]	11, EG
13. (x)(∃y)[Ax ⊃ (Bxy · Dxy)]	12, UG

QED

A more complex proof:

1. (x)(Wx ⊃ Xx)		
2. (x)[(Yx · Xx) ⊃ Zx]		
3. (x)(∃y)(Yy · Ayx)		
4. (x)(y)[(Ayx · Zy) ⊃ Zx]	/(x)[(y)(Ayx ⊃ Wy) ⊃ Zx]	
│5. (y)(Ayx ⊃ Wy)	ACP	
│6. (∃y)(Yy · Ayx)	3, UI	
│7. Ya · Aax	6, EI	
│8. Aax ⊃ Wa	5, UI	
│9. Aax	7, Com, Simp	
│10. Wa	8,9, MP	
│11. Wa ⊃ Xa	1, UI	
│12. Xa	11, 10, MP	
│13. Ya	7, Simp	
│14. Ya · Xa	13, 12, Conj	
│15. (Ya · Xa) ⊃ Za	2, UI	
│16. Za	15, 14, MP	You can't use this as the conclusion!
│17. (y)[(Ayx · Zy) ⊃ Zx]	4, UI	Keep the x, that's the point.
│18. (Aax · Za) ⊃ Zx	17, UI	
│19. Aax · Za	9, 16, Conj	
│20. Zx	18, 19, MP	
21. (y)(Ayx ⊃ Wy) ⊃ Zx	5-20, CP	
22. (x)[(y)(Ayx ⊃ Wy) ⊃ Zx]	21, UG	

QED

VII. **Exercises C**. Derive the conclusions of each of the following arguments.

1)

1. $(x)(Cax \supset Dxb)$
2. $(\exists x)Dxb \supset (\exists y)Dby$ / $(\exists x)Cax \supset (\exists y)Dby$

2)

1. $(x)[Ex \supset (y)(Fy \supset Gxy)]$
2. $(\exists x)[Ex \cdot (\exists y){\sim}Gxy]$ / $(\exists x){\sim}Fx$

3)

1. $(\exists x)Ax \supset (\exists x)Bx$ / $(\exists y)(x)(Ax \supset By)$

4)

1. $(x)[Mx \supset (y)(Ny \supset Oxy)]$
2. $(x)[Px \supset (y)(Oxy \supset Qy)]$ / $(\exists x)(Mx \cdot Px) \supset (y)(Ny \supset Qy)$

Solutions may vary.

VIII. **Translating to English**

Use the translation key on the left to translate the formulas on the right into English sentences. For example, the answer to the first one is given.

Ax: x is silver
Bxy: x belongs to y
Cx: x is a cloud
Cxy: x keeps company with y
Dx: x is a dog
Ex: x is smoke
Fx: x is fire
Fxy: x is fair for y
g: God
Gx: x is glass
Gxy: x gathers y
Hx: x is home
Hxy: x helps y
Ixy: x is in y
Jxy: x is judged by y
Kxy: x is a jack of y
Lx: x is a lining
Lxy: x is like y
Mx: x is moss
Mxy: x is master of y
Px: x is a person
Qx: x is a place
Rx: x rolls
Sx: x is a stone
Tx: x is a trade
Txy: x should throw y
Ux: x is a house
Uxy: x comes to y
Vxy: x ventures y
Wx: x waits
Yx: x is a day

1. $(x)[Dx \supset (\exists y)(Yy \cdot Byx)]$
For all x, if x is a dog, then there exists a day which belongs to that dog.
Or, every dog has its day!

2. $(x)[(\exists y)(Py \cdot Fxy) \supset (z)(Pz \supset Fxz)]$
3. $(x)[(Rx \cdot Sx) \supset (y)(My \supset {\sim}Gxy)]$
4. $(x)[(Px \cdot Wx) \supset (y)Uyx]$
5. $(x)[(Px \cdot Hxx) \supset Hgx]$
6. $(x)[Hx \supset (y)(Qy \supset {\sim}Lyx)]$
7. $(x)\{Cx \supset (\exists y)[(Ay \cdot Ly) \cdot Byx]\}$
8. $(x)[Px \supset (y)(Cxy \supset Jxy)]$
9. $(x)\{Qx \supset [(\exists y)(Ey \cdot Iyx) \supset (\exists z)(Fz \cdot Izx)]\}$
10. $(x)\{[Px \cdot (y)(Ty \supset Kxy)] \supset (z)(Tz \supset {\sim}Mxz)\}$
11. $(x)\{\{Px \cdot (\exists y)[(Gy \cdot Uy) \cdot Ixy]\} \supset (z)(Sz \supset {\sim}Txz)\}$
12. $(x)\{[Px \cdot (y){\sim}Vxy] \supset (z){\sim}Gxz\}$

This exercise is adapted from Copi, *Symbolic Logic*, 5th ed., MacMillan Publ., 1979.

IX. **Solutions**

Answers to Exercises A:

1. Ljm
2. $\sim Stn$
3. $Ipme$
4. $Tcam$

Answers to Exercises B:

1. $(x)[Px \supset (\exists y)Lxy]$
2. $(x)[Px \supset (\exists y)\sim Kxy]$ or $\sim(\exists x)[Px \cdot (y)Kxy]$
3. $(x)[Px \supset (\exists y)(Py \cdot \sim Kxy)]$ or $\sim(\exists x)[Px \cdot (y)(Py \supset Kxy]$
4. $(x)[Wx \supset (\exists y)(My \cdot Sxy)]$
5. $\sim(\exists x)[Cx \cdot (\exists y)(Hy \cdot Sxy)]$ or $(x)[Cx \supset (y)(Hy \supset \sim Sxy)]$
6. $(x)[(Dx \cdot Mx) \supset \sim(\exists y)(Ty \cdot Txy)]$
7. $(\exists x)(Cx \cdot Bnxw)$
8. $(x)[Px \supset (\exists y)(\exists z)(Pz \cdot Gyxz)]$
9. $(x)\{(\sim Ax \cdot Lx) \supset (y)[(Ay \cdot Dy) \supset Dxy]\}$
10. $(x)[(Lx \cdot Pxx) \supset (\exists y)(Fy \cdot Cxy)]$ or $(x)[(Lx \cdot Pxx) \supset Fx]$

Note that these two translations aren't equivalent.
The first translates the surface grammar.
The second translates the meaning.

Answers to Exercises D:

1. Every dog has its day.
2. What's fair for one is fair for all.
3. Rolling stones gather no moss.
4. All things come to those who wait.
5. God helps those who help themselves.
6. There's no place like home.
7. Every cloud has a silver lining.
8. A person is judged by the company he keeps. (But note the error in meaning, here.)
9. Where there's smoke, there's fire.
10. A jack of all trades is a master of none.
11. People who live in glass houses shouldn't throw stones.
12. Nothing ventured, nothing gained.